1. Note that if two sets are disjoint, then their union is the same as their symmetric difference. Let \( x \) and \( y \) be two sets. Then \( x \setminus y = x \cap (x \Delta y) \), and \( x \cup y \) is the disjoint union of \( x \setminus y \) and \( y \), so

\[
x \cup y = y \Delta (x \cap (x \Delta y))
\]

2. Recall that we have seen that each \( n \in \omega \) is a transitive set, \( n \notin n \), and \( < \) is a total ordering of \( \omega \).

   (a) If \( n < m \) then \( n \in m \) so by transitivity we have \( n \subseteq m \), and since we must have \( n \neq m \), we have \( n \not\subseteq m \). Conversely, if \( n \subseteq m \), we know \( n \neq m \), so we must either have \( n < m \) or \( m < n \). We can not have \( m < n \), since then \( m \subseteq n \), so we must have \( n < m \).

   (b) We can prove this by induction. It clearly holds for \( n = 0 = \emptyset \), so suppose we have \( n = \{m : m < n\} \) for some \( n \). Then \( n + 1 = n \cup \{n\} = \{m : m < n\} \cup \{n\} = \{m : m \leq n\} = \{m : m < n + 1\} \) as we wish.

3. Note that each \( V_n \) is transitive, as clearly \( \emptyset \) is, and if a set \( x \) is transitive then so is \( \varphi(x) \). Also, the three properties below are vacuously true of \( V_0 = \emptyset \).

   (a) Suppose \( x \in V_{n+1} \), so \( x \subseteq V_n \). We have \( y \in \cup x \) iff there is a \( z \in x \) with \( y \in z \). Such a \( z \) is then an element of \( V_n \); since \( V_n \) is transitive then \( y \in V_n \) as well. Hence \( \cup x \subseteq V_n \), so \( y \in \varphi(V_n) = V_{n+1} \).

   (b) If \( x, y \in V_n \) then \( \{x, y\} \subseteq V_n \) and hence is in \( \varphi(V_n) = V_{n+1} \).

   (c) Suppose \( x \in V_{n+1} \). Then \( x \subseteq V_n \), so \( y \subseteq V_n \) and hence \( y \in \varphi(V_n) = V_{n+1} \).

   (d) Extensionality is clear since we are dealing with pure sets. Pairing holds since for any two sets \( x \) and \( y \) in \( V_\omega \), there is some \( V_n \) containing them both, and hence their pair is in \( V_{n+1} \) by part (b) and thus in \( V_\omega \). Union is similar using part (a). Comprehension follows from part (c). Power set follows, since if \( x \in V_n \) then \( x \subseteq V_n \) and hence \( \varphi(x) \subseteq \varphi(V_n) = V_{n+1} \) so \( \varphi(x) \in \varphi(V_{n+1}) = V_{n+2} \).

4. Suppose \( Z \) proved \((*)\). Take \( x \) to be \( V_\omega \), so that for each \( n, f(n) = V_{\omega+n} \). But then \( Z \) proves the existence of the set \( \text{range}(f) = \{V_{\omega+n} : n \in \omega\} \) and hence of the set \( \cup\{V_{\omega+n} : n \in \omega\} = V_{\omega+\omega} \).

As we saw in class, \( Z \) can not prove the existence of this set, since it is a model of \( Z \).

This doesn’t violate the principle of definition by induction, because that depends on having a \textit{function} \( g \) such that \( f(n + 1) = g(f(n)) \), and the power set operation is not a function (its domain is a proper class). We can show that \( \varphi \) restricted to any set \textit{is} a function, but the set we would need here is the set \( V_{\omega+\omega} \) itself.