Oleg Ivrii

Research Statement

I enjoy solving problems in complex analysis by exploiting connections with dynamical systems, probability and non-linear partial differential equations. I began my mathematical career by studying an analogue of the Weil-Petersson metric in complex dynamics proposed by C. McMullen. The metric is naturally related to dimensions of quasicircles and to integral means of univalent functions. One can study the Weil-Petersson metric in two essentially different ways:

1. **Dynamics perspective.** One can fix the degree and study metric completions of particular parameter spaces. In this approach, the Weil-Petersson metric illuminates various structures and symmetries in complex dynamics involving pinching deformations, rescaling limits and parabolic implosion.

2. **Complex analysis perspective.** Alternatively, one can let the degree go to infinity and use fractal approximation techniques to solve non-dynamical problems using dynamical methods. Building on my work with K. Astala, A. Perälä and I. Prause and using a recent estimate of H. Hedenmalm, I was able to improve Smirnov's $1 + k^2$ bound for the dimension of a k-quasicircle, which was widely expected to be sharp. Together with I. Kayumov, we applied these ideas to the study of Makarov's law of the iterated logarithm which describes the fine-scale geometry of harmonic measure. I also exploited an analogy with the Feynman-Kac formula from the theory of stochastic processes to give an estimate for the dimension of sparse quasicircles.

Recently, I started working on an unrelated project where I try to understand inner functions in terms of their critical points. My main result is a surprising characterization of the space of inner functions of finite entropy, answering a question posed by K. Dyakonov. Although the problem originates in complex analysis, by a classical theorem of Liouville on conformal metrics of constant negative curvature, it is equivalent to a question in non-linear elliptic PDE that has not been studied before: to describe the *nearly-maximal* solutions of the Gauss curvature equation $\Delta u = e^{2u} + \nu$ on the unit disk with a measure-valued singularity. In a sequel to this work, I gave a concrete description of the natural topology on the space of inner functions of finite entropy (equivalently, nearly-maximal solutions of the Gauss curvature equation) that is obtained by deforming the critical structure (the measure ν). In the future, my hope is to parameterize the space of all inner functions by exhibiting a bijection with certain shift-invariant subspaces of a weighted Bergman space.

1 Describing Blaschke products by their critical sets

It is well known that a complex polynomial $p(z)$ is a proper holomorphic self-map of the complex plane. It is possible to define a finite Blaschke product $F(z)$ as a proper holomorphic self-map of the unit disk. Due to the similiarity in the definitions, it is reasonable to expect that polynomials and finite Blaschke products share many similar properties. This is indeed the case. Just to give one example, the Gauss-Lucas theorem asserts that the critical points of a polynomial (the zeros of the derivative) lie in the convex hull of the zeros, while Walsh showed that the critical points of a Blaschke product lie in the *hyperbolic* convex hull of the zeros.

The most common way to study Blaschke products is by examining their zero sets. It is not difficult to show that a finite Blaschke product $F(z)$ is uniquely determined by its zero set up to a rotation:

$$F(z) = e^{i\psi} \prod_{i=1}^{d} \frac{z - a_i}{1 - \overline{a_i} z}, \qquad a_1, a_2, \dots, a_d \in \mathbb{D},$$

where $d \geq 1$ is the degree of F. This approach allows one to factor zeros of bounded analytic functions and leads to Beurling's invariant subspace theorem, which is the cornerstone of modern complex analysis and function theory.

My work takes a less traveled path of examining the *critical sets* of Blaschke products, initiated by Heins in 1962.

Theorem 1 (Heins)**.** *A finite Blaschke product is uniquely determined by the set of its critical points up to post-composition with a Möbius transformation* $m \in \mathrm{Aut}(\mathbb{D})$*, and furthermore, any set of* $d-1$ *points in the unit disk arises as the critical set of some Blaschke product of degree* d*.*

The most natural analogue of a finite Blaschke product in infinite degree is an inner function, that is, a holomorphic self-map of the unit disk such that for almost every $\theta \in [0, 2\pi)$, the radial limit $\lim_{r\to 1} F(re^{i\theta})$ exists and has absolute value 1.

If one wants to generalize Heins' result to inner functions, one is confronted with the following obstacle: different inner functions can have the same critical set. For example,

$$F_1(z) = z, \qquad F_2(z) = \exp\left(\frac{z+1}{z-1}\right)$$

have no critical points. In order to distinguish F_1 and F_2, one must record some additional information. A natural setting where one can do this is in the realm of inner functions of finite entropy.

1.1 Inner function of finite entropy

Let Inn denote the space of all inner functions and $\mathscr{J} \subset \mathrm{Inn}$ be the collection of inner functions which satisfy

$$\lim_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} \log^+ |F'(re^{i\theta})| d\theta < \infty, \tag{1}$$

that is, with F' in the Nevanlinna class. The work of Ahern and Clark [AC74] implies that if $F \in \mathscr{J}$, then F' admits an "inner-outer" decomposition

$$F' = \mathrm{Inn}\, F' \cdot \mathrm{Out}\, F' = BSO.$$

Intuitively, $\mathrm{Inn}\, F' = BS$ describes the "critical structure" of the map F – the Blaschke factor records the locations of the critical points of F in the unit disk, while the singular inner factor describes the "boundary critical structure."

The main result of [Ivr18b] says that an inner function $F \in \mathscr{J}$ is uniquely determined up to a post-composition with a holomorphic automorphism of the disk by its critical structure and describes all possible critical structures of inner functions:

Theorem 2. *Let $\mathscr{J}$ be the set of inner functions whose derivative lies in the Nevanlinna class. The natural map*

$$F \to \mathrm{Inn}(F') \quad : \quad \mathscr{J} / \mathrm{Aut}(\mathbb{D}) \to \mathrm{Inn} / \mathbb{S}^1$$

is injective. The image consists of all inner functions of the form BS_μ where B is a Blaschke product and S_μ is the singular factor associated to a measure μ whose support is contained in a countable union of Beurling-Carleson sets.

Here, a *Beurling-Carleson set* is a closed subset of the unit circle of zero Lebesgue measure whose complement is a union of arcs $\bigcup_k I_k$ with $\sum |I_k| \log \frac{1}{|I_k|} < \infty$.

Theorem 2 answers a question posed by K. Dyakonov [Dya14, Dya15]. In [Dya14], Dyakonov showed that $\mathrm{Inn}\, F'$ is trivial if and only if F is a Möbius transformation. After reading Dyakonov's work, I noticed that a theorem of D. Kraus [Kra13] is equivalent to the statement "$F \to \mathrm{Inn}\, F'$ is a bijection from Maximal Blaschke Products in $\mathscr{J}$ to the space of all Blaschke Products." Understanding the role of singular factors introduces many new challenges since the mapping $F \to \mathrm{Inn}\, F'$ is not surjective.

1.2 Connections with the Gauss curvature equation

Given a conformal pseudometric $\lambda(z)|dz|$ on the unit disk with an upper semicontinuous density, its *Gaussian curvature* is given by

$$k_\lambda = -\frac{\Delta \log \lambda}{\lambda^2},$$

where the Laplacian is taken in the sense of distributions. It is well known that the Poincaré metric $\lambda_{\mathbb{D}}(z) = \frac{2}{1-|z|^2}$ has constant curvature -1. For a holomorphic self-map of the unit disk $F \in \mathrm{Hol}(\mathbb{D}, \mathbb{D})$, consider the pullback

$$\lambda_F := F^* \lambda_{\mathbb{D}} = \frac{2|F'|}{1 - |F|^2}.$$

Since curvature is a conformal invariant, $k_{\lambda_F} = -1$ on $\mathbb{D} \setminus \mathrm{crit}(F)$ where $\mathrm{crit}(F)$ denotes the critical set of F. However, on the critical set, $\lambda_F = 0$ while its curvature has δ-masses. Introducing the change of variables $u_F = \log \lambda_F$, we naturally arrive at the PDE

$$\Delta u = e^{2u} + 2\pi\nu, \qquad \nu \geq 0, \tag{2}$$

where $\nu = \sum_{c \in \mathrm{crit}(F)} \delta_c$ is an integral sum of point masses.

A theorem of Liouville [KR13, Theorem 5.1] states that the correspondence $F \to u_F$ is a bijection between

$$\mathrm{Hol}(\mathbb{D}, \mathbb{D}) \,/\, \mathrm{Aut}(\mathbb{D}) \quad \Longleftrightarrow \quad \big\{\text{solutions of (2) with } \nu \text{ integral}\big\}.$$

Liouville's theorem allows one to translate questions about holomorphic self-maps of the disk to problems in PDE. It turns out that the question of describing inner functions with derivative in the Nevanlinna class is related to studying the Gauss curvature equation with *nearly-maximal* boundary values

$$\begin{cases} \Delta u = e^{2u} + 2\pi\nu, & \text{in } \mathbb{D}, \\ u_{\mathbb{D}} - u = \mu, & \text{on } \mathbb{S}^1, \end{cases} \tag{3}$$

where $u_{\mathbb{D}} = \log \lambda_{\mathbb{D}}$ is the pointwise *maximal* solution of (2) in the sense that it dominates every solution of (2) with any $\nu \geq 0$. The maximality of $u_{\mathbb{D}}$ is just Ahlfors' restatement of the Schwarz lemma.

In (3), we may allow ν to be any positive measure on the unit disk which satisfies the *Blaschke condition*

$$\int_{\mathbb{D}} (1 - |z|) d\nu(z) < \infty, \tag{4}$$

and μ to be any finite positive measure on the unit circle. The first equality in (3) is understood weakly in the sense of distributions while the second equality expresses the fact that the measures $(u_{\mathbb{D}} - u)(d\theta/2\pi)|_{\{|z|=r\}}$ converge weakly to μ as $r \to 1$.

In this language, Theorem 2 states:

Theorem 3. *Let (μ, ν) be a pair of measures as above. If μ is supported on a countable union of Beurling-Carleson sets, then the Gauss curvature equation (3) admits a unique solution. Otherwise, no solution exists.*

We may endow $\mathscr{J}$ with the topology of *stable convergence* where $F_n \to F$ if the the critical structures of the F_n converge to that of F. The main result of [Ivr18c] gives a concrete description of this topology. Roughly speaking, it says that $F_n \to F$ if the critical structures of the F_n are uniformly concentrated on Korenblum stars – a Korenblum star $K_E \subset \overline{\mathbb{D}}$ is the union of Stolz angles emanating from a Beurling-Carleson set E.

1.3 On L^1 bounded solutions

It is interesting to compare Dyakonov's question with another problem in non-linear elliptic PDE involving measure-valued singularities. For this purpose, we slightly tweak the PDE and move to the unit ball in $\mathbb{B} \subset \mathbb{R}^N$, $N \ge 2$:

$$\Delta u = |u|^{q-1}u, \qquad u : \mathbb{B} \to \mathbb{R}, \quad q > 1.$$

We say that u is an L^1 *bounded solution* if

$$\limsup_{r \to 1} \int_{\mathbb{B}} |u(r\xi)| d\sigma_\xi < \infty.$$

Taking the weak limit of $u(r\xi)\, d\sigma$ as $r \to 1$, one obtains an embedding of L^1 bounded solutions into measures on the unit sphere. It is natural to ask which measure occur. The answer depends on the parameter q.

Theorem 4. *(A. Gmira & L. Véron, 1991) In the* **subcritical case**, *$q < q_c = \frac{N+1}{N-1}$, every measure is the boundary measure of some L^1 bounded solution.*

Theorem 5. *(J. F. Le Gall, $q = 2$, 1993, E. B. Dynkin & S. E. Kuznestov, $q_c \le q \le 2$, 1996, M. Marcus & L. Véron, $q > 2$, 1998.) In the* **supercritical case**, *$q \ge q_c$, a measure is the boundary measure of an L^1 bounded solution if and only if it is diffuse with respect to the Sobolev capacity* $\mathrm{cap}_{W^{2/q,q'}}$.

Note that in the above problem, a measure occurs if it is sufficiently diffuse, while in Dyakonov's question, a measure occurs if it is sufficiently concentrated.

1.4 Invariant subspaces of Bergman space

For a fixed $\alpha > -1$ and $1 \le p < \infty$, consider the weighted Bergman space $A^p_\alpha(\mathbb{D})$ which consists of holomorphic functions on the unit disk satisfying the norm boundedness condition

$$\|f\|_{A^p_\alpha} = \left(\int_{\mathbb{D}} |f(z)|^p \cdot (1 - |z|)^\alpha |dz|^2 \right)^{1/p} < \infty. \tag{5}$$

For a function $f \in A^p_\alpha$, let $[f]$ denote the (closed) z-invariant subspace generated by f, that is the closure of the set $\{p(z)f(z)\}$, where $p(z)$ ranges over polynomials. In the work [Kor81], Korenblum equipped subspaces of $A^p_\alpha(\mathbb{D})$ with the *strong topology* where $X_n \to X$ if any $x \in X$ can be obtained as a limit of a converging sequence of $x_n \in X_n$ and visa versa.

We focus our attention on a small but important subclass of invariant subspaces which are generated by a single inner function (here, we mean a usual Hardy-inner function rather than a Bergman-inner function). Such subspaces are called subspaces of *κ-Beurling-type*. According to a classical theorem of Korenblum [Kor81] and Roberts [Rob85], the equality $[BS_{\mu_1}] = [BS_{\mu_2}]$ holds if and only if $\mu_1 - \mu_2$ does not charge Beurling-Carleson sets. Comparing with Theorem 2, we see that the subspaces of κ-Beurling-type are in bijection with elements of $\mathscr{J} / \operatorname{Aut}(\mathbb{D})$. The following theorem appears in [Ivr18c]:

Theorem 6. *Fix $\alpha > -1$ and $1 \le p < \infty$. The correspondence between $\mathscr{J} / \operatorname{Aut}(\mathbb{D})$ and the collection of subspaces of κ-Beurling-type is a homeomorphism.*

1.5 Directions for further work

I will briefly mention three promising directions that I have been thinking about:

1. It would be desirable to parametrize all inner functions by their critical structures, not just the relatively small subset which have finite entropy. The starting point is a very elegant result of D. Kraus [Kra13] which states that critical sets of Blaschke products coincide with zero sets of functions in the weighted Bergman space A^2_1. Therefore, it makes sense to look for a bijection between $\operatorname{Inn} / \operatorname{Aut}(\mathbb{D})$ and certain invariant subspaces of A^2_1 which admit approximate spectral synthesis.

2. L. Brown and A. Shields [BS91] asked if one can describe the cyclic inner functions in the Bloch space when equipped with the weak-$*$ topology. Since the Bloch space is the dual of the Bergman space A^1, this can be viewed as an endpoint version of the Korenblum-Roberts theorem.

3. The work of Korenblum is based on his analysis of the space $A^{-\infty}$ which consists of holomorphic functions $f(z)$ on the unit disk which satisfy a growth bound of the form $|f(z)| \le C(1-|z|)^{-N}$ for some $N > 0$. In a series of two *Acta* papers [Kor75, Kor77], Korenblum classified the zero sets of $A^{-\infty}$ functions and the z-invariant subspaces of $A^{-\infty}$ (zero structures of $A^{-\infty}$ functions). Even though these results are of central importance in the theory of Bergman spaces, Korenblum's proofs are very complicated and not constructive. It would be desirable to find simple constructive proofs using the methods of Roberts [Rob85].

2 Definition of the Weil-Petersson metric

Let $\mathcal{T}_g$ denote the Teichmüller space of Riemann surfaces of genus g and $\mathcal{B}_d$ be the space of degree d Blaschke products that have an attracting fixed point at the origin:

$$z \to f_{\mathbf{a}}(z) = z \cdot \prod_{i=1}^{d-1} \frac{z + a_i}{1 + \overline{a_i} z}, \qquad a_1, a_2, \ldots, a_{d-1} \in \mathbb{D}. \tag{6}$$

Suppose μ is a **dynamical Beltrami coefficient** on the unit disk, invariant under either a co-compact Fuchsian group or a Blaschke product. In this case, it represents a (possibly trivial) tangent vector in $T_X\mathcal{T}_g$ or $\mathcal{T}_f\mathcal{B}_d$. Without loss of generality, we can assume that $\|\mu\|_\infty \le 1$. By solving the Beltrami equation $\overline{\partial}\varphi_t = t\mu\,\partial\varphi_t$, we obtain a holomorphic family of conformal maps

$$\varphi_t : \mathbb{D}^* \to \mathbb{C}, \qquad \varphi_0(z) = z, \qquad \text{where } \mathbb{D}^* = \{z : |z| > 1\}.$$

For these special families, McMullen [McM08] showed the remarkable formula

$$2\frac{d^2}{dt^2}\bigg|_{t=0} \mathrm{H.}\dim \varphi_t(\mathbb{S}^1) = \sigma^2(\mathcal{S}\mu), \tag{7}$$

where

$$\sigma^2(b) = \frac{1}{2\pi} \limsup_{R \to 1^+} \frac{1}{|\log(R-1)|} \int_{|z|=R} |b(z)|^2 \, |dz| \tag{8}$$

is the *asymptotic variance* of a Bloch function on $\mathbb{D}^*$ and

$$\mathcal{S}\mu = -\frac{1}{\pi} \int_{\mathbb{D}} \frac{\mu(\zeta)}{(\zeta - z)^2} |d\zeta|^2 \tag{9}$$

is the Beurling transform of μ. It is said that every great theorem in mathematics tends to become a definition. In the Fuchsian case, McMullen observed that

$$\sigma^2(\mathcal{S}\mu) = (8/3)\|\mu\|^2_{\mathrm{WP}} \tag{10}$$

is a constant multiple of the Weil-Petersson metric on Teichmüller space. In view of this, it is natural to take (7) as the definition of the Weil-Petersson metric on $\mathcal{B}_d$.

Remark. (i) Since μ is supported on the unit disk, $\mathcal{S}\mu$ is a holomorphic function on $\mathbb{D}^*$. Furthermore, it is easy to see that is satisfies a Bloch bound of the form

$$\|\mathcal{S}\mu\|_{\mathcal{B}^*} := \sup_{z \in \mathbb{D}^*} (|z|^2 - 1)|(\mathcal{S}\mu)'(z)| < \infty.$$

This ensures us that the asymptotic variance is finite, e.g. $\sigma^2(b) \le \|b\|^2_{\mathcal{B}_*}$.

(ii) For arbitrary families of conformal maps (that is, for general μ, not necessarily invariant), McMullen's identity need not hold as simple examples demonstrate.

(iii) An astute reader may observe that invariant Beltrami coefficients do not exhaust all tangent vectors at the parameters in $\mathcal{B}_d$ that have critical relations. One can fix this by considering *eventually-invariant* Beltrami coefficients, satisfying $f^*\mu = \mu$ in some neighbourhood of the unit circle.

3 Dimensions of quasicircles

A well-known problem in quasiconformal geometry is to find $D(k)$, the maximal Hausdorff dimension of a k-quasicircle, the image of the unit circle under a k-quasiconformal mapping of the plane. The first non-trivial bound (with the right growth rate) was given by Becker and Pommerenke [BP87] who showed that $1 + 0.36\,k^2 \le D(k) \le 1 + 37\,k^2$ if k is small. In his landmark work [Ast94] on the area distortion of quasiconformal mappings, K. Astala suggested that the correct bound was

$$D(k) \le 1 + k^2, \qquad 0 \le k < 1. \tag{11}$$

Using a clever variation of Astala's argument, S. Smirnov [Smi10] showed that (11) indeed holds but could not decide if it was sharp. Since the sharpness of (11) seemed to fit nicely together with a number of conjectures on conformal mappings, the general consensus of mathematicians in the field was that it was likely to be true.

Motivated by the connections to the Weil-Petersson metric, in [AIPP15], we introduced the quantity

$$\Sigma^2 := \sup_{|\mu| \le \chi_{\mathbb{D}}} \sigma^2(\mathcal{S}\mu), \tag{12}$$

taken over all Beltrami coefficients on the unit disk, invariant or otherwise. One of our results was the following estimate:

Theorem 7. [AIPP15]

$$1 \,\ge\, \liminf_{k \to 0} \frac{D(k) - 1}{k^2} \,\ge\, \Sigma^2 \,\ge\, 0.879. \tag{13}$$

Upper bound. In [AIPP15], we gave two different proofs of the upper bound in (13). The first approach used **complex interpolation**, similar to Smirnov's original argument. The second route was via **fractal approximation**. I will outline the second approach since it more closely follows my interests.

Let M_{I} denote the collection of Beltrami coefficients on the exterior unit disk which are eventually-invariant under $z \to z^d$ for some $d \ge 2$, i.e. satisfying $(z^d)^*\mu = \mu$ in some neighbourhood of the unit circle.

Theorem 8. [AIPP15]

$$\Sigma^2 = \sup_{\mu \in M_I, |\mu| \le \chi_{\mathbb{D}}} \sigma^2(\mathcal{S}\mu) \tag{14}$$

Since McMullen's identity holds for Beltrami coefficients in M_{I}, Smirnov's bound immediately gives $\sigma^2(\mathcal{S}\mu) \le 1$.

Somewhat surprisingly, it turns out that **Fuchsian coefficients are insufficient for this purpose**: if μ is a Beltrami coefficient invariant under a co-compact Fuchsian group, then classical duality arguments comparing the Weil-Petersson and Teichmüller metrics on $\mathcal{T}_g$ show that

$$\sigma^2(\mathcal{S}\mu) \le \frac{2}{3}\frac{\|\mu\|^2_{\mathrm{WP}}}{X} \le 2/3.$$

which is clearly less than 0.879. (In [AIPP15], we showed the strict inequality $\Sigma_F^2 < 2/3$.)

Remark. The reason fractal approximation fails in the Fuchsian case is because in complex dynamics one uses the metric $|dz|^2/(1-|z|^2)$ which satisfies a certain isoperimetric property that the hyperbolic metric $|dz|^2/(1-|z|^2)^2$ does not.

Lower bound. For the lower bound on Σ^2, we observed that the Julia sets of $P_t(z) = z^{20}+tz$ are quasicircles with $\mathrm{H.dim}\,\mathcal{J}(P_t) = 1+0.301\,t^2$ when t is small. Using the λ-lemma, it is easy to represent $\mathcal{J}(P_t)$ as $k(t)$-quasicircles with $k(t) = |t|$. Somewhat surprisingly, one can obtain a more efficient estimate $k(t) \approx 0.585|t|$ on the quasiconformal distortion using an explicit construction, which translates to

$$\mathrm{H.dim}\,\mathcal{J}(P_t) = 1 + 0.879\,k(t)^2.$$

The previous record for the maximal dimension of a k-quasicircle was $1 + 0.69k^2$, for k small [ARS].

Our hope was to prove the sharpness of Smirnov's bound by showing that $\Sigma^2 = 1$. After reading our manuscript, H. Hedenmalm surprised us by proving that actually $\Sigma^2 < 1$ [Hed17]. While Hedenmalm's estimate suggests that Smirnov's bound is not sharp, by itself, it is not conclusive. There were two difficulties:

1. Can $D(k)$ be computed by taking the supremum of the Hausdorff dimensions of "dynamical" k-quasicircles?

2. Even if $\Sigma^2 < 1$, a priori the equation

$$2(d^2/dt^2)|_{t=0}\,\mathrm{H.dim}\, w^{t\mu}(\mathbb{S}^1) = \Sigma^2,$$

which is valid for dynamical k-quasicircles, may not survive after taking the supremum over all Beltrami coefficients $\mu \in M_{\mathrm{I}}$ with $|\mu| \le \chi_{\mathbb{D}}$.

In [Ivr16], I was able to resolve these difficulties by exploiting connections with the theory of conformal mappings:

Theorem 9. [Ivr16]

$$D(k) = 1 + \Sigma^2 k^2 + \mathcal{O}(k^{8/3-\varepsilon}). \tag{15}$$

Taken together the trilogy [AIPP15, Hed17, Ivr16] shows that Smirnov's dimension bound is not sharp, at least when k is small. István Prause informed me (private communication) that this implies that $D(k) < 1 + k^2$ for all $0 < k < 1$.

It would be desirable to understand the structure of extremals in (12). Hedenmalm [Hed17] found an interesting connection between this problem and Abrikosov's conjecture from superconductivity where the extremal is expected to have invariance properties with respect to the hexagonal lattice. Thus one would expect that even though the extremal in (12) is far from unique (since it can be arbitrarily modified on a small set), its large scale behaviour should be essentially unique, e.g. it should be almost periodic with respect to some hyperbolic lattice. In [Ivr17], I showed that any extremal of (12) satisfies a central limit theorem, which gives some plausibility to this conjecture.

3.1 Connections with conformal mappings

The Riemann mapping theorem states that any simply-connected domain $\Omega \subset \mathbb{C}$ is the image of a conformal map $f : \mathbb{D}^* \to \Omega$ (unless $\Omega = \mathbb{C}$ itself). Since simply-connected domains can be very wild, it is reasonable to expect that the complexity of the boundary $\partial\Omega$ is manifested in the complexity of the Riemann map. For domains with rough boundaries, the relationship between f and $\partial\Omega$ is typically quantified by the *integral means spectrum*

$$\beta_f(p) = \limsup_{r\to 1} \frac{\log \int_{|z|=r} |f'(z)|^p d\theta}{\log \frac{1}{R-1}}, \qquad p \in \mathbb{C}.$$

The importance of $\beta_f(p)$ lies in the fact that is the Legendre-dual to the multifractal spectrum of harmonic measure [Mak99, Bin08]. Taking the supremum of $\beta_f(p)$ over bounded simply-connected domains, one obtains the *universal integral means spectrum*

$$B(p) = \beta_f(p).$$

It is clear from Hölder's inequality that $B(p)$ is a convex function with a minimum at $B(0) = 0$. Even though $B(p)$ is a central object in geometric function theory, apart from various estimates [HS08, Jon05], not much is rigorously known about its qualitative features, for instance whether or not $B(p)$ is differentiable.

More generally, one can consider the spectra $B_k(p) := \sup_{\varphi \in \mathbf{\Sigma}_k} \beta_\varphi(p)$, where we maximize over the collection of conformal mappings of the exterior unit disk, normalized so that $\varphi(z) = z + \mathcal{O}(1/z)$ near infinity, that admit k-quasiconformal extensions to the complex plane with dilatation at most k. The main results of [Ivr16, IK17] describe an equality of universal characteristics of conformal maps:

Theorem 10. *(Global version)*

$$\Sigma_k^2 \;=\; \lim_{p \to 0} \frac{B_k(p)}{|p|^2/4} \;=\; \sup_{\varphi \in \mathbf{\Sigma}_k} \sigma^2(\log \varphi') = \sup_{\varphi \in \mathbf{\Sigma}_k} C^2_{\mathrm{LIL}}(\log \varphi'). \tag{16}$$

(Infinitesimal version)

$$\Sigma^2 \;=\; \lim_{p \to 0, kp \to 0} \frac{B_k(p)}{|p|^2/4} \;=\; \sup_{|\mu| \le \chi_{\mathbb{D}}} \sigma^2(\mathcal{S}\mu) = \sup_{|\mu| \le \chi_{\mathbb{D}}} C^2_{\mathrm{LIL}}(\mathcal{S}\mu). \tag{17}$$

The final characteristic $C_{\mathrm{LIL}}(b)$ in the above theorem is the *constant in Makarov's law of the iterated logarithm* defined as the essential supremum of

$$C_{\mathrm{LIL}}(b, \theta) = \limsup_{r \to 1} \frac{|b(re^{i\theta})|}{\sqrt{\log \frac{1}{1-r} \log \log \log \frac{1}{1-r}}}$$

over $\theta \in [0, 2\pi)$.

Remark. To see why (17) is the infinitesimal form of (16), observe that in the case of a holomorphic family of quasiconformal maps $\{H_t, t \in \mathbb{D}\}$, with $\overline{\partial} H_t = t\mu\, \partial H_t$ and $|\mu| \le \chi_{\mathbb{D}}$, $(d/dt)\big|_{t=0} \log H_t' = \mathcal{S}\mu$ for $z \in \mathbb{D}^*$.

In light of Hedenmalm's estimate, Theorem 10 contradicts the conjecture "$B_k(p) = k^2p^2/4$ for all $k \in [0, 1)$ and $p \in [-2/k, 2/k]$ from [Jon05, PS11]. However, since we do not know whether $\lim_{k \to 1} \Sigma^2(k) =_? 1$, we cannot rigorously rule out Kraetzer's conjecture which asserts the more limited statement that "$B(p) = p^2/4$ for all $p \in [-2, 2]$." It is currently known that

$$0.93 < \lim_{k \to 1^-} \Sigma^2(k) < (1.24)^2.$$

The lower lower bound was given in [AIPP15] while the upper bound is due to [HK07, Corollary 2.3]. The number 0.93 is a significant improvement over the previously best known bound $(0.91)^2$ established in [HK07]. In addition, I believe that 0.93 is close to optimal which makes it likely that Kraetzer's conjecture is also false.

3.2 Hyperbolic Brownian motion

In [Lyo90], T. Lyons gave a simple four page proof of Makarov's celebrated law of the iterated logarithm by using hyperbolic Brownian motion to explore conformal mappings. Lyons' argument has the advantage that it is *intrinsic*, for instance, Makarov's approach with dyadic martingales forces the reader to make a non-canonical choice of a dyadic structure on the unit circle. In [Ivr18a], I adapted Lyons' approach to study integral means of conformal mappings, and this led me to study the Feynman-Kac formula. As an application, I was able to estimate the Minkowski dimension of a "sparse" quasicircle, that is, the image of the unit circle under a k-quasiconformal mapping of the plane for which the support of the dilatation is very spread out. (In my paper, I considered the case when the support is contained in the union of disjoint horoballs, any two of which are at least a hyperbolic distance R apart. However, the argument also works for other models of sparse coefficients.) My estimate is sharp up to determining the multiplicative constant: $\mathrm{H.}\dim w^{\mu}(\mathbb{S}^1) \le 1 + Ck^2 e^{-R/2}$ (for small $k > 0$).

4 A glimpse of my thesis

Perhaps the most famous object in complex dynamics is the Mandelbrot set $\mathcal{M}$. It is the set of parameters $c \in \mathbb{C}$ for which the Julia set of $z^2 + c$ is connected. I am interested in one component of the interior of the Mandelbrot set: the main cardioid. It consists of parameters c for which the Julia set of $z^2 + c$ has an attracting fixed point. More precisely, I am interested in understanding the main cardioid by studying **dynamically-intrinsic metrics**, such as the *Weil-Petersson metric*.

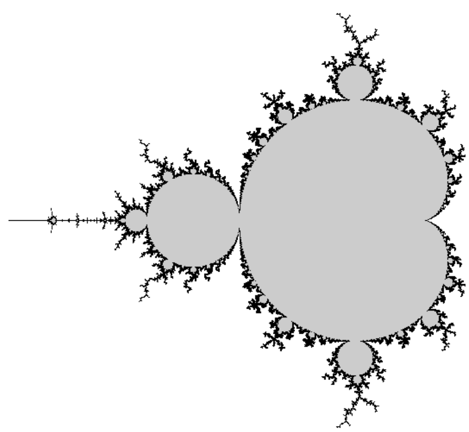

Figure 1: The Mandelbrot set.

The main result of my PhD thesis is:

Theorem 11. *The main cardioid is incomplete in the Weil-Petersson metric. Its completion contains the geometrically finite parameters from the Euclidean boundary (the cusp of the main cardioid and the roots at which the adjacent components are attached).*

More precisely, I proved that in a neighbourhood of the cusp and each root, the Weil-Petersson metric is **bi-Lipschitz equivalent to an incomplete cone**.

Conjecture. *Let $\mathscr{P}_d$ be the space of polynomials of degree $d \geq 3$. The completion of the Weil-Petersson metric on the main cardioid of $\mathscr{P}_d$ attaches the geometrically finite parameters on the Euclidean boundary.*

The space of geometrically finite parameters on the boundary of the main cardioid is quite complicated: it is a stratified space of dimension $(d-2)$. One should think of it as an analogue of the **Deligne-Mumford compactification** in complex dynamics. At the moment, I can rigorously identify only some of the strata in the completion.

While one visualizes the completion of the Weil-Petersson metric by adding points to the boundary of the main cardioid, in reality, one studies the dynamics of Blaschke products. The connection between Blaschke products and polynomials is given by the Bers embedding $\mathcal{B}_d \to \mathscr{P}_d$ which takes a Blaschke product in $\mathcal{B}_d$ and mates it with $z \to z^d$. For instance, when $d = 2$, the Bers embedding identifies $\mathcal{B}_2$ with the main cardioid.

4.1 Incompleteness of the Weil-Petersson metric

Recall that $\mathcal{B}_2$ is the space of Blaschke products of degree 2 with an attracting fixed point at the origin:

$$f_a(z) = z \cdot \frac{z+a}{1+\overline{a}z}, \qquad a \in \mathbb{D}.$$

Suppose f_a is a Blaschke product of degree 2, other than $z \to z^2$. Since the multiplier of the attracting fixed point is non-zero, the quotient of the disk by f_a is naturally a torus $T_{f_a} \in \mathcal{T}_{1,1}$ with one marked point (to take the quotient, we must puncture out the grand orbits of the attracting fixed point and the critical point). The work of McMullen and Sullivan identifies the tangent spaces $T_{f_a}\mathcal{B}_2^{\times} \cong T_{T_{f_a}}\mathcal{T}_{1,1} \cong M(T_{f_a})/\sim\; M(\mathbb{D})^{f_a}/\sim$.

Using the definition of the Weil-Petersson metric (10), it is not hard to show the following estimate:

Theorem 12. *If μ is an invariant Beltrami coefficient supported on the unit disk, then*

$$\|\mu\|^2_{\mathrm{WP}} \lesssim \|\mu\|^2_\infty \cdot \limsup_{r\to 1^-} \big|\operatorname{supp}\mu \cap \{z : |z| = r\}\big|.$$

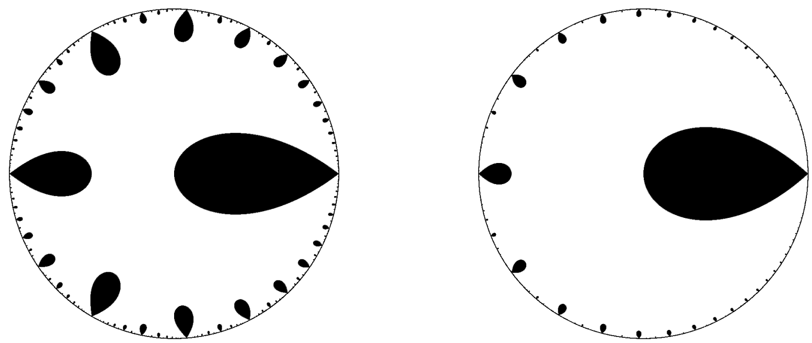

Figure 2: The supports of the half-optimal pinching coefficients for $a = 0.5$ and 0.8.

If one uses this estimate naively, one would see that the Weil-Petersson metric is bounded by a constant multiple of the hyperbolic metric. This is not terribly exciting since the hyperbolic metric is complete. To see that the Weil-Petersson length of the line segment $[0, 1)$ is finite, we instead use "half-optimal" pinching coefficients which take up half of the quotient torus at the attracting fixed point but are sparse near the unit circle. With this special choice of coefficients, the above theorem gives $\|\mu\|^2_{\mathrm{WP}} \lesssim \|\mu\|^2_\infty \cdot (1-a)^{1/2}$ which translates to the estimate $d\omega_{\mathrm{WP}} \le (1-a)^{-3/4}da$ as $a \to 1$ radially. The same argument shows that the Weil-Petersson length of any line segment $e(p/q) \cdot [0, 1)$ is finite.

4.2 Precise rate of decay

The find the precise rate of decay of the Weil-Petersson metric, we uses renewal theory to count the number of flowers which intersect a circle $\{z : |z| = r\}$ with r close to 1. In our context, the results of Lalley [Lal89] say that if $f \in \mathcal{B}_d$ is a Blaschke product other than $z \to z^d$ and $x \in \mathbb{S}^1$, the number $n(x, R)$ of repeated preimages y (i.e. points $y \in \mathbb{S}^1$ such that $f^{\circ n}(y) = x$ for some n) for which $|(f^{\circ n})'(y)| < R$ satisfies

$$n(x, R) \sim \frac{e^R}{\int \log |f'| dm}, \qquad \text{as} \quad R \to \infty. \tag{18}$$

One can extend formula (18) to points inside the disk using the identity

$$\log \frac{1}{|z|} = \sum_{f(w)=z} \log \frac{1}{|w|}. \tag{19}$$

Theorem 13. *Let $\mathcal{N}(z, R)$ be the number of repeated preimages of z that lie in the disk centered at the origin of hyperbolic radius R. Then,*

$$\mathcal{N}(z, R) \sim \frac{1}{2} \cdot \log \frac{1}{|z|} \cdot \frac{e^R}{\int \log |f'| dm}, \qquad \text{as } R \to \infty. \tag{20}$$

Remark. The expression in the denominator is called the *entropy* and it is responsible for the exponent in the rate of the decay of the Weil-Petersson metric. The analysis of the equidistribution of preimages is intimately connected with a certain object attached to a Blaschke product called a *Riemann surface lamination* which is locally a product of the unit disk and a cantor set.

4.3 Horocyclic degenerations

In degree 2, horocyclic degenerations are given by

$$f_a = z \cdot \frac{z+a}{1+\overline{a}z}, \qquad a = \Big((1-t) + t \cdot e^{i\theta}\Big) \cdot e(p/q) \text{ with } \theta \to 0.$$

One way to extract limits of $f_{a(\theta)}$ as $\theta \to 0$ is to rescale by a Möbius transformation to put the critical point at the origin: $\tilde{f}_a = m_{c\to 0} \circ f \circ m_{0\to c}$, where $m_{0\to c} = \frac{z+c}{1+\bar{c}z}$. Remarkably, A. Epstein [Eps00] showed that the maps $\tilde{f}_a^{\circ q}$ converge to parabolic Blaschke products of degree 2, independent of $q \geq 1$ (that is, if $q > 1$, the degree drops in the limit).

In light of Epstein's observation, one would expect that along horocyclic degenerations, the Weil-Petersson metric is asymptotically constant. However, this is not true: it is only asymptotically perioidic [Ivr15b]. This result is connected to the phenomenon of *parabolic implosion* which allows the Julia set to not vary continuously with respect to parameters. In fact, horocyclic degenerations give rise to what I call *Epstein-Lavaurs boundaries* that consists of non-degenerate parabolic Blaschke products endowed with a Lavaurs phase. Furthermore, the Lavaurs-Epstein boundaries attached to every $e(p/q)$ are isometric (carry the same limiting Weil-Petersson metric).

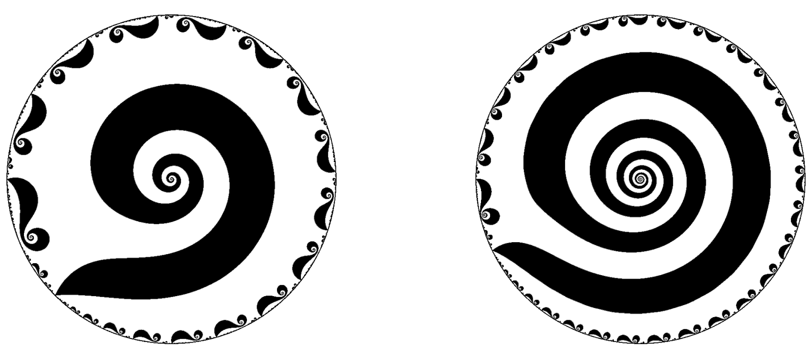

Figure 3: A horocyclic degeneration (above: $a = 0.5 + 0.5e^{i\theta}$ with $\theta = 2\pi/8$ and $2\pi/14$).

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