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Research Statement

My research interests lie in **complex analysis, ergodic theory and quasiconformal geometry**. The organizing theme of my work is the study of an analogue of the Weil-Petersson metric in complex dynamics proposed by C. McMullen. The metric is naturally related to dimensions of quasicircles and to integral means of univalent functions. One can study the Weil-Petersson metric in two essentially different ways:

1. **Dynamics perspective.** [Ivr1, Ivr2]

   One can *fix the degree* and study metric completions of particular parameter spaces. In this approach, the Weil-Petersson metric illuminates various structures and symmetries in complex dynamics involving pinching deformations, rescaling limits and parabolic implosion.

2. **Complex analysis perspective.** [AIPP, Ivr3, IK]

   Alternatively, one can *let the degree go to infinity* and use fractal approximation techniques to solve non-dynamical problems using dynamical methods. Building on previous work, joint with K. Astala, A. Perälä and I. Prause and using a recent estimate of H. Hedenmalm [H], I was able to improve Smirnov's $1 + k^2$ bound for the dimension of a k-quasicircle, which was widely expected to be sharp. Together with I. Kayumov, we applied these ideas to the study of Makarov's law of the iterated logarithm which describes the fine-scale geometry of harmonic measure.

1 An introduction to my work

Perhaps the most famous object in holomorphic dynamics is the Mandelbrot set $\mathcal{M}$. It is the set of parameters $c \in \mathbb{C}$ for which the Julia set of $z^2 + c$ is connected.

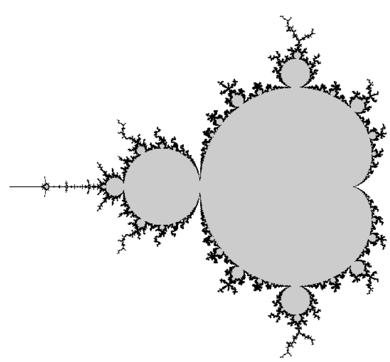

Fig. 1. The Mandelbrot set.

I am interested in one component of the interior of the Mandelbrot set: the main cardioid. It consists of parameters c for which the Julia set of z^2+c has an attracting fixed point. More precisely, I am interested in understanding the main cardioid by studying **dynamically-intrinsic metrics**, in particular, the *Weil-Petersson metric*, which will be defined in the next section. The central result of my PhD thesis is:

Theorem 1. *The main cardioid is incomplete in the Weil-Petersson metric. Its completion contains the geometrically finite parameters from the Euclidean boundary.*

In the picture above, we add the cusp of the main cardioid and the roots at which the adjacent components are attached. More precisely, I proved that in a neighbourhood of the cusp and each root, the Weil-Petersson metric is **bi-Lipschitz equivalent to an incomplete cone**.

Higher degree

Apart from the main cardioid in the classical Mandelbrot set, one has "main cardioids" in other parameter spaces. The most natural generalization is to polynomials of higher degree $\mathscr{P}_d$.

Conjecture. *The completion of the Weil-Petersson metric on the main cardioid of* $\mathscr{P}_d$ *attaches the geometrically finite parameters on the Euclidean boundary.*

The space of geometrically finite parameters on the boundary of the main cardioid is quite complicated: it is a stratified space of dimension $(d-2)$. One should think of it as an analogue of the **Deligne-Mumford compactification** in complex dynamics. At the moment, I can rigorously identify only some of the strata in the completion.

Blaschke products

Let $\mathcal{B}_d$ be the space of degree d Blaschke products that have an attracting fixed point at the origin:

$$z \to f_{\mathbf{a}}(z) = z \cdot \prod_{i=1}^{d-1} \frac{z+a_i}{1+\overline{a_i}z}, \qquad a_1, a_2, \ldots, a_{d-1} \in \mathbb{D}. \tag{1}$$

Given two Blaschke products $f_{\mathbf{a}}, f_{\mathbf{b}}$, one can find a rational map $f_{\mathbf{a},\mathbf{b}}(z)$ – the *mating* of $f_{\mathbf{a}}, f_{\mathbf{b}}$ – whose Julia set is a quasicircle $\mathcal{J}_{\mathbf{a},\mathbf{b}}$ which separates the Riemann sphere into two domains Ω_-, Ω_+ such that on one side $f_{\mathbf{a},\mathbf{b}}(z)$ is conformally conjugate to $f_{\mathbf{a}}$, and to $f_{\overline{\mathbf{b}}}$ on the other. The mating is unique up to conjugation by Möbius transformations. The existence of the mating can be proved using quasiconformal surgery.

Remark. The relation between spaces of Blaschke products and main cardioids is via the *Bers embedding* $\mathcal{B}_d \to \mathscr{P}_d$, given by mating $f_{\mathbf{a}} \in \mathcal{B}_d$ with $z \to z^d$. While one visualizes the completion of the Weil-Petersson metric by adding points to the boundary of the main cardioid, in reality, one studies the dynamics of Blaschke products.

2 Definition of the Weil-Petersson metric

Suppose μ is a **dynamical Beltrami coefficient** on the unit disk, invariant under either a co-compact Fuchsian group or a Blaschke product. In this case, it represents a (possibly trivial) tangent vector in $T_X\mathcal{T}_g$ and $T_f\mathcal{B}_d$. Without loss of generality, we can assume that $\|\mu\|_\infty \le 1$. By solving the Beltrami equation $\overline{\partial}\varphi_t = t\mu\,\partial\varphi_t$, we obtain a holomorphic family of conformal maps

$$\varphi_t\colon \mathbb{D}^* \to \mathbb{C}, \qquad \varphi_0(z) = z, \qquad \text{where } \mathbb{D}^* = \{z : |z| > 1\}.$$

For these special families, McMullen showed that the remarkable formula

$$2\,\frac{d^2}{dt^2}\bigg|_{t=0} \mathrm{H.\,dim}\ \varphi_t(\mathbb{S}^1) = \sigma^2(\mathcal{S}\mu), \tag{2}$$

where

$$\sigma^2(b) = \frac{1}{2\pi}\limsup_{R\to 1^+} \frac{1}{|\log(R-1)|}\int_{|z|=R} |b(z)|^2\,|dz| \tag{3}$$

is the *asymptotic variance* of a Bloch function on $\mathbb{D}^*$ and

$$\mathcal{S}\mu = -\frac{1}{\pi}\int_{\mathbb{D}} \frac{\mu(\zeta)}{(\zeta - z)^2}|d\zeta|^2 \tag{4}$$

is the Beurling transform of μ.

It is said that every great theorem in mathematics tends to become a definition. In the Fuchsian case, McMullen observed that

$$\sigma^2(\mathcal{S}\mu) = (8/3)\,\|\mu\|^2_{\mathrm{WP}} \tag{5}$$

is a constant multiple of the Weil-Petersson metric on Teichmüller space. In view of this, it is natural to take (2) as the *definition* of the Weil-Petersson metric on $\mathcal{B}_d$.

Remark. (i) Since μ is supported on the unit disk, $\mathcal{S}\mu$ is a holomorphic function on $\mathbb{D}^*$. Furthermore, it is easy to see that it satisfies a Bloch bound of the form

$$\|\mathcal{S}\mu\|_{\mathcal{B}^*} := \sup_{z\in\mathbb{D}^*} (|z|^2 - 1)|(\mathcal{S}\mu)'(z)| < \infty.$$

This ensures us that the asymptotic variance is finite, e.g. $\sigma^2(b) \le \|b\|^2_{\mathcal{B}_*}$.

(ii) For arbitrary families of conformal maps (that is, for general μ, not necessarily invariant), McMullen's identity need not hold as simple examples demonstrate.

(iii) An astute reader may observe that invariant Beltrami coefficients do not exhaust all tangent vectors at the parameters in $\mathcal{B}_d$ that have critical relations. One can fix this by considering *eventually-invariant* Beltrami coefficients, satisfying $f^*\mu = \mu$ in some neighbourhood of the unit circle.

3 A glimpse of my thesis [Ivr1, Ivr2]

In my PhD thesis, I studied the Weil-Petersson metric on $\mathcal{B}_2$, the space Blaschke products of degree 2:

$$f_a(z) = z \cdot \frac{z+a}{1+\overline{a}z}, \qquad a \in \mathbb{D}.$$

Suppose f_a is a Blaschke product of degree 2, other than $z \to z^2$. Since the multiplier of the attracting fixed point is non-zero, the quotient of the disk by f_a is naturally a torus T_{f_a} with one marked point (to take the quotient, we must puncture out the grand orbits of the attracting fixed point and the critical point). The work of McMullen and Sullivan identifies the tangent spaces $T_{f_a}\mathcal{B}_2 \cong T_{T_{f_a}}\mathcal{T}_{1,1} \cong M(T_{f_a})/\sim$.

We now describe two specific deformations of Blaschke products: *radial degenerations* and *horocyclic degenerations*. In $\mathcal{T}_{1,1}$, they are geodesics and horocycles respectively. General estimates may be found in [Ivr1].

Basic incompleteness. One can express radial degenerations of Blaschke products by "half-optimal" pinching coefficients which take up half of the quotient torus at the attracting fixed point but are sparse near the unit circle. The incompleteness of the Weil-Petersson metric follows from the following estimate:

Theorem 2. *If μ is an invariant Beltrami coefficient supported on the unit disk, then*

$$\|\mu\|^2_{\mathrm{WP}} \lesssim \frac{9}{4} \cdot \|\mu\|^2_\infty \cdot \limsup_{R\to 1^+} \big|\operatorname{supp}\mu \cap \{z : |z| = R\}\big|. \tag{6}$$

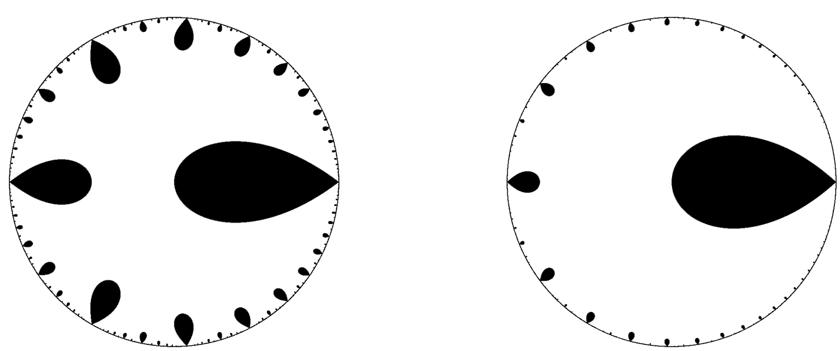

Fig. 2. The support of μ during a radial degeneration (above: $a = 0.5$ and 0.8).

In particular, the above theorem shows that the Hausdorff dimension of $w^{t\mu}(\mathbb{S}^1)$ is at most $1 + Ct^2 \cdot \limsup_{r\to 1^-} \big|\operatorname{supp}\mu \cap \{z : |z| = r\}\big|$, if μ is an *invariant* Beltrami coefficient. (Recently, I proved that this estimate for Hausdorff dimension holds for all Beltrami coefficients, assuming only that the support has an invariance property [Ivr3].)

Incompleteness with a precise rate of decay. To obtain a stronger version of incompleteness with a precise rate of decay, one uses renewal theory, which is the study of the distribution of repeated pre-images of a point. For hyperbolic dynamical systems, this has been developed by Lalley [La]. In our context, he showed that given a Blaschke product $f(z)$ other than $z \to z^d$, if $x \in \mathbb{S}^1$, the number $n(x, R)$ of repeated pre-images y (i.e. $f^{\circ n}(y) = x$ for some n) for which $|(f^{\circ n})'(y)| < R$ satisfies

$$n(x, R) \sim \frac{e^R}{\int \log |f'| dm} \qquad \text{as} \quad R \to \infty. \tag{7}$$

One can extend formula (7) to points *inside* the disk using the identity

$$\log \frac{1}{|z|} = \sum_{f(w)=z} \log \frac{1}{|w|}. \tag{8}$$

Theorem 3. *Let $\mathcal{N}(z, R)$ be the number of repeated pre-images of z that lie in the disk centered at the origin of hyperbolic radius R. Then,*

$$\mathcal{N}(z, R) \sim \frac{1}{2} \cdot \log \frac{1}{|z|} \cdot \frac{e^R}{\int \log |f'| dm} \qquad \textit{as} \quad R \to \infty. \tag{9}$$

Remark. The expression in the denominator is called the *entropy* and it is responsible for the exponent in the rate of the decay of the Weil-Petersson metric. The analysis of the equidistribution of pre-images is intimately connected with a certain object attached to a Blaschke product called a *Riemann surface lamination* which is locally a product of the unit disk and a cantor set.

Completing the cardioid. To find the *completion* of $\mathcal{B}_2$ in the Weil-Petersson metric, one needs to give lower bounds for the integral average (2). These come from repelling periodic orbits on the unit circle:

Theorem 4. *If ξ_0 is repelling periodic point of f_0 with multiplier $(f_0^{\circ q})'(\xi_0) < 1 + \epsilon$, with $\epsilon > 0$ sufficiently small, then $\|\dot{f}_t\|_{\mathrm{WP}} \gtrsim (d/dt)|_{t=0} \log |(f^{\circ q})'(\xi_{0,t})|$.*

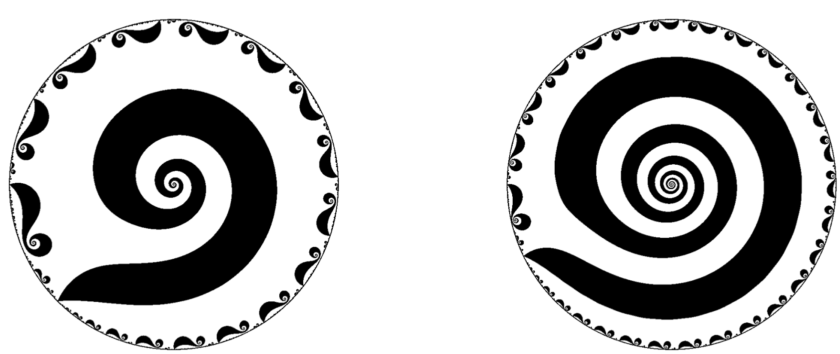

Fig. 3. A horocyclic degeneration (above: $a = 0.5 + 0.5e^{i\theta}$ with $\theta = 2\pi/8$ and $2\pi/14$).

Horocyclic degenerations. In degree 2, horocyclic degenerations are given by

$$f_a = z \cdot \frac{z+a}{1+\overline{a}z}, \qquad a = \Big((1-t) + t \cdot e^{i\theta}\Big) \cdot e(p/q) \text{ with } \theta \to 0.$$

One way to extract limits is to rescale by a Möbius transformation to put the critical point at the origin: $\tilde{f}_a = m_{c\to 0} \circ f_a \circ m_{0\to c}$, where $m_{0\to c} = \frac{z+c}{1+\bar{c}z}$ and $m_{c\to 0} = \frac{z-c}{1-\bar{c}z}$. Somewhat surprisingly, A. Epstein showed that the maps $\tilde{f}_a^{\circ q}$ converge to degree 2 parabolic Blaschke products [E].

It turns out that along horocyclic degenerations, the Weil-Petersson metric is asymptotically periodic [Ivr2]. This result is connected to an interesting phenomenon called *parabolic implosion* that a Julia set may not vary continuously with respect to parameters. In fact, horocyclic degenerations give rise to what I call "Epstein-Lavaurs" boundaries (a certain blow-up of the completion of the cardioid) that consists of non-degenerate parabolic Blaschke products endowed with a Lavaurs phase. In fact, one can define the Weil-Petersson metric directly on the Epstein-Lavaurs boundary by mating these objects. This uses the work of M. Urbański.

4 Dimensions of quasicircles [AIPP]

A well-known problem in quasiconformal geometry is to find $D(k)$, the maximal Hausdorff dimension of a k-quasicircle, the image of the unit circle under a k-quasiconformal mapping. The first non-trivial bound (with the right growth rate) was given in 1987 by Becker and Pommerenke [BP] who showed that $1 + 0.36\,k^2 \le D(k) \le 1 + 37\,k^2$ if k is small. In 1994, in his landmark work [Ast] on the area distortion of quasiconformal mappings, K. Astala suggested that the correct bound was

$$D(k) \le 1 + k^2, \qquad 0 \le k < 1. \tag{10}$$

Using a clever variation of Astala's argument, S. Smirnov [Sm] showed that the bound (10) indeed holds but the sharpness remained open (until recently).

A natural way to investigate the sharpness of Smirnov's bound is to use the infinitesimal approach offered by the Weil-Petersson metric. Motivated by these connections, we introduced the quantity

$$\Sigma^2 := \sup_{|\mu| \le \chi_{\mathbb{D}}} \sigma^2(\mathcal{S}\mu), \tag{11}$$

taken over all Beltrami coefficients supported on the exterior unit disk, invariant or otherwise. One of our results was the following estimate:

Theorem 5. [AIPP]

$$1 \;\ge\; \liminf_{k\to 0} \frac{D(k)-1}{k^2} \;\ge\; \Sigma^2 \;\ge\; 0.879. \tag{12}$$

Upper bound. In [AIPP], we gave two different proofs of the upper bound in (12). The first approach used **complex interpolation**, similar to Smirnov's original argument. The second route was via **fractal approximation**. I will outline the second approach, since it more closely follows my interests.

Let M_{I} denote the collection of Beltrami coefficients on the exterior unit disk, which are eventually-invariant under $z \to z^d$ for some $d \geq 2$, i.e. satisfying $(z^d)^*\mu = \mu$ in some neighbourhood of the unit circle.

Theorem 6. [AIPP]

$$\Sigma^2 = \sup_{\mu \in M_{\mathrm{I}},\, |\mu| \leq \chi_{\mathbb{D}}} \sigma^2(\mathcal{S}\mu).$$

Since McMullen's identity holds for Beltrami coefficients in M_{I}, Smirnov's bound immediately gives $\sigma^2(\mathcal{S}\mu) \leq 1$. Somewhat surprisingly, **Fuchsian coefficients are insufficient for this purpose**: if μ is a Beltrami coefficient invariant under a co-compact Fuchsian group, then classical duality arguments comparing the Weil-Petersson and Teichmüller metrics on $\mathcal{T}_g$ show that

$$\sigma^2(\mathcal{S}\mu) \;\leq\; \frac{2}{3}\,\frac{\|\mu\|^2_{\mathrm{WP}}}{\|\mu\|^2_T} \;\leq\; 2/3,$$

which is clearly less than 0.879. (In [AIPP], we showed that $\Sigma^2_{\mathrm{F}} < 2/3$.)

Remark. The reason fractal approximation fails in the Fuchsian case, is because in complex dynamics, one uses the metric $|dz|^2/(1-|z|^2)$ which satisfies a certain **isoperimetric property** that the hyperbolic metric $|dz|^2/(1-|z|^2)^2$ does not.

Lower bound. For the lower bound on Σ^2, we observed that the Julia sets of

$$P_t(z) = z^{20} + tz$$

are quasicircles with $\mathrm{H.dim}\,\mathcal{J}(P_t) = 1 + 0.879\,k(t)^2$ when t is small. According to the λ-lemma, it is easy to represent $\mathcal{J}(P_t)$ as $k(t)$-quasicircles with $k(t) \leq |t|$. Somewhat surprisingly, one can obtain a more efficient estimate $k(t) \approx 0.585\,|t|$ on the quasiconformal distortion using an explicit construction (given in [AIPP]).

Remark. The previous record for the maximal dimension of a k-quasicircle was $1 + 0.69k^2$, for k small [ARS].

Further developments. Our hope was to prove the sharpness of Smirnov's bound by showing that $\Sigma^2 = 1$. After reading our manuscript, H. Hedenmalm surprised us by proving that actually $\Sigma^2 < 1$ [H]. While Hedenmalm's estimate suggests that Smirnov's bound is not sharp, by itself, it is not conclusive. There are two difficulties:

1. It is (still) an open problem whether $D(k)$ is the supremum of Hausdorff dimensions of "dynamical" k-quasicircles.

2. Even if $\Sigma^2 < 1$, one still needs uniform estimates: e.g. the equation

$$2(d^2/dt^2)|_{t=0}\,\mathrm{H.dim}\,w^{t\mu}(\mathbb{S}^1) \le \Sigma^2 < 1,$$

true for dynamical k-quasicircles may not survive after taking the supremums.

Nevertheless, recently, I was able to navigate around these difficulties.

Theorem 7. [Ivr3]

$$D(k) = 1 + \Sigma^2 k^2 + \mathcal{O}(k^{2.5}). \tag{13}$$

Taken together the trilogy [AIPP, H, Ivr3] **shows that Smirnov's dimension bound is not sharp**, at least when k is small. István Prause informed me (private communication) that this implies that $D(k) < 1 + k^2$ for all $0 < k < 1$.

One problem I am currently working on is to show an analogous result in the Fuchsian case, i.e.

$$D_{\mathrm{F}}(k) =_? 1 + \Sigma_{\mathrm{F}}^2\, k^2 + o(k^2).$$

5 Conformal maps [Ivr3, IK]

The Riemann mapping theorem states that any simply-connected domain $\Omega \subset \mathbb{C}$ is the image of a conformal map $f : \mathbb{D}^* \to \Omega$ (unless $\Omega = \mathbb{C}$ itself). Since simply-connected domains can be very wild, it is reasonable to expect that the complexity of the boundary $\partial\Omega$ is manifested in the complexity of the Riemann map. For domains with rough boundaries, the relationship between f and $\partial\Omega$ may be quantified using several geometric characteristics. One notable characteristic is the *integral means spectrum* given by

$$\beta_f(p) = \limsup_{R \to 1^-} \frac{\log \int_{|z|=R} |f'(z)^p|\, d\theta}{\log \frac{1}{R-1}}, \qquad p \in \mathbb{C}.$$

The importance of $\beta_f(p)$ lies in the fact that it is Legendre-dual to the multifractal spectrum of harmonic measure [Mak, Bin]. Taking the supremum of $\beta_f(p)$ over bounded simply-connected domains, one obtains the *universal integral means spectrum*

$$B(p) = \sup \beta_f(p).$$

It is clear from Hölder's inequality that $B(p)$ is a convex function, with a minimum at $B(0) = 0$. Even though $B(p)$ is a central object in geometric function theory, apart from various estimates [HS, J], not much is rigorously known about its qualitative features.

Let $\mathbf{\Sigma}_k$ denote the collection of conformal mappings of the exterior unit disk, normalized so that $\varphi(z) = z + \mathcal{O}(1/z)$ near infinity, that admit k-quasiconformal extensions to the complex plane with dilatation at most k. Maximizing over $\mathbf{\Sigma}_k$, we obtain the spectra $B_k(p) := \sup_{\varphi \in \mathbf{\Sigma}_k} \beta_\varphi(p)$. The main results of [Ivr3, IK] describe an **equality of universal characteristics** of conformal maps:

Theorem 8. *(Global version)*

$$\lim_{p\to 0} \frac{B_k(p)}{|p|^2/4} \;=\; \sup_{\varphi\in\mathbf{\Sigma}_k} \sigma^2(\log\varphi') \;=\; \sup_{\varphi\in\mathbf{\Sigma}_k} C^2_{\mathrm{LIL}}(\log\varphi'). \tag{14}$$

(Infinitesimal version)

$$\lim_{k\to 0,\, kp\to 0} \frac{B_k(p)}{|p|^2/4} \;=\; \sup_{|\mu|\le\chi_{\mathbb{D}}} \sigma^2(\mathcal{S}\mu) \;=\; \sup_{|\mu|\le\chi_{\mathbb{D}}} C^2_{\mathrm{LIL}}(\mathcal{S}\mu). \tag{15}$$

Above, $C_{\mathrm{LIL}}(b)$ is the *constant in Makarov's law of iterated logarithm* defined as the essential supremum of

$$C_{\mathrm{LIL}}(b,\theta) = \limsup_{r\to 1} \frac{|b(re^{i\theta})|}{\sqrt{\log\frac{1}{1-r}\log\log\log\frac{1}{1-r}}} \tag{16}$$

over $\theta\in[0,2\pi)$. The applications to $C_{\mathrm{LIL}}(\log\varphi')$ are joint with I. Kayumov.

To see why (15) is the infinitesimal form of (14), observe that in the case of a holomorphic family of quasiconformal maps $\{H_t,\, t\in\mathbb{D}\}$, with $\overline{\partial}H_t = t\mu\,\partial H_t$ and $|\mu|\le\chi_{\mathbb{D}}$, $(d/dt)\big|_{t=0}\log H_t' = \mathcal{S}\mu$ for $z\in\mathbb{D}$.

Some immediate consequences

One advantage of Theorem 8 is that asymptotic variance is much easier to estimate than the other two characteristics. We give two applications:

(1) The asymptotic expansion of $D(k)$ (Theorem 7) is a straightforward consequence of Theorem 8, together with the relation

$$\beta_f(p) = p-1 \quad\Longleftrightarrow\quad p = \mathrm{M.}\dim f(\mathbb{S}^1), \qquad f\in\mathbf{\Sigma}_k\,. \tag{17}$$

(2) In light of Hedenmalm's estimate, Theorem 8 contradicts the conjecture "$B_k(p) = k^2p^2/4$ for all $k\in[0,1)$ and $p\in[-2/k,2/k]$" from [J, PS]. However, since we do not know whether $\lim_{k\to 1^-}\Sigma^2(k) \stackrel{?}{=} 1$, we cannot rigorously rule out Kraetzer's conjecture which asserts the more limited statement that "$B(p) = p^2/4$ for all $p\in[-2,2]$."

Remark. It is currently known that

$$0.93 < \lim_{k\to 1^-}\Sigma^2(k) < (1.24)^2.$$

The interested reader can consult [AIPP] for the lower bound and [HK, Corollary 2.3] for the upper bound. The number 0.93 is a significant improvement over the previously best known bound $(0.91)^2$ established in [HK]. In addition, I believe that 0.93 is close to optimal, which makes it likely that Kraetzer's conjecture is also false.

Additional applications

The proof of Theorem 8 is remarkably robust, and has many other applications besides the universal bounds:

- Gives an elementary proof of McMullen's identity without the use of thermodynamic formalism, which also handles the "parabolic" cases.
- Applies in higher dimensions, e.g. in $\mathbb{H}^n$ or $\mathbb{CH}^n$.
- Gives improved dimensions bounds for sparse coefficients.
- Works with any norm on the Bloch space, not just the L^∞ norm.

Future work

At the moment, my arguments are purely analytic; however, for additional applications, one needs a slightly more robust way to estimate integral means. Currently, I am working on a very promising approach involving integration over Brownian paths in the spirit of R. Bañuelos [Ban]. These ideas can also be applied to the familiar setting of dynamical systems; where one replaces the traditional approach of thermodynamic formalism with the study of heat diffusion processes, which I believe to be more visual. This is somewhat reminiscent of the ideas introduced by D. Sullivan in [Sul].

References

[AIPP] K. Astala, A. Perälä, O. Ivrii and I. Prause. *Asymptotic variance of the Beurling transform*, Geom. Funct. Anal., to appear.

[Ast] K. Astala. *Area distortion of quasiconformal mappings*, Acta Math. 173 (1994), 37–60.

[ARS] K. Astala, S. Rohde, O. Schramm, *Self-similar Jordan curves*, in preparation.

[Ban] R. Bañuelos. *Brownian motion and Area Functions*, Indiana Univ. Math. J. 35 (1986), no. 3, 643–668.

[Bin] I. Binder, *Harmonic measure and rotation spectra of planar domains*, preprint, 2008.

[BP] J. Becker, C. Pommerenke, *On the Hausdorff dimension of quasicircles*, Ann. Acad. Sci. Fenn. Ser. A I Math. 12 (1987), 329–333.

[E] Epstein, A. *Bounded hyperbolic components of quadratic rational maps*, Ergod. Th. & Dynam. Sys. 20 (2000), 727–748.

[H] H. Hedenmalm, *Bloch functions and asymptotic variance*, in preparation.

[HK] H. Hedenmalm, I. R. Kayumov, *On the Makarov law of integrated logarithm*, Proc. Amer. Math. Soc. 135 (2007), 2235–2248.

[HS] H. Hedenmalm, A. Sola, *Spectral notions for conformal maps: a survey*, Comput. Meth. Funct. Th. 8 (2008), no. 2, 447–474.

[J] P. W. Jones, *On scaling properties of harmonic measure*, Perspectives in analysis, Mathematical Physics Studies 27 (Springer, Dordrecht, 2005) 73–81.

[Ivr1] O. Ivrii. *The geometry of the Weil-Petersson metric in complex dynamics*, preprint, 2015.

[Ivr2] O. Ivrii. *Rescaling limits of Blaschke products*, in preparation.

[Ivr3] O. Ivrii. *Quasicircles of dimension* $1+k^2$ *do not exist*, in preparation.

[IK] O. Ivrii, I. R. Kayumov. *Asymptotic variance of conformal maps and Makarov's law of iterated logarithm*, in preparation.

[La] S. P. Lalley. *Renewal Theorems in symbolic dynamics, with applications to geodesic flows, non-Euclidean tessellations and their fractal limits.* Acta Math. 163 (1989), 1–55.

[Mak] N. G. Makarov, *Fine structure of harmonic measure*, St. Petersburg Math. J. 10 (1999), no. 2, 217–268.

[McM1] C. T. McMullen, *Thermodynamics, dimension and the Weil-Petersson metric*. Inv. math. 173 (2008), 365–425.

[McM2] C. T. McMullen. *Dynamics on the unit disk: Short geodesics and simple cycles*, Comm. Math. Helv. 85 (2010), 723–749.

[PS] I. Prause, S. Smirnov, *Quasisymmetric distortion spectrum*, Bull. Lond. Math. Soc. 43 (2011), 267–277.

[Sm] S. Smirnov. *Dimension of Quasicircles*, *Acta Math.*, 205 (2010), 189–197.

[Sul] D. Sullivan. *The density at infinity of a discrete group of hyperbolic motions*, *Pub. Math. de l'IHÉS*, 50 (1979), no. 1, 171–202.