Sparse Beltrami coefficients, integral means of conformal mappings and the Feynman-Kac formula

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Abstract

In this note, we give an estimate for the dimension of the image of the unit circle under a quasiconformal mapping whose dilatation has small support. We also prove an analogous estimate for the rate of growth of a solution of a second-order parabolic equation given by the Feynman-Kac formula with a sparsely supported potential and introduce a dictionary between the two settings.

1 Introduction

For a Beltrami coefficient μ defined on the complex plane with $\|\mu\|_\infty < 1$, let $\tilde{w}^\mu$ denote the normalized solution of the Beltrami equation $\overline{\partial} w = \mu\, \partial w$ which fixes the points $0, 1, \infty$. In the classical question on dimensions of quasicircles, one is interested in maximizing the Minkowski dimension of $\tilde{w}^\mu(\mathbb{S}^1)$ over all Beltrami coefficients on the plane with $\|\mu\|_\infty \le k$ for a fixed $0 \le k < 1$. This question has been studied by many authors, although a complete answer is currently out of reach. One notable result in this area is due to S. Smirnov [16] who gave the upper bound

$$D(k) = \sup_{\|\mu\|_\infty \le k} \mathrm{M.dim}\, \tilde{w}^\mu(\mathbb{S}^1) \le 1 + k^2, \qquad \text{for all } \ k \in [0, 1), \tag{1.1}$$

while recently it was observed that $D(k) = 1 + k^2\Sigma^2 + \mathcal{O}(k^{8/3-\varepsilon})$ for some constant $0.87913 < \Sigma^2 < 1$, see the works [2, 9, 11].

In the present paper, we are interested in the case when the support of μ is contained in a garden

$$\mathcal{G} = \bigcup_{j=1}^{\infty} B_j, \qquad d_{\mathbb{D}}(B_i, B_j) > R, \quad i \neq j, \tag{1.2}$$

made up of countably many horoballs $B_j \subset \mathbb{D}$, any two of which are at least a distance R apart in the hyperbolic metric. Equivalently, we require the horoballs $B_j^* = \{z \in \mathbb{D} : d_{\mathbb{D}}(z, B_j) \leq R/2\}$ to be disjoint.

Let $\mu^+(z) = \overline{\mu(1/\overline{z})} \cdot (z^2/\overline{z}^2)$ denote the reflection of μ in the unit circle. Since the support of μ^+ is contained in the exterior of the unit disk, $\tilde{w}^{\mu^+} : \mathbb{D} \to \mathbb{C}$ is conformal. Our first main theorem states:

Theorem 1.1. *Suppose μ is a Beltrami coefficient on the unit disk with $\|\mu\|_\infty \leq 1$. If μ has sparse support, that is, its support is contained in a garden $\mathcal{G}$ given by (1.2), then*

$$\operatorname{M.dim} \tilde{w}^{k\mu^+}(\mathbb{S}^1) \leq 1 + Ce^{-R/2}k^2, \quad k < \min\Big(0.49, \frac{c}{2R}\Big). \tag{1.3}$$

1.1 Integral means spectra

To prove Theorem 1.1, we will analyze integral means of conformal mappings. For a conformal mapping $f : \mathbb{D} \to \mathbb{C}$, its *integral means spectrum* is given by

$$\beta_f(p) = \limsup_{r \to 1^-} \frac{\log \int_{|z|=r} |f'(z)|^p \, d\theta}{\log \frac{1}{1-r}}, \qquad p > 0. \tag{1.4}$$

The connection between integral means and the Minkowski dimension of the boundary of the image domain comes from the relation

$$\beta_f(p) = p - 1 \quad \iff \quad p = \operatorname{M.dim} f(\mathbb{S}^1), \tag{1.5}$$

valid when $f(\mathbb{S}^1)$ is a quasicircle, see for instance [15, Corollary 10.18]. In view of the above identity, to prove Theorem 1.1, it suffices to show:

Theorem 1.2.

$$\beta_{\tilde{w}^{k\mu^+}}(p) \leq Ce^{-R/2}k^2p^2/4, \qquad k < 0.49, \quad kp < c/R. \tag{1.6}$$

The advantage of estimating integral means comes from the fact that they allow us to view Theorem 1.1 as a *growth problem*. To make this feature more visible, consider the *Brownian spectrum of a conformal mapping*:

$$\tilde{\beta}_f(p) := \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}_0 |f'(B_t)|^p, \tag{1.7}$$

$$= \limsup_{t\to\infty} \frac{1}{t} \log \int_{\mathbb{D}} |f'(z)|^p \cdot p_t(0,z) dA_{\text{hyp}}(z). \tag{1.8}$$

In the equations above, B_t is hyperbolic Brownian motion, that is, geometric Brownian motion in $\mathbb{D}$ equipped with the hyperbolic metric $\rho = \frac{2|dz|}{1-|z|^2}$. The subscript "0" in $\mathbb{E}_0$ indicates that Brownian motion is to be started at the origin. Finally, $p_t(0,z)$ is the hyperbolic heat kernel which measures the probability density that a Brownian particle travels from 0 to z in time t.

In Section 4, we will show the elementary estimate $\tilde{\beta}_f(p) \geq \beta_f(p)$ for any $p > 0$; this follows from the fact that the expected displacement of hyperbolic Brownian motion is linear in time. Therefore, to prove Theorem 1.2, we may estimate the Brownian spectrum instead.

Remark. The use of hyperbolic Brownian motion is inspired by the work [13] of T. Lyons who give an alternative perspective on Makarov's *law of the iterated logarithm* for Bloch functions. As noted by I. Kayumov in [12], the study of the behaviour of the integral means spectrum at the origin is slightly more general, so our considerations may be viewed as an extension of Lyons' ideas.

1.2 Growth of solutions of PDEs

Our proof of Theorem 1.2 is inspired by an analogous statement from parabolic PDEs whose solution is given by the Feynman-Kac formula. Let $\Delta_{\text{hyp}} = \rho(x)^{-2}\Delta$ denote the hyperbolic Laplacian and $A_{\text{hyp}} = \rho(x)^2 dA$ be the hyperbolic area element. For a positive and bounded potential V, we consider the second order parabolic differential equation

$$\frac{\partial u}{\partial t} = \frac{1}{2} \cdot \Delta_{\text{hyp}} u + V(x)u(x,t), \qquad (x,t) \in \mathbb{D} \times (0,\infty), \tag{1.9}$$

where the initial condition $u_0(x) = u(x,0) = \lim_{t\to 0^+} u(x,t)$ is a smooth compactly supported function. If the potential $V \equiv 0$, then (1.9) reduces to the heat equation.

As is well known, the unique weak solution to (1.9) which is bounded on $[0,T] \times \mathbb{D}$ for any $T > 0$, is given by *Feynman-Kac formula*

$$u_t(x) = \mathbb{E}_x \left\{ u_0(B_t) \exp \int_0^t V(B_s) ds \right\}. \tag{1.10}$$

In this paper, we define (1.10) to be the solution of (1.9) even if it does not possess sufficient regularity to be considered a genuine solution. For example, if V is not continuous, then u_t cannot be simultaneously C^2 in x and C^1 in t.

There are many possible ways to measure the growth of solutions of PDEs but, for our purposes, the *Lyapunov exponent*

$$\beta_V := \limsup_{t \to \infty} \frac{1}{t} \log \int_{\mathbb{D}} u_t(x) dA_{\text{hyp}}(x) \tag{1.11}$$

is the most natural. Since the initial condition u_0 was positive, u_t will remain positive for all time and the *total mass* of the solution $\int_{\mathbb{D}} u_t(x) dA_{\text{hyp}}(x)$ will be increasing in t. Our second main theorem is an analogue of Theorem 1.2 for sparse potentials:

Theorem 1.3. *For a potential V of sparse support, in that $V = \chi_{\mathcal{G}}$ where $\mathcal{G}$ is of the form (1.2),*

$$\beta_V(p) \,:=\, \beta_{p^2 V} \,\le\, C e^{-R/2} p^2,$$

for $0 < p < p_0(R)$ sufficiently small.

Remark. The choice of notation is justified by the observation that the Lyapunov spectra $\beta_V(p)$ possess many of the same properties as integral means spectra of conformal mappings, for instance, they are increasing and convex functions in p.

1.3 Alternative ideas and remarks

The main difficulty in Theorem 1.1 is to make use of the separation condition between horoballs. Perhaps the most obvious attempt is to take a Beltrami coefficient μ supported on $\mathcal{G}$ and cook up a Beltrami coefficient ν supported on $\mathbb{D}$ with $\|\nu\|_\infty < \|\mu\|_\infty$ and $\tilde{w}^\mu = \tilde{w}^\nu$ on $\mathbb{S}^1$ and then apply the general bounds on dimensions of quasicircles mentioned above. However, it is easily seen that such an approach is impossible: there are sparse k-quasicircles which are genuine k-quasicircles. In fact, this is true even if $\mathcal{G} \subset \mathbb{D}$ is composed of a single horoball – we only need $\mathcal{G}$ to contain round balls of arbitrarily large hyperbolic diameter. This easily follows from the fact that the Teichmüller norm can be described by the dual pairing

$$\|\mu\|_T = \inf_{\nu \sim \mu} \|\mu\|_\infty = \sup_{\|q\|=1} \int_{\mathbb{D}} \mu \cdot q,$$

where the supremum is taken over all integrable quadratic differentials q on the unit disk with $\|q\| = \int_{\mathbb{D}} |q| = 1$.

The reader may also try to improve on the arguments of Smirnov [16] who used complex interpolation techniques to give an elegant proof of the bound $D(k) \le 1 + k^2$ suggested by Astala [1]. However, in this set of ideas, it seems unlikely that one can make use of the sparsity assumption on the support. Another natural approach is to extend the arguments of C. Bishop [4, Lemma 6.4] which involve a corona-type construction. While these ideas do utilize the sparsity of the support, it is not really clear how to exploit the martingale nature of Bloch functions in this context, which is necessary to obtain quadratic growth.

In [11, Section 9], an analogue of Theorem 1.1 was proved using the techniques of Becker and Pommerenke for estimating integral means of univalent functions for gardens $\mathcal{G}$ that are unions of thickened geodesics (unit neighbourhoods of hyperbolic geodesics) with the same separation condition. The case of horoballs introduces non-uniformity and therefore requires new ideas. A priori, it was not clear to the author how to extend these arguments either, although it can be done – see Section 5. This approach can be most easily generalized to gardens composed of objects other than horoballs.

Another perspective was offered by N. Michalache via the notion of *mean wiggly sets* from [8]. Here, one seeks to decompose $\mathbb{S}^1 = B \sqcup G$ so that $\tilde{w}^{k\mu^+}(G)$ satisfies a mean wiggliness condition while B has dimension less than $1 - \varepsilon$. Hölder properties of quasiconformal mappings guarantee that for small $k > 0$, $\mathrm{M.}\dim \tilde{w}^{k\mu^+}(\mathbb{S}^1) = \mathrm{M.}\dim \tilde{w}^{k\mu^+}(G)$, from which point the mean wiggly machinery can be applied. A possible definition of G could be $G = \left\{ e^{i\theta} \in \mathbb{S}^1 : \beta(\theta) < Ce^{-R/2} \right\}$ where

$$\beta(\theta) = \limsup_{r \to 1} \frac{\ell_{\mathrm{hyp}}\big([0, re^{i\theta}] \cap \mathcal{G}\big)}{\ell_{\mathrm{hyp}}\big([0, re^{i\theta}]\big)}, \tag{1.12}$$

with ℓ_{hyp} being the hyperbolic length.

The author's original proof of Theorem 1.1, which will be presented in Section 4, was motivated by the observation that hyperbolic Brownian motion started at a point $z_0 \notin \mathcal{G}$ spends little time in the garden: for almost every Brownian path B_t,

$$\limsup_{t \to \infty} \frac{\int_0^t \chi_{\mathcal{G}}(B_s) ds}{t} \le Ce^{-R/2}. \tag{1.13}$$

This may be viewed as a stochastic analogue of mean wiggliness. Since we will not actually use (1.13) in this paper, we will not give a proof.

1.4 Dynamical considerations

Suppose Γ is a cofinite area Fuchsian group with at least one cusp. One may construct a Beltrami coefficient $\mu \in M(\mathbb{D})^\Gamma$ satisfying the hypotheses of the theorem by lifting a Beltrami coefficient on $\mathbb{D}/\Gamma$ supported on a collar neighbourhood of one of the cusps. For these special dynamical coefficients, one can give an alternative proof of Theorem 1.1 based on McMullen's identity [14, 10] which says that

$$\frac{d^2}{dt^2}\bigg|_{t=0} \operatorname{M.dim} \tilde{w}^{t\mu}(\mathbb{S}^1) = \frac{4}{3} \cdot \lim_{r \to 1^-} \frac{1}{2\pi} \int_{|z|=r} \left| \frac{v'''_{\mu^+}}{\rho^2}(re^{i\theta}) \right|^2 d\theta, \tag{1.14}$$

where

$$v'''_{\mu^+}(z) = -\frac{6}{\pi} \int_{|z|>1} \frac{\mu^+(\zeta)}{(\zeta - z)^4} |d\zeta|^2.$$

As explained in [14], since v'''_{μ^+} is naturally a quadratic differential, to measure its size, one should divide by the square of the Poincaré metric. According to [10, Section 2], one has

$$\frac{d^2}{dt^2}\bigg|_{t=0} \operatorname{M.dim} \tilde{w}^{t\mu}(\mathbb{S}^1) \le C \cdot \lim_{r \to 1^-} |\mathcal{G} \cap S_r|, \tag{1.15}$$

$$\le C \cdot \limsup_{r \to 1^-} \frac{1}{|\log(1-r)|} \int_0^r |\mathcal{G} \cap S_s| \frac{ds}{1-s}, \tag{1.16}$$

where $S_r = \{z : |z| = r\}$. It is not difficult to see that for a garden $\mathcal{G}$ which is the union of horoballs a hyperbolic distance R apart, the quantity (1.16) is $\le Ce^{-R/2}$, which is essentially the estimate we want. (To be honest, it is not clear how to use thermodynamic formalism to obtain a uniform estimate for $\mathrm{M.dim}\,\tilde{w}^{t\mu}(\mathbb{S}^1)$ independent of μ and the underlying dynamical system.)

In any case, McMullen's identity fails for general Beltrami coefficients μ, and the bound (1.16) is not enough since one can concentrate the Beltrami coefficient to expand a small arc of the unit circle and still obtain maximal dimension distortion.

Notation

We write $\mathbb{C}$ for the complex plane, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ for the unit disk and $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ for the unit circle. To denote balls, circles and annuli, we use $B(x,r) = \{z : |z-x| = r\}$, $S_r = \{z : |z| = r\}$ and $A(r,R) = \{z : r < |z| < R\}$ respectively. To compare quantities, we use $A \lesssim B$ to denote $A < \mathrm{const} \cdot B$, while $A \asymp B$ means $A \lesssim B \lesssim A$.

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2 Background in probability

Consider the *hyperbolic heat equation*

$$\frac{\partial u}{\partial t} = \frac{1}{2} \cdot \Delta_{\mathrm{hyp}} u, \qquad (x,t) \in \mathbb{D} \times (0,\infty). \tag{2.1}$$

Its fundamental solution, the *hyperbolic heat kernel* $p_t(x,y)$, is characterized by the property that if $u_0(x)$ is a bounded continuous function on the disk then

$$u_t(x) = \int_{\mathbb{D}} p_t(x,y) u_0(y) dA_{\mathrm{hyp}}(y) \tag{2.2}$$

is the unique bounded solution of (2.1) with $\lim_{t\to 0^+} u_t(x) = u_0(x)$. Since this reproducing property determines $p_t(x,y)$ uniquely, the heat kernel is conformally invariant and symmetric, that is, $p_t(\phi(x),\phi(y)) = p_t(x,y) = p_t(y,x)$ for any $\phi \in \mathrm{Aut}\,\mathbb{D}$.

As noted in the introduction, $p_t(x,y)$ measures the probability density for a Brownian particle to go from x to y in time t (in particular, it is strictly positive). This means that if $E \subset \mathbb{D}$ is a measurable set, then

$$\mathbb{P}_x(B_t \in E) = \int_E p_t(x,y)\, dA_{\mathrm{hyp}}(y). \tag{2.3}$$

Here, the subscript "x" in $\mathbb{P}_x$ denotes the fact that Brownian motion is started at x. In fact, (2.3) may be taken as the definition of hyperbolic Brownian motion.

For $t > 0$, we define the *partial Green's function* as $g_t(x,y) := \int_0^t p_s(x,y)ds$. Taking $t = \infty$ gives the usual *Green's function* $g_\infty(x,y)$. The Green's function measures the occupation density of Brownian motion, that is, the integral $\int_E g_\infty(x,y)dA_{\mathrm{hyp}}(y)$ computes the expected amount of time Brownian motion starting at x spends in E.

When $y = 0$, we will shorten notation to $p_t(x) := p_t(0,x)$ and $g_t(x) := g_t(0,x)$. We have the explicit formula

$$g_\infty(x) = \frac{1}{\pi} \log \frac{1}{|x|}. \tag{2.4}$$

The above formula should not be surprising. According to [6, Chapter 4.8],

$$g_\infty(x) dA(x) = \frac{1}{\pi} \log \frac{1}{|x|} dA(x)$$

is the occupation measure for the Euclidean Brownian motion in $\mathbb{C}$ started at the origin and simulated until it hits $\partial\mathbb{D}$. This coincidence is explained by the fact that hyperbolic Brownian motion is a time change of the Euclidean Brownian motion by $\rho(x)^2$, that is, if one wants to simulate hyperbolic Brownian motion up to time t, one can instead simulate two-dimensional Euclidean Brownian motion W_2 up to time τ defined by $t = \int_0^\tau \rho^2(W_2(s))ds$.

The existence of the Green's function implies that hyperbolic Brownian motion is transient: any path tends to the unit circle. In fact, the hyperbolic distance from the starting point to B_t grows linearly with time, i.e. for any $\varepsilon > 0$,

$$\mathbb{P}_0\Big((1-\varepsilon)t \,<\, d_{\mathbb{D}}(0, B_t) \,<\, (1+\varepsilon)t\Big) \to 1, \qquad \text{as } t \to \infty. \tag{2.5}$$

At times, we will use the following precise estimate from [5, Theorem 3.1]:

$$p_t(x,y) \sim \frac{1+\rho}{t\,(1+\rho+t)^{1/2}} \cdot \exp\left(-\frac{t}{4} - \frac{\rho^2}{4t} - \frac{\rho}{2}\right), \qquad d_{\mathbb{D}}(x,y) = \rho, \tag{2.6}$$

uniform for $0 \le \rho < \infty$ and $0 < t < \infty$.

2.1 Brownian motion escapes from horoballs

The theme of the next two lemmas is that if a Brownian motion enters a horoball, it does not want to stay there for very long.

Lemma 2.1. *Let B be a horoball in the unit disk and consider hyperbolic Brownian motion started at a point $x \in \mathbb{D}$. Let $\ell(B) = \int_0^\infty \chi_B(B_s)ds$ denote the amount of time Brownian motion spends in B. Then,*

(i) *If $x \in \partial B$, $\mathbb{E}_x\big(\ell(B)\big) \asymp 1$.*

(ii) *If $x \in \partial B^* = \partial\{y \in \mathbb{D} : d_{\mathbb{D}}(y, B) \le R/2\}$, then $\mathbb{E}_x\big(\ell(B)\big) \asymp e^{-R/2}$.*

Proof. (i) We compute the expected amount of time that Brownian motion spends in B by integrating the Green's function g_∞:

$$\mathbb{E}_x(\ell(B)) = \int_B g_\infty(x, y)dA_{\mathrm{hyp}}(y).$$

By the conformal invariance of Brownian motion, this integral is independent of the choice of horoball B and the point $x \in \partial B$. For a horoball B passing through $x = 0$, this is

$$\mathbb{E}_0(\ell(B)) = \frac{1}{\pi}\int_B \log\frac{1}{|y|}dA_{\mathrm{hyp}}(y) < \infty.$$

(ii) Similarly, conformal invariance allows us to consider the case when the initial point $x = 0$ and $B \subset \mathbb{D}$ is a horoball that rests on 1. The assumption on the hyperbolic distance from x to B implies that the Euclidean diameter of B is $\asymp e^{-R/2}$. Integrating, we obtain

$$\mathbb{E}_0(\ell(B)) = \frac{1}{\pi}\int_B \log\frac{1}{|y|}dA_{\mathrm{hyp}}(y) \asymp e^{-R/2}$$

as desired. □

Lemma 2.2. *Let B be a horoball in the unit disk and consider hyperbolic Brownian motion started at a point $x \in \partial B$. Then, $\mathbb{P}_x(B_t \in B) < Ce^{-\gamma t}$ for some $\gamma > 0$.*

Proof. As in the proof of Lemma 2.1(i), we may assume that the horoball B passes through $x = 0$. To see the lemma, note that $\mathbb{P}\big(d_{\mathbb{D}}(0, B_t) < t/2\big)$ and $|B \cap S_{r_t}|$ decay exponentially in t, where the latter quantity is just the length of the intersection of B and $S_{r_t} = \{z \in \mathbb{D} : d_{\mathbb{D}}(0, z) = t\}$. For the decay of the first quantity, the reader may consult (2.6). □

2.2 Monotonicity of the partial Green's functions

For a horoball B in the disk which does not contain the origin, we denote its top point (the one closest to the origin) by z_B and its Euclidean center by z_B^{mid}. The following lemma will be useful in the sequel:

Lemma 2.3. *We have:*

(i) *For a fixed $t > 0$, the quotient $g_t(r)/g_\infty(r)$ is decreasing in $r \in [0, 1)$.*

(ii) *For a horoball B in the unit disk contained in $\{z \in \mathbb{D} : 1/2 < |z| < 1\}$,*

$$g_t(z_B^{\mathrm{mid}}) \lesssim \int_B g_t(x) dA_{\mathrm{hyp}}(x) \lesssim g_t(z_B).$$

Proof. (i) Let us show that $g_t(r_1)/g_\infty(r_1) > g_t(r_2)/g_\infty(r_2)$ if $r_1 < r_2$. Consider two very thin disjoint annuli $A_1 = A(r_1, r_1')$ and $A_2 = A(r_2, r_2')$. From the probabilistic interpretation of the Green's function, it is clear that the function

$$G_1(x) = \int_{A_1} g_\infty(x, y) dA_{\mathrm{hyp}}(y)$$

only depends on $|x|$, is constant on $B(0, r_1)$ and is decreasing for $|x| \ge r_1$. We denote the value of G_1 on $B(0, r_1)$ by E_1. We define G_2 and E_2 similarly using the annulus A_2 in place of A_1.

We claim that for any $x \in \mathbb{D}$,

$$\frac{1}{E_2} \cdot \int_{A_2} g_\infty(x, y) dA_{\mathrm{hyp}}(y) \ge \frac{1}{E_1} \cdot \int_{A_1} g_\infty(x, y) dA_{\mathrm{hyp}}(y). \tag{2.7}$$

There are three possibilities: either $x \in B(0, r_1)$, $A(r_1, r_2)$ or $A(r_2, 1)$. We examine the three cases separately. In the first case,

$$\int_{A_1} g_\infty(x, y) dA_{\mathrm{hyp}}(y) = E_1, \qquad \int_{A_2} g_\infty(x, y) dA_{\mathrm{hyp}}(y) = E_2,$$

so (2.7) is an equality. In the second case,

$$\int_{A_1} g_\infty(x, y) dA_{\mathrm{hyp}}(y) < E_1, \qquad \int_{A_2} g_\infty(x, y) dA_{\mathrm{hyp}}(y) = E_2,$$

so (2.7) holds with strict inequality.

Finally, in the third case when $x \in A(r_2, 1)$, if a path of Brownian motion started at x hits A_1, it must first cross the circle $S_{r_2} = \{z : |z| = r_2\}$, so the third case can only be worse than the second case. This proves (2.7).

Let $g_{t,\infty} = g_\infty - g_t$. From (2.7) and the Markov property of Brownian motion, it is clear that

$$\frac{1}{E_2} \cdot \int_{A_2} g_{t,\infty}(0, y) dA_{\text{hyp}}(y) \geq \frac{1}{E_1} \cdot \int_{A_1} g_{t,\infty}(0, y) dA_{\text{hyp}}(y). \tag{2.8}$$

Subtracting both sides of the previous inequality from 1, we get

$$\frac{1}{E_2} \cdot \int_{A_2} g_t(0, y) dA_{\text{hyp}}(y) \leq \frac{1}{E_1} \cdot \int_{A_1} g_t(0, y) dA_{\text{hyp}}(y). \tag{2.9}$$

Letting the thickness of the annuli A_1 and A_2 tend to zero, we arrive at (i).

(ii) For $t = \infty$, this is a simple computation based on the explicit expression for g_∞. For $t < \infty$, we use (i). The upper bound is immediate, while for the lower bound, we only need to estimate the integral over the top half of B. □

We will also need:

Lemma 2.4. *Suppose $B_1 \subsetneq B_2$ are two horoballs which rest on the same point of the unit circle and $x \in \partial B_2$. Then, for any $t > 0$,*

$$\frac{\int_{B_1} g_t(x, y) dA_{\text{hyp}}(y)}{\int_{B_1} g_\infty(x, y) dA_{\text{hyp}}(y)} < \frac{\int_{B_2} g_t(x, y) dA_{\text{hyp}}(y)}{\int_{B_2} g_\infty(x, y) dA_{\text{hyp}}(y)}. \tag{2.10}$$

Proof. By conformal invariance, the ratio on the right side of (2.10) does not depend on the choice of starting point $x \in \partial B_2$. We can therefore denote this ratio by $\mathcal{Q}_t$. Clearly, $\mathcal{Q}_t$ is increasing in t since $g_t(x, y)$ is. In order for a Brownian path emanating from $x \in \partial B_2$ to contribute to the numerator of the left side of (2.10), it must cross ∂B_1 before time t. Let Π denote the collection of all such paths. We partition Π into disjoint collections $\Pi(x', t')$, indexed by $x' \in \partial B_1$ and $0 < t' < t$, where x' and t' are respectively the location and time of first entry into B_1. By the Markov property of Brownian motion and the conformal equivalence of horoballs, the ratio

$$\frac{\text{expected time Brownian motion spends in } B_1 \text{ during } [0, t]}{\text{expected time Brownian motion spends in } B_1}$$

over the bundle $\Pi(x', t')$ is $\mathcal{Q}_{t-t'}$. Since the ratio over any bundle is less than $\mathcal{Q}_t$, the ratio on the left side of (2.10) must also be less than $\mathcal{Q}_t$. This proves the lemma. □

3 Feynman-Kac formula

Consider a potential $V : \mathbb{D} \to \mathbb{R}$, which we assume to be positive and bounded. We are interested in studying the growth of solutions of the second order parabolic differential equation

$$\frac{\partial u}{\partial t} = \frac{1}{2} \cdot \Delta_{\mathrm{hyp}} u + V(x)u(x,t), \qquad (x,t) \in \mathbb{D} \times (0,\infty), \tag{3.1}$$

where the initial condition $u_0(x) = u(x,0)$ is a smooth positive compactly supported function.

3.1 Basic properties

Since $p_t(x,y) \to 0$ as $d_{\mathbb{D}}(x,y) \to \infty$, the Feynman-Kac formula (1.10) guarantees that u_t vanishes on the unit circle, i.e. $u_t(x) \to 0$ as $|x| \to 1$. In fact, since $p_t(x,y) \to 0$ super-exponentially quickly by (2.6), u_t must vanish on the unit circle to infinite order. Applying Green's formula, we find

$$\int_{\mathbb{D}} \Delta_{\mathrm{hyp}} u_t(x) dA_{\mathrm{hyp}}(x) \;=\; \int_{\mathbb{D}} \Delta u_t(x) dA(x) \;=\; \int_{\mathbb{S}^1} \partial_{\mathbf{n}} u_t(x) |dx| \;=\; 0,$$

where $\partial_{\mathbf{n}}$ denotes differentiation with respect to the outward pointing unit normal. In light of the above identity, if we integrate (3.1) over the unit disk, we obtain the crucial formula

$$\frac{d}{dt} \int_{\mathbb{D}} u_t(x) dA_{\mathrm{hyp}}(x) = \int_{\mathbb{D}} V(x) u_t(x) dA_{\mathrm{hyp}}(x), \tag{3.2}$$

which says that locally near a point $x \in \mathbb{D}$, the mass of u_t grows at rate V. The following theorem provides a convenient way to compute the Lyapunov exponent (1.11):

Theorem 3.1. *The rate of growth of the solution is given by*

$$\beta_V = \limsup_{t\to\infty} \frac{1}{t} \cdot \log \mathbb{E}_0 \Big\{ \exp \int_0^t V(B_s) ds \Big\}, \tag{3.3}$$

irrespective of the initial condition.

Proof. For the proof, we consider the auxiliary function

$$v_t(x) := \mathbb{E}_x\Big\{\exp\int_0^t V(B_s)ds\Big\}. \tag{3.4}$$

Using the Markov property of Brownian, it is easy to see that

$$p_1(x_1, y) < \phi\big(d_{\mathbb{D}}(x_1, x_2)\big) \cdot p_2(x_2, y)$$

for some increasing function $\phi : (0, \infty) \to (0, \infty)$. Another application of the Markov property of Brownian motion shows

$$\frac{v_{t+1}(x_1)}{v_{t+2}(x_2)} < \phi\big(d_{\mathbb{D}}(x_1, x_2)\big) \cdot e^{\|V\|_\infty}, \qquad t > 0.$$

Reversing the roles of x_1 and x_2 gives an inequality in the other direction. Therefore, the growth rates of all functions $v_t(x)$, $x \in \mathbb{D}$, are the same.

From the Feynman-Kac formula and the symmetry of the Brownian transition function $p_t(x, y) = p_t(y, x)$, it follows that

$$\begin{aligned}\int_{\mathbb{D}} u_t(x)dA_{\text{hyp}}(x) &= \int_{\mathbb{D}} u_0(x) \cdot \mathbb{E}_x\Big\{\exp\int_0^t V(B_s)ds\Big\}dA_{\text{hyp}}(x) \\ &= \int_{\mathbb{D}} u_0(x)v_t(x)\, dA_{\text{hyp}}(x). \end{aligned} \tag{3.5}$$

The above equation says that $\int_{\mathbb{D}} u_t(x)dA_{\text{hyp}}(x)$ is a weighted average of $v_t(x)$ over a compact subset of the disk and therefore it must have the same growth rate as well. This proves (3.3). □

Before continuing further, we note that since the total mass of u_t is increasing in time, the rate of growth of u_t is the same as that of

$$\hat{u}_t = \int_0^t u_s(x)ds, \tag{3.6}$$

i.e.

$$\beta_V = \limsup_{t\to\infty} \frac{1}{t} \log \int_{\mathbb{D}} \hat{u}_t(x)dA_{\text{hyp}}(x). \tag{3.7}$$

In practice, we prefer to work with $\hat{u}_t$ since it is slightly easier to estimate.

3.2 Some useful measures

For each $x \in \mathbb{D}$ and $t \in (0, \infty)$, we may disintegrate (3.4) to obtain a measure $v_t(x, y)dA_{\text{hyp}}(y)$ on the unit disk with the property that

$$\int_E v_t(x, y)dA_{\text{hyp}}(y) = \mathbb{E}_x\left\{\chi_E(B_t) \cdot \exp \int_0^t V(B_s)ds\right\} \tag{3.8}$$

holds for any measurable set $E \subset \mathbb{D}$. Indeed, the right hand side of (3.8) defines a measure on the unit disk, which is absolutely continuous since the exponential term is bounded and $\mathbb{E}_x\big(\chi_E(B_t)\big) = \mathbb{P}_x(B_t \in E) = \int_E p_t(x, y)dA_{\text{hyp}}(y)$. We set

$$\hat{v}_t(x, y) := \int_0^t v_s(x, y)ds.$$

Using the symmetry of the Brownian transition function as before shows

$$\int_E \hat{u}_t(x)dA_{\text{hyp}}(x) = \int_{\mathbb{D}} u_0(x)\left\{\int_E \hat{v}_t(x, y)dA_{\text{hyp}}(y)\right\}dA_{\text{hyp}}(x). \tag{3.9}$$

3.3 Potentials supported on gardens

We now turn our attention to Theorem 1.3. For the proof, we may assume that $\operatorname{supp} u_0 \cap \mathcal{G} = \emptyset$. In view of (3.2), in order to obtain an upper bound for β_V, it suffices to prove a *non-concentration estimate* – i.e. to show that most of the mass of $u_t(x)$ is located outside of $\mathcal{G}$.

Lemma 3.1. *For a parameter $p > 0$, consider the potential $V = p^2 \cdot \chi_{\mathcal{G}}$ where $\mathcal{G}$ is of the form (1.2), and let $u_t(x)$ be any solution of (3.1). If $p < p_0(R)$ is sufficiently small, then for any horoball $B \subset \mathcal{G}$ and any $t > 0$,*

$$\frac{\int_B \hat{u}_t(x)dA_{\text{hyp}}(x)}{\int_{B^*} \hat{u}_t(x)dA_{\text{hyp}}(x)} \le Ce^{-R/2}, \tag{3.10}$$

where as usual $B^* = \{z \in \mathbb{D} : d_{\mathbb{D}}(z, B) \le R/2\}$.

Temporarily assuming Lemma 3.1, note that since $V = 0$ on $\mathbb{D} \setminus \mathcal{G}$,

$$\int_{\mathbb{D}} V(x)\hat{u}_t(x)dA_{\text{hyp}}(x) \le Ce^{-R/2}p^2 \int_{\mathbb{D}} \hat{u}_t(x)dA_{\text{hyp}}(x), \qquad t > 0. \tag{3.11}$$

Combining with (3.2), we arrive at the inequality

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{D}} \hat{u}_t(x)dA_{\text{hyp}}(x) &= \int_{\mathbb{D}} u_t(x)dA_{\text{hyp}}(x),\\
&= \int_0^t \left(\frac{d}{ds}\int_{\mathbb{D}} u_s(x)dA_{\text{hyp}}(x)\right)ds + \int_{\mathbb{D}} u_0(x)dA_{\text{hyp}}(x),\\
&= \int_0^t \left(\int_{\mathbb{D}} V(x)u_s(x)dA_{\text{hyp}}(x)ds\right) + C(u_0),\\
&= \int_{\mathbb{D}} V(x)\hat{u}_t(x)dA_{\text{hyp}}(x) + C(u_0),\\
&\leq Ce^{-R/2}p^2\int_{\mathbb{D}} \hat{u}_t(x)dA_{\text{hyp}}(x) + C(u_0),
\end{aligned} \tag{3.12}$$

from which Theorem 1.3 follows after integration.

3.4 Proof of the non-concentration estimate

Recall that by (3.9), for any measurable set $E \subset \mathbb{D}$, the integral $\int_E \hat{u}_t(x)dA_{\text{hyp}}(x)$ is a weighted average of $\int_E \hat{v}_t(x,y)dA_{\text{hyp}}(y)$. Therefore, it is sufficient to show the non-concentration estimate for each $\hat{v}_t(x,y)$, $x \in \operatorname{supp} u_0 \subset \mathbb{D} \setminus \mathcal{G}$ instead.

Lemma 3.2. *For any $x \notin B^*$ and $t > 0$, the ratio*

$$\mathcal{Q}_{x,t}(B, B^*) \;=\; \frac{\int_B \hat{v}_t(x,y)dA_{\text{hyp}}(y)}{\int_{B^*} \hat{v}_t(x,y)dA_{\text{hyp}}(y)} \;\leq\; Ce^{-R/2}. \tag{3.13}$$

More precisely, the implication (Lemma 3.2 $\Rightarrow$ Lemma 3.1) follows from (3.9) and the following elementary observation: if (X,μ) is a measure space and $f, g : X \to [0,\infty)$ are non-negative functions on X, then

$$\frac{\int_X f(\xi)d\mu}{\int_X g(\xi)d\mu} \;\leq\; \operatorname*{ess\,sup}_{\xi\in X} \frac{f(\xi)}{g(\xi)}.$$

Similar reasoning shows that it is sufficient to consider the case when $x \in \partial B^*$. Inspecting the definition of $\hat{v}_t(x,y)$, we see that in order for a Brownian path started at $x \notin B^*$ to contribute to either the numerator or denominator of (3.13), it must cross ∂B^* before time t. Let Π denote the collection of all such paths. We partition

Π into disjoint collections $\Pi(x', t')$, indexed by $x \in \partial B^*$ and $0 < t' < t$, where x' and t' are respectively the location and time of first entry into B^*. By the Markov property of Brownian motion, the ratio over each bundle $\Pi(x', t')$ is at most $Ce^{-R/2}$, so the same must be true for $\mathcal{Q}_{x,t}(B, B^*)$.

Proof of Lemma 3.2, assuming $x \in \partial B^*$. According to Lemma 2.2, $\mathbb{P}(B_t \in B^*) \lesssim e^{-\gamma t}$ for some $\gamma > 0$. Utilizing the crude estimate $V \leq p^2$ gives

$$\int_{B^*} v_t(x,y) dA_{\text{hyp}}(y) \;=\; \mathbb{E}_x \left\{ \chi_{B^*}(B_t) \cdot \exp \int_0^t V(B_s) ds \right\} \;\lesssim\; e^{-(\gamma - p^2)t}.$$

Therefore, if the exponent $p^2 < \gamma - \delta$, then the contribution of long paths is very small: for any $\varepsilon > 0$, we can find $L(\varepsilon)$ sufficiently large to ensure that

$$\left| \int_{B^*} \hat{v}_t(x,y) dA_{\text{hyp}}(y) - \int_{B^*} \hat{v}_{L(\varepsilon)}(x,y) dA_{\text{hyp}}(y) \right| < \varepsilon, \quad \text{for } t > L(\varepsilon).$$

Since $\mathbb{P}(B_t \in B) \leq \mathbb{P}(B_t \in B^*)$, we must also have

$$\left| \int_{B} \hat{v}_t(x,y) dA_{\text{hyp}}(y) - \int_{B} \hat{v}_{L(\varepsilon)}(x,y) dA_{\text{hyp}}(y) \right| < \varepsilon, \quad \text{for } t > L(\varepsilon).$$

Hence, it is sufficient to prove (3.13) for $t \leq L(\varepsilon)$.

By making $p > 0$ sufficiently small, we can ensure that $1 < e^{p^2 L(\varepsilon)} < 1 + \varepsilon$ which means that the exponential term in (3.8) is essentially frozen if $t \leq L(\varepsilon)$. Thus $\mathcal{Q}_{x,t}(B, B^*)$ can be estimated by

$$\mathcal{Q}_{x,t}(B, B^*) \leq (1 + \varepsilon) \cdot \frac{\int_B g_t(x,y) dA_{\text{hyp}}(y)}{\int_{B^*} g_t(x,y) dA_{\text{hyp}}(y)}. \tag{3.14}$$

In view of Lemma 2.4 and Lemma 2.1, this is $\leq Ce^{-R/2}$ as desired. □

4 Brownian integral means spectrum

We now return to the original problem involving conformal mappings. To prove Theorem 1.1, we translate the Feynman-Kac argument from the previous section. In this section, we give a direct translation which mimics the previous section as much as possible. Later, we will give a slightly simplified account of this argument that does not involve Brownian motion.

Let $f : \mathbb{D} \to \mathbb{C}$ be a conformal mapping. Fix $p > 0$ and consider the functions

$$u_t(x) = |f'(x)|^p \cdot p_t(x) \tag{4.1}$$

and

$$\hat{u}_t(x) = \int_0^t u_s(x) ds = |f'(x)|^p \cdot g_t(x). \tag{4.2}$$

Differentiating, we discover

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{D}} |f'(x)|^p \cdot p_t(x) dA_{\text{hyp}}(x) &= \frac{1}{2} \int_{\mathbb{D}} |f'(x)|^p \cdot \Delta_{\text{hyp}}[p_t(x)] dA_{\text{hyp}}(x), \\ &= \frac{1}{2} \int_{\mathbb{D}} \Delta_{\text{hyp}} |f'(x)|^p \cdot p_t(x) dA_{\text{hyp}}(x), \\ &= \int_{\mathbb{D}} V(x) |f'(x)|^p \cdot p_t(x) dA_{\text{hyp}}(x), \end{aligned} \tag{4.3}$$

where

$$V = \frac{1}{2} \cdot p^2 |n_f/\rho|^2 \qquad \text{and} \qquad n_f := f''/f'. \tag{4.4}$$

By [15, Proposition 4.1], the "potential" $V(z) \le 9p^2/2$ is bounded. The above identity suggests, by comparison with (3.2), that estimating integral means

$$\tilde{\beta}_f(p) := \limsup_{t \to \infty} \frac{1}{t} \log \int_{\mathbb{D}} u_t(x) dA_{\text{hyp}}(x), \tag{4.5}$$

is quite similar to studying the growth rate of solutions of parabolic equations given by Feynman-Kac formula. However, now the mass flows differently: in the Feynman-Kac setting, the mass increases at rate V near a point equally in all directions, while in the conformal setting, the mass is spread out unevenly.

Similarly to (3.7) and (3.12), we have

$$\tilde{\beta}_f(p) = \limsup_{t\to\infty} \frac{1}{t} \log \int_{\mathbb{D}} \hat{u}_t(x) dA_{\text{hyp}}(x) \tag{4.6}$$

and

$$\frac{d}{dt} \int_{\mathbb{D}} \hat{u}_t(x) dA_{\text{hyp}}(x) \le C(p,f) + \int_{\mathbb{D}} V(x)\hat{u}_t(x) dA_{\text{hyp}}. \tag{4.7}$$

The other property of the "Brownian spectrum" (1.8) of a conformal mapping that we need is that it is larger than the usual integral means spectrum (1.4):

Lemma 4.1. *The inequality $\beta_f(p) \le \tilde{\beta}_f(p)$ holds for any conformal map f.*

Proof. Fix an $\varepsilon > 0$. Since the integral means $\int_{S_r} |f'(z)|^p \, d\theta$ are increasing in r, (2.5) shows that

$$\int_{\mathbb{D}} u_t(x) dA_{\text{hyp}}(x) \ge \frac{1}{2} \int_{S_{r_t}} |f'(z)|^p \, d\theta, \qquad t \ge t_0(\varepsilon), \tag{4.8}$$

where r_t is chosen so that $d_{\mathbb{D}}(0, S_{r_t}) = (1-\varepsilon)t$. Hence, $\tilde{\beta}_f(p) \ge (1-\varepsilon)\beta_f(p)$. Since $\varepsilon > 0$ was arbitrary, the proof is complete. □

In reality, with exponentially small probability, Brownian particles can travel farther than expected, so the two characteristics need not be equal. Using (2.6), it is not difficult to show the precise relation $\tilde{\beta}_f(p) = \beta_f(p) + \beta_f(p)^2$, for any conformal map f, but we will not need this fact.

4.1 Sparse conformal mappings

We now recall the setting of Theorems 1.1 and 1.2. We begin with a Beltrami coefficient μ with $\|\mu\|_\infty \le 1$, supported on a garden $\mathcal{G} \subset \mathbb{D}$ satisfying the sparsity condition (1.2). We reflect μ in the unit circle to obtain a Beltrami coefficient μ^+, and then solve the Beltrami equation $\overline{\partial} w = k\mu^+ \, \partial w$ for some $k < 0.49$ to obtain a quasiconformal map $f = \tilde{w}^{k\mu^+}$ which is conformal on $\mathbb{D}$.

We now fix a $p > 0$. To estimate $\tilde{\beta}_f(p)$, we consider the potential $V(z)$ from (4.4). Since f is holomorphic, $V = \frac{1}{2} \cdot p^2 |n_f/\rho|^2$ cannot be identically zero on $\mathbb{D} \setminus \mathcal{G}$ (unless f is linear). Nevertheless, by carefully estimating the non-linearity $n_f = f''/f'$ (see Appendix A), one can show that V decays exponentially quickly away from $\mathcal{G}$, with a sufficiently large exponent.

Lemma 4.2. *Let $S(z) := d_{\mathbb{D}}(z, \mathcal{G})$ denote the hyperbolic distance from a point $z \in \mathbb{D}$ to the garden $\mathcal{G}$. For any $\delta > 0$, we can find an $0 < r_\delta < 1$ so that*

$$V(z) \,\le\, V_2^\delta(z) \,:=\, \begin{cases} Ck^2p^2e^{-1.01\,S(z)} + \delta, & |z| > r_\delta,\ S(z) < R/2, \\ Ck^2p^2e^{-1.01\,R/2} + \delta, & |z| > r_\delta,\ S(z) \ge R/2, \\ M, & |z| < r_\delta. \end{cases} \tag{4.9}$$

Remark. The bound on the compact set $\{z : |z| < r_\delta\}$ is unimportant since it plays no role in determining the integral means spectrum.

We will write V_2 for $V_2^\delta - \delta$. Proceeding by analogy with Section 3, the following non-concentration estimate is likely to be useful:

Lemma 4.3. *Suppose $B \subset \mathcal{G}$ is one of the horoballs in the garden. If the enlarged horoball $B^* \subset \{z : r_\delta < |z| < 1\}$, then for any $t > 0$,*

$$\frac{\int_{B^*} V_2^\delta(x)\hat{u}_t(x)dA_{\mathrm{hyp}}(x)}{\int_{B^*} \hat{u}_t(x)dA_{\mathrm{hyp}}(x)} \le Ck^2p^2e^{-R/2} + \delta, \qquad kp < c/R. \tag{4.10}$$

Assuming Lemma 4.3 for the moment, we have

$$\frac{\int_{\mathbb{D}} V(x)\hat{u}_t(x)dA_{\mathrm{hyp}}(x)}{\int_{\mathbb{D}} \hat{u}_t(x)dA_{\mathrm{hyp}}(x)} \le Ck^2p^2e^{-R/2} + \delta, \qquad t > t_0. \tag{4.11}$$

Indeed, by the transience of Brownian motion, the contribution of $B(0, r_\delta)$ to the integrals in the numerator and denominator of (4.11) is negligible for $t > t_0$. Similarly, Lemma 2.2 allows us to neglect finitely many large horoballs B_i^* which happen to intersect $\{z : |z| = r_\delta\}$. Finally, the analogue of (4.10) for the complementary region $A(r_\delta, 1) \cap \{z : S(z) \ge R/2\}$ is trivial since $V_2^\delta(z) \le Ck^2p^2e^{-R/2} + \delta$ there.

Recalling (4.7) and integrating, we arrive at $\tilde{\beta}_{\tilde{w}^{k\mu^+}}(p) \le Ce^{-R/2}k^2p^2 + \delta$. Since $\delta > 0$ was arbitrary, we conclude that $\tilde{\beta}_{\tilde{w}^{k\mu^+}}(p) \le Ce^{-R/2}k^2p^2$. Applying Lemma 4.1 proves $\beta_{\tilde{w}^{k\mu^+}}(p) \le Ce^{-R/2}k^2p^2$, which is the statement of Theorem 1.2. As mentioned in the introduction, Theorem 1.1 can be obtained from Theorem 1.2 using (1.5).

4.2 Proof of the non-concentration estimate

To prove the non-concentration estimate, we use bounds on the Bloch norm of conformal maps. Recall that the Bloch space $\mathcal{B}$ consists of all holomorphic functions on the unit disk which satisfy

$$\|g\|_{\mathcal{B}} := \sup_{z\in\mathbb{D}} |g'(z)|(1-|z|^2) = 2\cdot \sup_{z\in\mathbb{D}} |(g'/\rho)(z)| < \infty.$$

By [15, Proposition 4.1], if $f:\mathbb{D}\to\mathbb{C}$ is a conformal mapping, then $\|\log f'\|_{\mathcal{B}} \le 6$. If we know that f admits a k-quasiconformal extension to the plane, then by Lehto's majorant principle [15, Chapter 5.6], this bound can be improved to $\|\log f'\|_{\mathcal{B}} \le 6k$. We will use this bound in the integrated form $|\log f'(x) - \log f'(y)| \le 3k\cdot d_{\mathbb{D}}(x,y)$, for $x,y\in\mathbb{D}$. Before proving Lemma 4.3, we first show the weaker statement

$$\frac{\int_{B^*} V_2^{\delta}(x)\hat{u}_\infty(x)dA_{\mathrm{hyp}}(x)}{\int_{B^*}\hat{u}_\infty(x)dA_{\mathrm{hyp}}(x)} \le Ck^2p^2e^{-R/2}+\delta, \qquad kp<c/R. \tag{4.12}$$

This relies on the "freezing lemma" which says that for the purpose of estimating integral means, $|f'(z)|^p$ is essentially constant on horoballs:

Lemma 4.4. *Suppose $f:\mathbb{D}\to\mathbb{C}$ is a conformal mapping which admits a k-quasiconformal extension to the plane. If $6kp<0.49$, then*

$$\int_{B^*} |f'(z)|^p\, g_\infty(z)dA_{\mathrm{hyp}}(z) \asymp \operatorname{diam} B^*\cdot |f'(z_{B^*})|^p. \tag{4.13}$$

If additionally $k<0.49$ and $kp<c/R$, then

$$\int_{B^*} V_2(z)|f'(z)|^p\, g_\infty(z)dA_{\mathrm{hyp}}(z) \asymp Ck^2p^2e^{-R/2}\cdot \operatorname{diam} B^*\cdot |f'(z_{B^*})|^p. \tag{4.14}$$

Proof. We begin with (4.13). For the lower bound, it suffices to observe that the integral over the top half of B^* is comparable to $\operatorname{diam} B^*\cdot|f'(z_{B^*})|^p$. For the upper bound, we use

$$\int_{B^*} |f'(z)|^p\, g_\infty(z)dA_{\mathrm{hyp}}(z) \lesssim |f'(z_{B^*})|^p\cdot \int_{B^*}\left(\frac{1-|z_{B^*}|}{1-|z|}\right)^{6kp}\frac{|dz|^2}{1-|z|}.$$

The right hand side is integrable provided that $6kp<1/2$. By asking for $6kp<0.49$, we ensure that the integral over the top half of B^* controls the integral over the bottom half.

We turn to (4.14). It is useful to decompose B^* into shells. For $1\le m\le n=\lceil R/2\rceil$, set $B^m=\{z\in B^*: d_{\mathbb{D}}(z,B)\le m\}$, $S^m=B^m\setminus B^{m-1}$ and $S^0=B$. Thus

$B^* = \bigcup_{m=0}^n S^m$. The condition $kp < c/R$ ensures that $|f'(z_{B^m})|^p \asymp |f'(z_{B^*})|^p$ for all $m = 0, 1, \ldots, n$. According to (4.9), $V_2(z) \asymp Ck^2p^2e^{-1.01\,m}$ for $z \in S^m$. As m increases, the contributions of S^m decay exponentially (due to the exponent 1.01), so the integral in (4.14) is dominated by the integral over $S^0 = B$. □

Equation (4.12) follows after dividing (4.14) by (4.13). In order to show Lemma 4.3, it remains to replace "∞" with "t." This last step is not necessary when one uses the Becker-Pommerenke method described in the next section, nevertheless it is quite easy with help of Lemma 2.3.

Proof of Lemma 4.3. Since $V_2 \le Ck^2p^2e^{-R/2}$ is small on shells n and $n-1$, we need not worry about them in the numerator. Estimating the numerator (with two shells removed) from above and the denominator from below, we get

$$\begin{aligned}\int_{B^{n-2}} V_2(z)|f'(z)|^p\, g_t(z) dA_{\text{hyp}}(z) &\lesssim |f'(z_{B^*})|^p \cdot \sum_{m=0}^{n-2} V_2(z_{B^m}) \cdot g_t(z_{B^m}) \\ &\lesssim |f'(z_{B^*})|^p \cdot g_t(z_{B^{n-2}}) \cdot \sum_{m=0}^{n-2} k^2p^2e^{-1.01\,m} \cdot \frac{e^m}{e^{n-2}} \\ &\lesssim |f'(z_{B^*})|^p \cdot g_t(z_{B^{n-2}}) \cdot k^2p^2e^{-R/2}\end{aligned}$$

and

$$\int_{B^*} |f'(z)|^p\, g_t(z) dA_{\text{hyp}}(z) \gtrsim |f'(z_{B^*})|^p \cdot g_t(z_{B^{n-2}}).$$

Hence, (4.10) follows after division. □

5 Becker-Pommerenke argument

The reader may suspect that Brownian motion is not truly required to prove Theorems 1.1 and 1.2. In this section, we give a slightly simplified account of the above argument using the framework of Becker and Pommerenke for estimating integral means as presented in [11]. This deterministic approach is based on the study of the growth rate of the function

$$\hat{u}(r) := \int_{B(0,r)} |f'(z)|^p \cdot g_\infty(z) dA_{\text{hyp}},$$

as $r \to 1$. The reader may compare the above function with the one in (4.2). As in [11], we prefer to work in the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ since it greatly simplifies the computation.

5.1 Growth of functions in the upper half-plane

We write $z = x + iy$ and $\rho_{\mathbb{H}} = 1/y$. For a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ satisfying the periodicity condition $f(z+1) = f(z) + 1$, define

$$u(y) := \int_0^1 |f'(x+iy)|^p \, dx, \tag{5.1}$$

and

$$\hat{u}(y) := \int_{A(y)} |f'(x+it)|^p \, \frac{dxdt}{t}, \tag{5.2}$$

where $y \in (0,1]$ and $A(y)$ is the rectangle $[0,1] \times [y,1] \subset \mathbb{H}$. Taking the second derivative as in [11, Section 7], we obtain:

Lemma 5.1.

$$u''(y) = \frac{p^2}{4y^2} \int_0^1 |f'(x+iy)|^p \left| \frac{2n_f}{\rho_{\mathbb{H}}} \right|^2 dx \tag{5.3}$$

$$\hat{u}''(y) = \frac{p^2}{4y^2} \int_{A(y)} |f'(x+it)|^p \left| \frac{2n_f}{\rho_{\mathbb{H}}} \right|^2 \frac{dxdt}{t} + \mathcal{O}_{p,f}(1/y^2). \tag{5.4}$$

Proof. The periodicity condition implies that

$$\partial_x^2 \int_0^1 |f'(x+iy+s)|^p ds = 0 \qquad \Longrightarrow \qquad u''(y) = \Delta \int_0^1 |f'(x+iy+s)|^p ds.$$

Expanding $\Delta = 4\partial\overline{\partial}$ proves (5.3). For the second statement, note

$$\hat{u}(y) \;=\; \int_y^1 u(t)\, \frac{dt}{t} \;=\; \int_1^{1/y} u(yt)\, \frac{dt}{t}.$$

Differentiation shows

$$\hat{u}'(y) = \int_1^{1/y} u'(yt)\, dt - u(1)/y,$$

$$\hat{u}''(y) \;=\; \int_1^{1/y} u''(yt)\, tdt - u'(1)/y^2 + u(1)/y^2 \;=\; \frac{1}{y^2} \int_y^1 u''(t)\, tdt + \mathcal{O}(1/y^2).$$

After some rearranging, we arrive at (5.4). $\square$

5.2 A non-concentration estimate

Let $\mathcal{G} = \bigcup B_j$ be a collection of horoballs in $\mathbb{H}$ such that $d_{\mathbb{H}}(B_i, B_j) > R$ for $i \neq j$, and suppose μ is a Beltrami coefficient with $\|\mu\|_\infty \leq 1$ whose support is contained in the reflected garden $\overline{\mathcal{G}} \subset \overline{\mathbb{H}}$. Without loss of generality, we may assume that $\mathcal{G}$ is contained in $\{z \in \mathbb{C} : 0 < \operatorname{Im} z < 2\}$ and that $\mathcal{G}$ and μ are invariant under $z \to z+1$ so that $f = \tilde{w}^{k\mu}$ satisfies the periodicity condition $f(z+1) = f(z)+1$. For a horoball B in the upper half-plane, let $B^* = \{z \in \mathbb{H} : d_{\mathbb{H}}(z, B) < R/2\}$, where $d_{\mathbb{H}}(\cdot,\cdot)$ denotes the hyperbolic distance in $\mathbb{H}$. The separation condition ensures us that the horoballs B_j^* are disjoint. In this setting, the statement of Theorem 1.2 becomes:

Theorem 5.1. *For $f = \tilde{w}^{k\mu}$ as above,*

$$\beta_f(p) \leq Ce^{-R/2}k^2p^2/4, \qquad k < 0.49, \quad kp < c/R, \tag{5.5}$$

where the integral means spectrum of $f : \mathbb{H} \to \mathbb{C}$ is given by

$$\beta_f(p) = \limsup_{y \to 0^+} \frac{\log \int_0^1 |f'(x+iy)|^p \, dx}{|\log y|}, \qquad p > 0.$$

We define $\hat{u}(y)$ by (5.2). From the definition, it is clear that $\hat{u}(y) \geq 0$ and $\hat{u}'(y) \leq 0$ for $y \in (0,1]$. Our aim is to show the estimate

$$\hat{u}''(y) \leq \Big[\tilde{\beta} \cdot \hat{u}(y) + \mathcal{O}(1)\Big] \cdot (1/y^2), \qquad \tilde{\beta} = Ce^{-R/2}k^2p^2/4. \tag{5.6}$$

From the differential inequality (5.6), it is not difficult to deduce Theorem 5.1: if we define β as the unique positive root of $\beta^2 + \beta = \tilde{\beta}$, then by [11, Lemma 7.1], we have $\hat{u}(y) \leq Cy^{-\beta}$ which easily implies that $\beta_f(p) \leq \beta$. This is fine since β and $\tilde{\beta}$ are comparable when either quantity is small. Note that the $\mathcal{O}(1)$ term in (5.6) is pretty harmless in this discussion.

The proof of (5.6) is quite simple and most of the heavy lifting has been done in Section 4. For a set $K \subset \mathbb{H}$, let us write

$$\mathcal{Q}(K) := \frac{\int_K |f'(x+iy)^p| \left|\frac{2n_f}{\rho_{\mathbb{H}}}\right|^2 \frac{|dz|^2}{y}}{\int_K |f'(x+iy)^p| \frac{|dz|^2}{y}}. \tag{5.7}$$

In this formalism, we must show that

$$\mathcal{Q}(A(y)) \le Ce^{-R/2}k^2, \tag{5.8}$$

provided $y > 0$ is sufficiently small. By an analogue of the freezing lemma for the upper half-plane (Lemma 4.4, with $|dz|^2/y$ replacing $g_\infty dA_{\text{hyp}}$), we have

$$\mathcal{Q}\big(B_j^* \cap A(y)\big) \le Ce^{-R/2}k^2$$

for each enlarged horoball B_j^*. Outside $\mathcal{G}^* = \bigcup B_j^*$, the weight $\big|\frac{2n_f}{\rho_{\mathbb{H}}}\big|^2$ is negligible, and so the quotient $\mathcal{Q}\big(A(y) \setminus \mathcal{G}^*\big) \le Ce^{-R/2}k^2$ is small as well. Putting these estimates together proves (5.8), (5.6) and Theorem 5.1.

Remark. Define a "flower" of order γ to be an image of

$$F(1,0,\gamma) = \big\{z \in \mathbb{H} \,:\, -1 < x < 1,\, x^\gamma < y < 1\big\}$$

under an affine mapping $z \to az + b$ with $a > 0$, $b \in \mathbb{R}$. The argument in this section is also applicable when the garden $\mathcal{G}$ is a union of flowers $\bigcup_{j=1}^\infty F(a_j, b_j, \gamma_j)$ which are located at least a hyperbolic distance R apart, provided the orders $\{\gamma_j\}$ are bounded. In this case, (5.5) still holds but the constants c and C could be different. The bound on the orders implies that there exists an $\varepsilon > 0$ sufficiently small so that $\int_{F_j} |dz|^2/y^{1+\varepsilon} < \infty$, for all $F_j \subset \mathcal{G}$, which is needed to establish an analogue of Lemma 4.4. We leave the details to the interested reader.

A Dyn'kin's estimate

In [7, Theorem 1], E. Dyn'kin proved a general estimate for the non-linearity $n_f = f''/f'$ of a conformal mapping $f : \mathbb{D} \to \mathbb{C}$ which admits a quasiconformal extension to a homeomorphism of the plane, mapping $\mathbb{C}$ onto itself:

Lemma A.1. *Suppose $0 < k < 1$ and $f : \mathbb{D} \to \mathbb{C}$ is a conformal mapping which has a k-quasiconformal extension to the plane with dilatation μ. Then,*

$$\left|\frac{n_f}{\rho}(z)\right| \le C_k(1-|z|)^{1-k}\left[1 + \int_{1-|z|}^1 \frac{\omega(z,t)}{t^{2-k}}dt\right], \qquad |z| < 1, \tag{A.1}$$

where

$$\omega(z,t) = \left(\frac{1}{\pi t^2} \int_{|\zeta - z| \le t} |\mu(\zeta)|^2 |d\zeta|^2 \right)^{1/2}.$$

Here, the constant C_k can be taken to be non-decreasing in $k \in (0,1)$.

If one is interested in utilizing only the support of μ, Dyn'kin's technique yields a slightly better estimate:

Lemma A.2. *Suppose $0 < k < 1$ and $f : \mathbb{D} \to \mathbb{C}$ is a conformal mapping which has a k-quasiconformal extension to the plane with dilatation μ. Then,*

$$\left| \frac{n_f}{\rho}(z) \right| \le C'_k (1 - |z|)^{1-k} \left[1 + \int_{1-|z|}^{1} \frac{\tilde{\omega}(z,t)}{t^{2-k}} dt \right], \qquad |z| < 1, \tag{A.2}$$

where

$$\tilde{\omega}(z,t) = k \cdot \left(\frac{|\operatorname{supp} \mu \cap B(z,t)|}{|B(z,t)|} \right)^{\frac{1}{1+k}}.$$

Again, the constant C'_k can be taken to be non-decreasing in $k \in (0,1)$.

The proof of the above lemma is nearly identical to that of Dyn'kin's theorem, so we only give a sketch of the argument and explain where the improvement comes from. By post-composing with a linear map, we may assume that $f(0) = 0$ and $f'(0) = 1$. Set $r = 1 - |z|$, $B_j = B(z, 2^j r)$ and $E_j = \operatorname{supp} \mu \cap B(z, 2^j r)$. By the Cauchy-Green formula and the elementary bound $|f(z)| \le C_k$ for $|z| \le 2$, we have

$$|f''(z)| \le \left| \frac{2}{\pi} \int_{1<|\zeta|<2} \frac{\partial f}{\partial \overline{\zeta}} \frac{|d\zeta|^2}{(\zeta - z)^3} \right| + C_k, \qquad |z| < 1.$$

Applying the Cauchy-Schwarz inequality, we see that the contribution of the annulus $\{\zeta : 2^j r < |\zeta - z| < 2^{j+1} r\}$ does not exceed

$$\frac{1}{(2^j r)^3} \left(\int_{E_j} |\mu(\zeta)|^2 |d\zeta|^2 \right)^{1/2} \left(\int_{E_j} \left| \frac{\partial f}{\partial \zeta} \right|^2 |d\zeta|^2 \right)^{1/2}.$$

Dyn'kin estimates the first term by $(2^j r) \cdot \omega(z, 2^j r)$; to estimate the second term, he replaces the integrand with the Jacobian of f and uses the coarse bound

$$|f(E_j)| \le |f(B_j)|.$$

In our setting, Astala's area distortion theorem [3, Theorem 13.1.5] yields the stronger estimate

$$|f(E_j)| \le C_k |f(B_j)| \left(\frac{|E_j|}{|B_j|} \right)^{\frac{1-k}{1+k}}.$$

The use of this stronger estimate explains why the exponent in Lemma A.2 is $\frac{1}{2} + \frac{1}{2} \cdot \frac{1-k}{1+k} = \frac{1}{1+k}$ compared to the exponent in Lemma A.1 which is only $\frac{1}{2}$.

In this paper, we utilize the above estimate in a slightly different form. Let $(\operatorname{supp}\mu)^+ \subset \mathbb{D}$ be the reflection of the support of μ in the unit circle. In terms of the hyperbolic distance from z to $(\operatorname{supp}\mu)^+$, the estimate says:

Corollary A.1. *Suppose* $0 < k < 1$, $d_{\mathbb{D}}\big(z, (\operatorname{supp}\mu)^+\big) > L$ *and* $d_{\mathbb{D}}(z, 0) > L$. *Then,*

$$\left| \frac{n_f}{\rho}(z) \right| \lesssim k \cdot L e^{-(1-k)L} + (1-|z|)^{1-k},$$

where the implicit constant can be taken to be non-decreasing in $k \in (0,1)$.

Proof. For $j = 0, 1, 2, \dots, \lfloor L \rfloor$, the assumptions on the support of μ imply that the Euclidean area

$$\big|\operatorname{supp}\mu \cap B\big(z, e^j(1-|z|)\big)\big| \le C(1-|z|)^2 e^{3j-L},$$

which gives the bounds $\omega\big(z, e^j(1-|z|)\big) \lesssim k \cdot e^{(j-L)/2}$ and $\tilde{\omega}\big(z, e^j(1-|z|)\big) \lesssim k \cdot e^{(j-L)/(1+k)}$. Hence, the right hand side of (A.2) is bounded above by

$$C(1-|z|)^{1-k} \left(1 + \int_{e^L(1-|z|)}^{\infty} \frac{k}{t^{2-k}} dt + \sum_{j=0}^{\lfloor L \rfloor} \int_{e^j(1-|z|)}^{e^{j+1}(1-|z|)} \frac{k \cdot C e^{(j-L)/(1+k)}}{t^{2-k}} dt \right). \tag{A.3}$$

After opening the brackets, the second term in (A.3) is

$$= \frac{Ck}{1-k}(1-|z|)^{1-k} \big[e^L (1-|z|) \big]^{-(1-k)} \asymp Ck \cdot e^{-L(1-k)}, \tag{A.4}$$

while the j-th term in the sum is comparable to

$$C(1-|z|)^{1-k} \int_{e^j(1-|z|)}^{e^{j+1}(1-|z|)} \frac{k \cdot e^{(j-L)/(1+k)}}{t^{2-k}} dt \asymp Ck \cdot e^{-j(1-k)} \cdot e^{(j-L)/(1+k)}.$$

Since $1/(1+k) - (1-k) > 0$ for any $0 < k < 1$, this is an increasing geometric series in j. In particular, each term in the sum is bounded by the last term with $j = L$. Putting these estimates together gives the corollary. $\square$

B Wiggly potentials

In this appendix, we give another entry in the dictionary between integral means spectra of conformal maps and perturbations of the Laplacian. Call a potential $V(x)$ *wiggly* if there exists $R, \alpha > 0$ such that $\int_B V(x) > \alpha$ over any ball of hyperbolic radius R.

Theorem B.1. *For a wiggly potential V, there exists a constant $c = c(\alpha, R) > 0$ such that $\beta_V(p) = \beta_{p^2 V} > cp^2$ for $0 < p \le 1$.*

This is reminiscent of a variant of a theorem of P. Jones where we use non-linearity instead of the Schwarzian derivative:

Theorem B.2. *Suppose $f : \mathbb{D} \to \mathbb{C}$ is a conformal mapping onto a bounded domain with quasicircle boundary, such that any point $z \in \mathbb{D}$ is located within hyperbolic distance R of a point $w \in \mathbb{D}$ with*

$$\left| \frac{n_f}{\rho}(w) \right| > \alpha.$$

Then, there exists a constant $c(\alpha, R) > 0$ so that

$$\beta_f(p) > c(\alpha, R)p^2, \qquad 0 < p \le 1.$$

Theorem B.2 is a special case of [11, Theorem 1.5]. We leave the proof of Theorem B.1 as an exercise for the reader.

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