Asymptotic variance of shell-type coefficients

In [AIPP], we constructed a Beltrami coefficient μ_{20} with $\sigma^2(\mathcal{S}\mu_{20}) \geq 0.87913$. In this note, we compute the supremum of $\sigma^2(\mathcal{S}\mu)$ over all Beltrami coefficients of *shell-type*, i.e. of the form

$$\mu = \sum_{j=0}^{\infty} \mu_j = \sum_{j=0}^{\infty} (\overline{z}/|z|)^{j-2} \chi_{A(r_j,R_j)}. \tag{1}$$

We show:

Theorem 1. *The supremum of $\sigma^2(\mathcal{S}\mu)$ over Beltrami coefficients of shell-type is given by*

$$\sup_{\mu \in M_{\text{shell}}} \sigma^2(\mathcal{S}\mu) = \max_{d\in\mathbb{R}} 4d^{\frac{2}{1-d}} \frac{(d-1)^2}{d^2 \log d} \approx 0.89714, \tag{2}$$

maximized for $d \approx 20.6$.

More generally, one may consider Beltrami coefficients of *generalized shell-type*,

$$\mu = \sum_{j=0}^{\infty} \mu_j = \sum_{j=0}^{\infty} (\overline{z}/|z|)^{j-2} \chi_{A_{E_j}}, \tag{3}$$

where $E_j \subset (0,1)$ are disjoint measurable subsets, and $A_{E_j} = \{z : |z| \in E_j\}$. However, we will see here that the use of this enlarged class of Beltrami coefficients does not increase the asymptotic variance.

For Beltrami coefficients of shell-type, one may compute the Beurling transform explicitly. On the exterior unit disk, we have

$$\begin{aligned} \mathcal{S}\mu &= \sum_{j=0}^{\infty} -\frac{n_j - 1}{n_j/2} \cdot z^{-n_j} \cdot (R_j^{n_j} - r_j^{n_j}) \\ &\sim \sum_{j=0}^{\infty} -2 \cdot z^{-n_j} \cdot (R_j^{n_j} - r_j^{n_j}) \end{aligned}$$

where $\sim$ denotes equality up to an additive factor in $\mathcal{B}_0$.

We first show the $(\leq)$ in (2). Using the infinitesimal fractal approximation principle of [AIPP], we may assume that μ is eventually-invariant under a map $z \to z^N$, for some $N \in \mathbb{N}$.

In this case, the asymptotic variance is given by

$$\sigma^2(\mathcal{S}\mu) = \sum_{j=1}^{J} \frac{4(R_j^{n_j} - r_j^{n_j})^2}{\log N}, \tag{4}$$

where the sum is taken over equivalence classes of annuli, with two annuli equivalent if one is the image of another by a map $z \to z^{N^k}$. We rewrite the above equation as

$$\sigma^2(\mathcal{S}\mu) = \sum_{j=1}^{J} \frac{\log(R_j/r_j)}{\log N} \cdot \frac{4(R_j^{n_j} - r_j^{n_j})^2}{\log(R_j/r_j)}. \tag{5}$$

For an annulus A_j, we define the width $w(A_j) := \log(R_j/r_j)$. To maximize (4), we need to use the width of each annulus as efficiently as possible. Therefore, it is natural to maximize the quantity

$$F(r, R, n) := \frac{4(R^n - r^n)^2}{\log(R/r)}, \qquad \text{over all } 0 < r < R < 1 \text{ and } n \in \mathbb{N}. \tag{6}$$

Observe that the quantity (6) is invariant under substitutions

$$(r, R, n) \to (r^{1/\ell}, R^{1/\ell}, \ell n).$$

Therefore, when taking the maximum of $F(r, R, n)$, we may set $n = 1$. With this choice of n, we seek to find the optimal choices of r and R and show that they are unique. It is easy to see that r must be bounded away from 0 and R must be bounded away from 1.

Simple calculus shows that for a fixed $d := \log(R/r) \in (0, \infty)$, $F(r, R, 1)$ is maximized for $r(d) = d^{d/(1-d)}$ and $R(d) = d^{1/(1-d)}$. Maximizing over $d \geq 0$ over real numbers gives the values r_*, R_*, d_* with $d_* \approx 20.6$, in which case $F(r_*, R_*, 1)$ is given by (2). Therefore,

$$\sigma^2(\mathcal{S}\mu) \leq F(r_*, R_*, 1),$$

being the average of terms, each of which is at most $F(r_*, R_*, 1)$.

Now we construct a Beltrami coefficient, with asymptotic variance arbitrarily close to (2). Take N to be a large integer. Consider the annulus $A = A(r_0, r_0^{1/N})$, with r_0 close to 1, to be chosen shortly. We will define the Beltrami coefficient μ on

A and extend it to $\{z : r_0^{1/N} < |z| < 1\}$ by z^N-invariance. In the disk $\{z : |z| < r_0\}$, we simply set $\mu = 0$. The annulus A may be partitioned into bands

$$A_1 = A(r_0, r_0^{1/d_*}),\ A_2 = A(r_0^{1/d_*},\ r_0^{1/d_*^2}),\ \ldots,\ A_n = A(r_0^{1/d_*^{n-1}}, r_0^{1/d_*^n})$$

and an "error band" A_{err}.

On A_{err}, we set μ to be 0. On A_j with $1 \le j \le n$, we take n_j so that $|r_j^{n_j} - r_*| < \varepsilon$. This may be achieved by taking r_0 very close to 1 to make the "step size" arbitrarily small. It is easy to see that the given Beltrami coefficient has the desired asymptotic variance as the error term is insignificant if N is large.

To complete the discussion, let us show that the generalized shell-type coefficients do not give a larger asymptotic variance. Indeed, it suffices to show that for a set $E \subset (0, 1)$, the quantity

$$4\Big(\int_E dx\Big)^2 \Big/ \Big(\int_E \frac{dx}{x}\Big)$$

is maximized for an interval. The point is that if E has gaps, we can rearrange E in such a way that $|E|$ is unchanged but $\int_E \frac{dx}{x}$ is decreased by either sliding the left part of E rightward, or the right part of E leftward.