Ph125c lecture notes, 5/22/01

Generalized measurement theory

Early on in this course (start of first term) we discussed the basic formalism of quantum measurement, in terms of projection operators. Today we'll develop a broader picture, introducing first the distinction between direct and indirect measurements and then building up the general measurement formalism of Effects and Operations. We'll restrict our discussion to the case of finite-dimensional Hilbert spaces, but rest assured that the general treatment can be adapted to continuous degrees of freedom as well.

Before jumping into formalism, let's take a minute to clarify what we mean when we talk about a 'measurement.' Basically we have some physical system of interest (an atom, an electromagnetic field mode, a many-body system of electrons, etc.), and some sequence of events occur (we shine lasers on it, drop it, place it in contact with a heat bath, etc.) such that the system ends up in some quantum state $\boldsymbol{\rho}_{\text{pre}}$. Depending on how much control we've had over this sequence of events, we may know exactly what $\boldsymbol{\rho}_{\text{pre}}$ is or we may have only a vague idea (for example we could let the system evolve according to its own Hamiltonian, which we may or may not know!). In any case, to perform a measurement we couple the system of interest to some sort of 'sensing apparatus' (traditionally called the 'meter') such as a photodetector or charge sensor, etc. The meter is typically a macroscopic device, and from it we suppose we can read off a *result* for the measurement, which hopefully gives us some interesting information about the state $\boldsymbol{\rho}_{\text{pre}}$.

What we want from quantum measurement *theory* is a way of representing this type of measurement procedure mathematically. In particular, we certainly want to know how to calculate the probabilities of the various possible measurement outcomes, given some particular $\boldsymbol{\rho}_{\text{pre}}$. In some cases we may also want to know how to compute the post-measurement system state $\boldsymbol{\rho}_{\text{post}}$, which in general can depend on both $\boldsymbol{\rho}_{\text{pre}}$ and the particular measurement outcome.

The range of different measurements that we currently know how to perform in the laboratory is restricted by technology and ingenuity. Quantum mechanics, however, places some fundamental restrictions on the types of measurements that are physically possible, via formal restrictions on their mathematical representations. It is of basic scientific interest to investigate these intrinsic quantum limits to measurement. Somewhat surprisingly, quantum measurement theory can also be used in the converse sense, in that one can specify the 'space' of all possible measurements (again via their mathematical representations) in such a way that mathematical reasoning can be used to guide the development of sophisticated laboratory measurement procedures. This is likely to become an increasingly important methodology in science and engineering over the next few decades, so pay attention!

Direct measurements

Say we have a quantum system that lives in an N_S –dimensional Hilbert space H_S,

$$|\Psi\rangle \in H_S.$$

Then a basic measurement axiom is that every possible *direct measurement* is represented by a complete set of projection operators $\{\mathbf{\Pi}_i\}$,

$$\sum_i \mathbf{\Pi}_i = \mathbf{1}^S,$$

$$\mathbf{\Pi}_i^2 = \mathbf{\Pi}_i,$$

where $\mathbf{1}^S$ denotes the identity operator on H_S. Note that we will associate a *set* of operators $\{\mathbf{\Pi}_0, \mathbf{\Pi}_1, \ldots\}$ with any given direct measurement, and by 'measurement' here we formally mean a set of rules for computing the relative probabilities of all possible outcomes (given a density operator representing the system state). Given a complete set of projectors, we have the standard probability rules:

$$\begin{aligned}\Pr(i) &= \mathrm{Tr}\left[\boldsymbol{\rho}_{\text{pre}} \mathbf{\Pi}_i\right] \\ &= \langle\Psi|\mathbf{\Pi}_i|\Psi\rangle,\end{aligned}$$

where the second line can be used if the system happens to be prepared in a pure state.

Hence we see that the specification of a set $\{\mathbf{\Pi}_i\}$ suffices to allow computation of the measurement outcome probabilities for any possible system state. Completeness of the set ensures that

$$\begin{aligned}\sum_i \Pr(i) &= \sum_i \mathrm{Tr}\left[\boldsymbol{\rho}_{\text{pre}} \mathbf{\Pi}_i\right] \\ &= \mathrm{Tr}\left[\boldsymbol{\rho}_{\text{pre}} \sum_i \mathbf{\Pi}_i\right] \\ &= \mathrm{Tr}\left[\boldsymbol{\rho}_{\text{pre}}\right] = 1.\end{aligned}$$

Geometrically, we can understand the action of $\mathbf{\Pi}_i$ on a pure state (vector) $|\Psi\rangle$ as projection into some subspace of H_S. If $\mathbf{\Pi}_i$ happens to be rank 1 (*i.e*. can be written as the outer product $|\varphi_i\rangle\langle\varphi_i|$ of a state with itself) then it projects onto an axis of H_S. If $\mathbf{\Pi}_i$ is rank 2 ($\mathbf{\Pi}_i = |\varphi_i^1\rangle\langle\varphi_i^1| + |\varphi_i^2\rangle\langle\varphi_i^2|$ with $\langle\varphi_i^1|\varphi_i^2\rangle = 0$) it projects into a plane, etc. So the completeness condition effectively ensures that every axis of H_S is accounted for exactly once. We see that the maximum number of projectors in a complete set is N_S (all projectors rank 1), in which case we are projecting onto the axes of some complete basis for H_S.

In the formal paradigm of direct measurements, the post-measurement state is given by

$$\begin{aligned}\boldsymbol{\rho}_{\text{post}} &= \frac{\mathbf{\Pi}_i \boldsymbol{\rho}_{\text{pre}} \mathbf{\Pi}_i^\dagger}{\mathrm{Tr}\left[\mathbf{\Pi}_i \boldsymbol{\rho}_{\text{pre}} \mathbf{\Pi}_i^\dagger\right]} \\ &= \frac{\mathbf{\Pi}_i \boldsymbol{\rho}_{\text{pre}} \mathbf{\Pi}_i^\dagger}{\mathrm{Tr}\left[\mathbf{\Pi}_i \boldsymbol{\rho}_{\text{pre}}\right]} \\ &= \frac{\mathbf{\Pi}_i \boldsymbol{\rho}_{\text{pre}} \mathbf{\Pi}_i^\dagger}{\Pr(i)},\end{aligned}$$

where i here represents the actual measurement outcome that is obtained in a given trial of the experiment and in going from the first to the second line we have used the cyclic property of Trace and the projector property $\mathbf{\Pi}^2 = \mathbf{\Pi} = \mathbf{\Pi}^\dagger$. Hence if the measurement is performed and outcome $i = 0$ is obtained, then the post-measurement state is $\mathbf{\Pi}_0 \boldsymbol{\rho}_{\text{pre}} \mathbf{\Pi}_0^\dagger / \Pr(0)$. If on the other hand $i = 1$ is obtained, then $\boldsymbol{\rho}_{\text{post}} = \mathbf{\Pi}_1 \boldsymbol{\rho}_{\text{pre}} \mathbf{\Pi}_1^\dagger \boldsymbol{\rho}_{\text{pre}} / \Pr(1)$. So for a given, fixed pre-measurement state $\boldsymbol{\rho}_{\text{pre}}$, the post-measurement state $\boldsymbol{\rho}_{\text{post}}$ cannot be predicted with complete certainty if more than one of the values $\Pr(i)$ is non-zero. Note that case of rank 1 projectors is particularly simple,

$$\begin{aligned}\boldsymbol{\rho}_{\text{post}} &= \frac{\mathbf{\Pi}_i \boldsymbol{\rho}_{\text{pre}} \mathbf{\Pi}_i^\dagger}{\Pr(i)} \\ &= \frac{|\varphi_i\rangle\langle\varphi_i|\boldsymbol{\rho}_{\text{pre}}|\varphi_i\rangle\langle\varphi_i|}{\Pr(i)} \\ &= \frac{|\varphi_i\rangle \Pr(i) \langle\varphi_i|}{\Pr(i)} \\ &= |\varphi_i\rangle\langle\varphi_i|.\end{aligned}$$

Hence the system is simply left in the pure state (vector) $|\varphi_i\rangle$, and all details of the initial state $\boldsymbol{\rho}_{\text{pre}}$ are wiped out save the fact that $\langle\varphi_i|\boldsymbol{\rho}_{\text{pre}}|\varphi_i\rangle > 0$.

Direct measurements are the canonical measurements of traditional quantum theory. In practice, many measurement procedures that achieve the statistics

$$\Pr(i) = \text{Tr}\left[\boldsymbol{\rho}_{\text{pre}} \mathbf{\Pi}_i\right]$$

are in fact constructed as indirect measurements (see below) or only function at the cost of *destroying* the system of interest (demolition measurements). For example, it is possible to 'count' the number of photons in a single electromagnetic field mode, such that

$$\Pr(n) = \text{Tr}\left[\boldsymbol{\rho}_{\text{pre}} |n\rangle\langle n|\right]$$

as long as we restrict to $n \leq 2$ photons. However, the detector that accomplishes this must absorb the photons in order to generate a signal!

Nevertheless direct measurements play the role of 'building' blocks in generalized measurement theory, in part because of their canonical (historical) status in quantum mechanics and in part because of the empirical fact that essentially every quantum measurement we know how to perform 'contains' at its heart a procedure whose statistics are given by $\Pr(i) = \text{Tr}\left[\boldsymbol{\rho}_{\text{pre}} \mathbf{\Pi}_i\right]$.

Indirect measurements

We can build more general types of measurements by coupling our system of interest to an ancillary quantum system prepared in a known initial state

$$|A\rangle \in H_A,$$

where H_A is an N_A –dimensional Hilbert space, and then performing a direct measurement on the *ancilla*.

The general situation is as follows. We first take the system in its pre-measurement state $\boldsymbol{\rho}_{\text{pre}}$ and combine it with the ancilla, such that their joint state can be written

$$\rho_{\text{pre}} \otimes |A\rangle\langle A| \in H_S \otimes H_A.$$

The joint system then evolves under some (possibly time-dependent) interaction

Hamiltonian $\mathbf{H}_{\text{int}}(t)$ for a fixed time interval, yielding

$$\mathbf{U}_t(\rho_{\text{pre}} \otimes |A\rangle\langle A|)\mathbf{U}_t^\dagger,$$

where

$$\mathbf{U}_t = \exp\left[\frac{-i}{\hbar}\int_0^t dt' \, \mathbf{H}_{\text{int}}(t')\right].$$

At this point the interaction is turned off, and in principle the ancilla can be taken away from the system. A direct measurement on the ancilla, specified by some set of *partial* projectors $\{\mathbf{\Pi}_i^A \otimes \mathbf{1}^S\}$, will then have statistics

$$\Pr(i) = \text{Tr}\left[\mathbf{\Pi}_i^A \otimes \mathbf{1}^S \mathbf{U}_t(\rho_{\text{pre}} \otimes |A\rangle\langle A|)\mathbf{U}_t^\dagger \mathbf{\Pi}_i^A \otimes \mathbf{1}^S\right],$$

and we expect to have post-measurement states

$$\rho_{\text{post}} = \frac{\mathbf{\Pi}_i^A \otimes \mathbf{1}^S \mathbf{U}_t(\rho_{\text{pre}} \otimes |A\rangle\langle A|)\mathbf{U}_t^\dagger \mathbf{\Pi}_i^A \otimes \mathbf{1}^S}{\Pr(i)}.$$

Note that the set $\{\mathbf{\Pi}_i^A \otimes \mathbf{1}^S\}$ qualifies as a complete set on the joint state space $H_S \otimes H_A$ since

$$\begin{aligned}\sum_i \left(\mathbf{\Pi}_i^A \otimes \mathbf{1}^S\right) &= \left(\sum_i \mathbf{\Pi}_i^A\right) \otimes \mathbf{1}^S \\ &= \mathbf{1}^A \otimes \mathbf{1}^S \\ &= \mathbf{1}^{A\otimes S},\end{aligned}$$

while the separability of these operators clearly indicates that this measurement can be performed by actions involving the ancilla only.

The essential idea here is that the interaction $\mathbf{U}_t$ should generate *entanglement* between the system and ancilla, such that their states become correlated. We have seen simple examples of this many times, *e.g.*,

$$\mathbf{U}_t = \mathbf{C}_{SA} \equiv |0_S\rangle\langle 0_S| \otimes \mathbf{1}^A + |1_S\rangle\langle 1_S| \otimes (|1_A\rangle\langle 0_A| + |0_A\rangle\langle 1_A|),$$

the controlled-not interaction between two two-dimensional quantum systems. If the initial system state is

$$|\Psi_S\rangle = c_0|0_S\rangle + c_1|1_S\rangle,$$

and we choose $|A\rangle = |0_A\rangle$, then we have the sequence

$$\begin{aligned}|\Psi_S\rangle &\mapsto |\Psi_S\rangle \otimes |0_A\rangle \\ &\mapsto \mathbf{C}_{SA}|\Psi_S\rangle \otimes |0_A\rangle \\ &= c_0|0_S\rangle \otimes |0_A\rangle + c_1|1_S\rangle \otimes |1_A\rangle, \quad \text{or in density operator notation:} \\ \boldsymbol{\rho}_{\text{pre}} &= |\Psi_S\rangle\langle\Psi_S| \\ &\mapsto |\Psi_S\rangle\langle\Psi_S| \otimes |0_A\rangle\langle 0_A| \\ &\mapsto \mathbf{C}_{SA}|\Psi_S\rangle\langle\Psi_S| \otimes |0_A\rangle\langle 0_A|\mathbf{C}_{SA}^\dagger \\ &= (c_0|0_S\rangle \otimes |0_A\rangle + c_1|1_S\rangle \otimes |1_A\rangle)(c_0^*\langle 0_S| \otimes \langle 0_A| + c_1^*\langle 0_S| \otimes \langle 0_A|).\end{aligned}$$

If now we measure the ancilla in its $\{|0_A\rangle, |1_A\rangle\}$ basis, the probabilities will be

$$\begin{aligned}\Pr(0) &= |c_0|^2, \\ \Pr(1) &= |c_1|^2.\end{aligned}$$

It can furthermore be seen that the post-measurement states are given by

$$
\begin{aligned}
|\Psi_{\text{post}}\rangle &= |0_S\rangle \qquad i = 0,\\
&= |1_S\rangle \qquad i = 1.
\end{aligned}
$$

We thus find that this controlled-not procedure leads to an indirect measurement whose statistics and post-measurement states are identical to those of a direct measurement of the $\{|0_S\rangle, |1_S\rangle\}$ basis.

Next let us consider a similar procedure, but with

$$|A\rangle = a_0|0_A\rangle + a_1|1_A\rangle.$$

Then

$$
\begin{aligned}
|\Psi_S\rangle &\mapsto |\Psi_S\rangle \otimes (a_0|0_A\rangle + a_1|1_A\rangle)\\
&\mapsto \mathbf{C}_{SA}(c_0a_0|0_S\rangle \otimes |0_A\rangle + c_0a_1|0_S\rangle \otimes |1_A\rangle + c_1a_0|1_S\rangle \otimes |0_A\rangle + c_1a_1|1_S\rangle \otimes |1_A\rangle)\\
&= c_0a_0|0_S\rangle \otimes |0_A\rangle + c_0a_1|0_S\rangle \otimes |1_A\rangle + c_1a_0|1_S\rangle \otimes |1_A\rangle + c_1a_1|1_S\rangle \otimes |0_A\rangle\\
&= (c_0a_0|0_S\rangle + c_1a_1|1_S\rangle) \otimes |0_A\rangle + (c_0a_1|0_S\rangle + c_1a_0|1_S\rangle) \otimes |1_A\rangle,
\end{aligned}
$$

and

$$
\begin{aligned}
\Pr(0) &= |c_0a_0|0_S\rangle + c_1a_1|1_S\rangle|^2\\
&= |c_0a_0|^2 + |c_1a_1|^2,\\
\Pr(1) &= |c_0a_1|0_S\rangle + c_1a_0|1_S\rangle|^2\\
&= |c_0a_1|^2 + |c_1a_0|^2.
\end{aligned}
$$

We can verify that

$$
\begin{aligned}
\Pr(0) + \Pr(1) &= |c_0a_0|^2 + |c_1a_1|^2 + |c_0a_1|^2 + |c_1a_0|^2\\
&= |c_0|^2\left(|a_0|^2 + |a_1|^2\right) + |c_1|^2\left(|a_0|^2 + |a_1|^2\right)\\
&= 1.
\end{aligned}
$$

The post-measurement states are now

$$
\begin{aligned}
|\Psi_{\text{post}}\rangle &= \frac{c_0a_0|0_S\rangle + c_1a_1|1_S\rangle}{\sqrt{\Pr(0)}} \qquad i = 0,\\
&= \frac{c_0a_1|0_S\rangle + c_1a_0|1_S\rangle}{\sqrt{\Pr(1)}} \qquad i = 1.
\end{aligned}
$$

Note that if $a_0 = a_1 = 1/\sqrt{2}$ the outcome probabilities are equal and independent of $|\Psi_S\rangle$, and both of the post-measurement states are equal to $|\Psi_{\text{pre}}\rangle$. Of course, our previous case of the equivalent-to-direct measurement was a special case of this one with $a_0 = 1,\ a_1 = 0$.

For our next trick, consider the modified interaction

$$\tilde{\mathbf{C}}_{SA} \equiv |1_S\rangle\langle 0_S| \otimes \mathbf{1}^A + |0_S\rangle\langle 1_S| \otimes (|1_A\rangle\langle 0_A| + |0_A\rangle\langle 1_A|).$$

It may be verified that this interaction is still unitary. With the general ancilla preparation

$$|A\rangle = a_0|0_A\rangle + a_1|1_A\rangle$$

this leads to

$$
\begin{aligned}
|\Psi_S\rangle &\mapsto |\Psi_S\rangle \otimes (a_0|0_A\rangle + a_1|1_A\rangle)\\
&\mapsto \tilde{\mathbf{C}}_{SA}(c_0a_0|0_S\rangle \otimes |0_A\rangle + c_0a_1|0_S\rangle \otimes |1_A\rangle + c_1a_0|1_S\rangle \otimes |0_A\rangle + c_1a_1|1_S\rangle \otimes |1_A\rangle)\\
&= c_0a_0|1_S\rangle \otimes |0_A\rangle + c_0a_1|1_S\rangle \otimes |1_A\rangle + c_1a_0|0_S\rangle \otimes |1_A\rangle + c_1a_1|0_S\rangle \otimes |0_A\rangle,\\
&= (c_1a_1|0_S\rangle + c_0a_0|1_S\rangle) \otimes |0_A\rangle + (c_1a_0|0_S\rangle + c_0a_1|1_S\rangle) \otimes |1_A\rangle,
\end{aligned}
$$

so that we still have

$$\Pr(0) = |c_0 a_0|^2 + |c_1 a_1|^2,$$
$$\Pr(1) = |c_0 a_1|^2 + |c_1 a_0|^2,$$

but now

$$|\Psi_{\text{post}}\rangle = \frac{c_1 a_1 |0_S\rangle + c_0 a_0 |1_S\rangle}{\sqrt{\Pr(0)}} \qquad i = 0,$$
$$= \frac{c_1 a_0 |0_S\rangle + c_0 a_1 |1_S\rangle}{\sqrt{\Pr(1)}} \qquad i = 1.$$

If we now set $a_0 = 1,\ a_1 = 0,$ this reproduces like the statistics of a direct measurement but with the opposite mapping of measurement result to post-measurement system state!

Finally, let us consider

$$\mathbf{C}_{AS} \equiv \mathbf{1}^S \otimes |0_A\rangle\langle 0_A| + (|1_S\rangle\langle 0_S| + |0_S\rangle\langle 1_S|) \otimes |1_A\rangle\langle 1_A|.$$

Then keeping $|A\rangle = a_0|0_A\rangle + a_1|1_A\rangle$ we have

$$|\Psi_S\rangle \mapsto |\Psi_S\rangle \otimes (a_0|0_A\rangle + a_1|1_A\rangle)$$
$$\mapsto \mathbf{C}_{AS}(c_0 a_0|0_S\rangle \otimes |0_A\rangle + c_0 a_1|0_S\rangle \otimes |1_A\rangle + c_1 a_0|1_S\rangle \otimes |0_A\rangle + c_1 a_1|1_S\rangle \otimes |1_A\rangle)$$
$$= c_0 a_0|0_S\rangle \otimes |0_A\rangle + c_0 a_1|1_S\rangle \otimes |1_A\rangle + c_1 a_0|1_S\rangle \otimes |0_A\rangle + c_1 a_1|0_S\rangle \otimes |1_A\rangle,$$
$$= (c_0 a_0|0_S\rangle + c_1 a_0|1_S\rangle) \otimes |0_A\rangle + (c_1 a_1|0_S\rangle + c_0 a_1|1_S\rangle) \otimes |1_A\rangle,$$

so that

$$\Pr(0) = |c_0 a_0|^2 + |c_1 a_0|^2$$
$$= |a_0|^2,$$
$$\Pr(1) = |c_0 a_1|^2 + |c_1 a_1|^2$$
$$= |a_1|^2,$$

and

$$|\Psi_{\text{post}}\rangle = \frac{c_0 a_0|0_S\rangle + c_1 a_0|1_S\rangle}{\sqrt{\Pr(0)}} = c_0|0_S\rangle + c_1|1_S\rangle \qquad i = 0,$$
$$= \frac{c_1 a_1|0_S\rangle + c_0 a_1|1_S\rangle}{\sqrt{\Pr(1)}} = c_0|1_S\rangle + c_1|0_S\rangle \qquad i = 1.$$

We thus have a situation where the statistics of the measurement are independent of $|\Psi_S\rangle$, but depending on the measurement outcome the post-measurement system state is either equal to the pre-measurement state or ‘flipped’ by the transformation

$$|0_S\rangle \mapsto |1_S\rangle, \quad |1_S\rangle \mapsto |0_S\rangle.$$

Our choice of the amplitudes a_0 and a_1 independently sets the relative likelihood of the two outcomes!

In summary, we see that indirect measurements can be used to ‘mimic’ direct measurements, but can also be used to construct measurement procedures in which the transformation from pre- to post-measurement system states is more general than a projection and can be decoupled from the information obtained about $\boldsymbol{\rho}_{\text{pre}}$.

Higher-dimensional ancillas

It is crucial to note that the ancillary system in an indirect measurement can have arbitrary dimension – that is, N_A can be much larger than N_S leading to a measurement procedure on an N_S –dimensional system with more than N_S outcomes!

For example, still considering a two-dimensional system of interest, we could use a three-dimensional ancilla to generate an indirect measurement with three distinct outcomes. One possible interaction operator is

$$\mathbf{C}_{\text{permute}} \equiv |0_S\rangle\langle 0_S| \otimes \mathbf{1}^A + |1_S\rangle\langle 1_S| \otimes \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^A,$$

which performs the permutation on H_A

$$|0_A\rangle \mapsto |1_A\rangle, \quad |1_A\rangle \mapsto |2_A\rangle, \quad |2_A\rangle \mapsto |0_A\rangle,$$

if and only if the system is in state $|1_S\rangle$. With

$$|A\rangle = a_0|0_A\rangle + a_1|1_A\rangle + a_2|2_A\rangle,$$

we thus have

$$\begin{aligned} |\Psi_S\rangle &\mapsto |\Psi_S\rangle \otimes (a_0|0_A\rangle + a_1|1_A\rangle + a_2|2_A\rangle) \\ &\mapsto \mathbf{C}_{\text{permute}} \begin{pmatrix} c_0a_0|0_S\rangle \otimes |0_A\rangle + c_0a_1|0_S\rangle \otimes |1_A\rangle + c_0a_2|0_S\rangle \otimes |2_A\rangle \\ +c_1a_0|1_S\rangle \otimes |0_A\rangle + c_1a_1|1_S\rangle \otimes |1_A\rangle + c_1a_2|1_S\rangle \otimes |2_A\rangle \end{pmatrix} \\ &= c_0a_0|0_S\rangle \otimes |0_A\rangle + c_0a_1|0_S\rangle \otimes |1_A\rangle + c_0a_2|0_S\rangle \otimes |2_A\rangle \\ &\quad + c_1a_0|1_S\rangle \otimes |1_A\rangle + c_1a_1|1_S\rangle \otimes |2_A\rangle + c_1a_2|1_S\rangle \otimes |0_A\rangle, \\ &= (c_0a_0|0_S\rangle + c_1a_2|1_S\rangle) \otimes |0_A\rangle + (c_0a_1|0_S\rangle + c_1a_0|1_S\rangle) \otimes |1_A\rangle \\ &\quad + (c_0a_2|0_S\rangle + c_1a_1|1_S\rangle) \otimes |2_A\rangle. \end{aligned}$$

So the outcome probabilites are

$$\begin{aligned} \Pr(0) &= |c_0a_0|^2 + |c_1a_2|^2, \\ \Pr(1) &= |c_0a_1|^2 + |c_1a_0|^2, \\ \Pr(2) &= |c_0a_2|^2 + |c_1a_1|^2, \end{aligned}$$

with post-measurement states

$$\begin{aligned} |\Psi_{\text{post}}\rangle &= \frac{c_0a_0|0_S\rangle + c_1a_2|1_S\rangle}{\sqrt{\Pr(0)}} \qquad i = 0, \\ &= \frac{c_0a_1|0_S\rangle + c_1a_0|1_S\rangle}{\sqrt{\Pr(1)}} \qquad i = 1, \\ &= \frac{c_0a_2|0_S\rangle + c_1a_1|1_S\rangle}{\sqrt{\Pr(2)}} \qquad i = 2. \end{aligned}$$

With appropriate choices for $a_0 \neq a_1 \neq a_2$ such that $|a_0|^2 + |a_1|^2 + |a_2|^2 = 1$ (for example $a_0 = \sqrt{1/2}$, $a_1 = \sqrt{1/3}$, $a_2 = \sqrt{1/6}$), we see that the probabilities and post-measurement states associated with the three possible outcomes are all distinct. It is important to note at this point, however, that the 'amount' of information obtained in a measurement with $N_A > N_S$ outcomes can never be greater than that which could be obtained in an optimal

N_S –dimensional measurement. In some sense however (and as we'll see next week), one can use them to access different 'kinds' of information and to play different strategies in the ubiquitous inference-disturbance tradeoff.

Effects (POVM's) and operations

From the uniformity of the above discussion, you may have already picked up on the point that it is possible to summarize the set of all possible indirect measurement procedures in a simple-looking formalism. In essence, we have

$$\begin{aligned} |\Psi_S\rangle &\mapsto |\Psi_S\rangle \otimes |A\rangle \\ &\mapsto \mathbf{U}_t|\Psi_S\rangle \otimes |A\rangle, \qquad \text{or for density operators} \\ \boldsymbol{\rho}_S &\mapsto \boldsymbol{\rho}_S \otimes \boldsymbol{\rho}_A \\ &\mapsto \mathbf{U}_t \boldsymbol{\rho}_S \otimes \boldsymbol{\rho}_A \mathbf{U}_t^\dagger, \end{aligned}$$

followed by direct measurements on H_A. Then the outcome probabilites can be written in the form

$$\Pr(i) = \mathrm{Tr}\left[\boldsymbol{\Pi}_i^A \otimes \mathbf{1}^S \mathbf{U}_t \boldsymbol{\rho}_S \otimes \boldsymbol{\rho}_A \mathbf{U}_t^\dagger \boldsymbol{\Pi}_i^A \otimes \mathbf{1}^S\right],$$

for arbitrary N_A.

The following extremely convenient result can be generally derived. For any indirect measurement procedure defined by $(H_A, |A\rangle, \mathbf{U}_t, \{\boldsymbol{\Pi}_i^A\})$ it is possible to find *positive* operators (having all real and positive eigenvalues) $\{\mathbf{E}_i^S\}$ acting on H_S only, such that

$$\begin{aligned} \Pr(i) &= \mathrm{Tr}_{S\otimes A}\left[\boldsymbol{\Pi}_i^A \otimes \mathbf{1}^S \mathbf{U}_t \boldsymbol{\rho}_S \otimes \boldsymbol{\rho}_A \mathbf{U}_t^\dagger \boldsymbol{\Pi}_i^A \otimes \mathbf{1}^S\right] \\ &= \mathrm{Tr}_S\left[\mathbf{E}_i^S \boldsymbol{\rho}_S\right] \end{aligned}$$

and

$$\sum_i \mathbf{E}_i^S = \mathbf{1}^S.$$

The $\mathbf{E}_i^S$ operators are known as 'effects' (for historical reasons), and the complete set $\{\mathbf{E}_i^S\}$ is referred-to as a Positive-Operator Valued Measure (POVM).

It is a theorem that the statistics of any possible direct/indirect measurement procedure can be represented in the form $\Pr(i) = \mathrm{Tr}_S\left[\mathbf{E}_i^S \boldsymbol{\rho}_S\right]$, where for direct measurements the $\mathbf{E}_i^S$ are simply projectors. Remarkably, it is also a theorem that every possible set of positive operators $\{\mathbf{E}_i^S\}$ (with arbitrary number of members) satisfying $\sum_i \mathbf{E}_i^S = \mathbf{1}^S$ can in principle be realized as an indirect measurement!

Now what about those post-measurement states? We can formally write

$$\boldsymbol{\rho}_{\text{pre}} \mapsto \frac{\boldsymbol{\Pi}_i^A \otimes \mathbf{1}^S \mathbf{U}_t (\rho_{\text{pre}} \otimes |A\rangle\langle A|) \mathbf{U}_t^\dagger \boldsymbol{\Pi}_i^A \otimes \mathbf{1}^S}{\Pr(i)},$$

but this state technically lives in $H_S \otimes H_A$. We notice that because of the separable form of the projection operators on the outsides, however, this joint state will always be factorizable:

$$\begin{aligned} \boldsymbol{\rho}_{\text{post}}^{SA} &= \boldsymbol{\rho}_{\text{post}}^S \otimes |i_A\rangle\langle i_A|, \\ |i_A\rangle\langle i_A| &= \boldsymbol{\Pi}_i^A, \end{aligned}$$

where we are assuming for simplicity that all $\mathbf{\Pi}_i^A$ are chosen to be of rank 1. Then it is an additional theorem that the formal mapping

$$\boldsymbol{\rho}_{\text{pre}} \mapsto \boldsymbol{\rho}_{\text{post}}^S$$

induced by an indirect measurement procedure can always be represented via

$$\boldsymbol{\rho}_{\text{post}}^S = \frac{\sum_j \mathbf{A}_{ij}^S \boldsymbol{\rho}_{\text{pre}} \left(\mathbf{A}_{ij}^S\right)^\dagger}{\mathrm{Tr}_S\left[\sum_j \mathbf{A}_{ij}^S \boldsymbol{\rho}_{\text{pre}} \left(\mathbf{A}_{ij}^S\right)^\dagger\right]},$$

where i here labels the measurement outcome and the $\mathbf{A}_{ij}^S$ are positive operators (called *operation elements*) acting on H_S only, satisfying

$$\sum_j \left(\mathbf{A}_{ij}^S\right)^\dagger \mathbf{A}_{ij}^S = \mathbf{E}_i^S.$$

Since

$$\begin{aligned}
\Pr(i) &= \mathrm{Tr}_S\left[\mathbf{E}_i^S \boldsymbol{\rho}_S\right] \\
&= \mathrm{Tr}_S\left[\sum_j \left(\mathbf{A}_{ij}^S\right)^\dagger \mathbf{A}_{ij} \boldsymbol{\rho}_S\right] \\
&= \mathrm{Tr}_S\left[\sum_j \mathbf{A}_{ij}^S \boldsymbol{\rho}_{\text{pre}} \left(\mathbf{A}_{ij}^S\right)^\dagger\right],
\end{aligned}$$

we see that specification of a complete set $\left\{\mathbf{A}_{ij}^S\right\}$ (collectively called an *operation*) is equivalent to specifying a POVM plus the decomposition of each of its constituent effects. For rank 1 $\mathbf{\Pi}_i^A$'s the summation is not required and we have simply

$$\begin{aligned}
\boldsymbol{\rho}_{\text{post}}^S &= \frac{\mathbf{A}_i^S \boldsymbol{\rho}_{\text{pre}} \left(\mathbf{A}_i^S\right)^\dagger}{\mathrm{Tr}_S\left[\left(\mathbf{A}_i^S\right)^\dagger \mathbf{A}_i \boldsymbol{\rho}_S\right]}, \\
\Pr(i) &= \mathrm{Tr}_S\left[\left(\mathbf{A}_i^S\right)^\dagger \mathbf{A}_i \boldsymbol{\rho}_S\right] \\
&= \mathrm{Tr}_S\left[\mathbf{A}_i^S \boldsymbol{\rho}_{\text{pre}} \left(\mathbf{A}_i^S\right)^\dagger\right].
\end{aligned}$$

It is an additional remarkable theorem that any set of $\left\{\mathbf{A}_{ij}^S\right\}$, where each $\mathbf{A}_{ij}^S$ is positive, the sum $\sum_j \left(\mathbf{A}_{ij}^S\right)^\dagger \mathbf{A}_{ij}$ is positive for each value of i, and $\sum_i \sum_j \left(\mathbf{A}_{ij}^S\right)^\dagger \mathbf{A}_{ij} = \mathbf{1}^S$, can be realized as some sort of indirect measurement.