

Ph125c lecture notes, 5/31/01

Selective evolution and the Stochastic Schrödinger Equation

Consider the following general form for a pair of operations $\{\Omega_0, \Omega_1\}$, which are shown to form a complete measurement:

$$\begin{aligned}\Omega_1 &= \sqrt{dt} \mathbf{c}, \\ \Omega_0 &= \mathbf{1} - \left(i\mathbf{H} + \frac{1}{2} \mathbf{c}^\dagger \mathbf{c} \right) dt, \\ \Omega_0^\dagger \Omega_0 + \Omega_1^\dagger \Omega_1 &= \left[\mathbf{1} + \frac{i}{\hbar} \mathbf{H} dt - \frac{1}{2} \mathbf{c}^\dagger \mathbf{c} dt \right] \left[\mathbf{1} - \frac{i}{\hbar} \mathbf{H} dt - \frac{1}{2} \mathbf{c}^\dagger \mathbf{c} dt \right] + \mathbf{c}^\dagger \mathbf{c} dt \\ &\simeq \mathbf{1} - \mathbf{c}^\dagger \mathbf{c} dt + \mathbf{c}^\dagger \mathbf{c} dt \\ &= \mathbf{1}.\end{aligned}$$

Here we assume that \mathbf{H} is a Hermitian operator, and that the scalar dt is sufficiently small that we may retain terms up to first order only. As the notations suggest, dt is meant to correspond to an infinitesimal time increment and \mathbf{H} represents a Hamiltonian for the quantum system of interest. The remaining operator \mathbf{c} (which must be positive) will be called the *jump operator*, for reasons that may become evident soon. Note that the Ω_0 operator may be assumed positive since it is infinitesimally close to the identity, as long as \mathbf{H} and $\mathbf{c}^\dagger \mathbf{c}$ have bounded spectra (all eigenvalues finite).

We know from results discussed previously that the measurement $\{\Omega_0, \Omega_1\}$ can be implemented via an indirect procedure involving an ancilla. We also know that the outcome probabilities for an arbitrary pre-measurement system state ρ will be

$$\begin{aligned}\text{Pr}(1) &= \text{Tr} \left[\Omega_1^\dagger \Omega_1 \rho \right] \\ &= \text{Tr} \left[\mathbf{c}^\dagger \mathbf{c} \rho \right] dt \\ &= \langle \mathbf{c}^\dagger \mathbf{c} \rangle dt, \\ \text{Pr}(0) &= 1 - \langle \mathbf{c}^\dagger \mathbf{c} \rangle dt.\end{aligned}$$

Note that the 1 outcome is highly unlikely following the assumption that dt is infinitesimal. The associated selective evolution rules are

$$\begin{aligned}1 : \quad \rho &\mapsto \frac{\Omega_1 \rho \Omega_1^\dagger}{\text{Pr}(1)} \propto \mathbf{c} \rho \mathbf{c}^\dagger, \\ 0 : \quad \rho &\mapsto \frac{\Omega_0 \rho \Omega_0^\dagger}{\text{Pr}(0)} \propto \left[\mathbf{1} - \frac{i}{\hbar} \mathbf{H} dt - \frac{1}{2} \mathbf{c}^\dagger \mathbf{c} dt \right] \rho \left[\mathbf{1} + \frac{i}{\hbar} \mathbf{H} dt - \frac{1}{2} \mathbf{c}^\dagger \mathbf{c} dt \right] \\ &\simeq \left[\mathbf{1} - \frac{i}{\hbar} \mathbf{H} dt - \frac{1}{2} \mathbf{c}^\dagger \mathbf{c} dt \right] \rho + \rho \left[\frac{i}{\hbar} \mathbf{H} dt - \frac{1}{2} \mathbf{c}^\dagger \mathbf{c} dt \right] \\ &= \rho + \left(-\frac{i}{\hbar} \mathbf{H} \rho + \frac{i}{\hbar} \rho \mathbf{H} \right) dt - \frac{1}{2} (\mathbf{c}^\dagger \mathbf{c} \rho + \rho \mathbf{c}^\dagger \mathbf{c}) dt \\ &= \rho - \frac{i}{\hbar} [\mathbf{H}_{\text{eff}}, \rho] dt,\end{aligned}$$

where

$$\mathbf{H}_{\text{eff}} \equiv \mathbf{H} - i\frac{\hbar}{2}\mathbf{c}^\dagger\mathbf{c}.$$

Note that \mathbf{H}_{eff} is not a Hermitian operator because of the i , and that the generalized commutator $[\mathbf{H}_{\text{eff}}, \rho]$ should be understood to mean

$$[\mathbf{H}_{\text{eff}}, \rho] \equiv \mathbf{H}_{\text{eff}}\rho - \rho\mathbf{H}_{\text{eff}}^\dagger.$$

For the 0 outcome case, we thus recognize

$$\rho \mapsto \rho - \frac{i}{\hbar}[\mathbf{H}_{\text{eff}}, \rho]dt$$

as being formally equivalent to the density operator form of a Schrödinger Equation,

$$\begin{aligned} \frac{d}{dt}\rho &\propto -\frac{i}{\hbar}[\mathbf{H}_{\text{eff}}, \rho], \\ i\hbar\frac{d}{dt}|\Psi\rangle &\propto \mathbf{H}_{\text{eff}}|\Psi\rangle. \end{aligned}$$

The “ \propto ” is necessary here because \mathbf{H}_{eff} is no longer Hermitian, and therefore does not preserve probability (norm of $|\Psi\rangle$) or trace of ρ).

Suppose we have a quantum system that is subjected to a *continuous* sequence of measurements $\{\Omega_0, \Omega_1\}$. By this we mean that in each infinitesimal time interval dt , we perform an indirect measurement procedure on the system that corresponds to the operations

$$\begin{aligned} \Omega_0 &= \mathbf{1} - \left(i\mathbf{H} + \frac{1}{2}\mathbf{c}^\dagger\mathbf{c}\right)dt, \\ \Omega_1 &= \sqrt{dt}\mathbf{c}. \end{aligned}$$

Pulling together what we derived above, we see that the selective evolution of the system state $\rho(t)$ will almost always be described by SE-like evolution with an effective non-Hermitian Hamiltonian

$$\rho(t+dt) = \frac{\rho(t) - \frac{i}{\hbar}\left[\mathbf{H} - i\frac{1}{2}\mathbf{c}^\dagger\mathbf{c}, \rho(t)\right]dt}{\text{Tr}\left[\rho(t) - \frac{i}{\hbar}\left[\mathbf{H} - i\frac{1}{2}\mathbf{c}^\dagger\mathbf{c}, \rho(t)\right]dt\right]},$$

except in rare timesteps when the outcome 1 is obtained. At such times, $\rho(t)$ will evolve according to

$$\rho(t+dt) = \frac{\mathbf{c}\rho(t)\mathbf{c}^\dagger}{\text{Tr}\left[\mathbf{c}\rho(t)\mathbf{c}^\dagger\right]}.$$

It should be noted that during periods of 0 outcome, ‘in between’ the rare events of outcome 1, the evolution of $\rho(t)$ is infinitesimally close to the identity operator and may therefore be considered continuous. When outcome 1 is obtained, however, a very large ‘instantaneous’ change in ρ can occur. For this reason \mathbf{c} is known as the jump operator for the continuous measurement scenario.

Note that if $\mathbf{c} = 0$, the system evolution will correspond exactly to a Schrödinger Equation with Hamiltonian \mathbf{H} . Hence we may think of \mathbf{c} as representing the actual ‘measurement’ that is being made, since the frequency of obtaining outcome 1 depends directly on $\langle\mathbf{c}^\dagger\mathbf{c}\rangle$ and the ‘disturbance’ (deviation from pure Schrödinger dynamics) is all tied to the form of \mathbf{c} .

Let's look at an example, with a two-level atom. Say the atomic states are a ground state $|g\rangle$ and an excited state $|e\rangle$, with Hamiltonian

$$\begin{aligned}\mathbf{H} &= \hbar\omega_0 \Pi_e \\ &\equiv \hbar\omega_0 |e\rangle\langle e|.\end{aligned}$$

Let the jump operator be

$$\begin{aligned}\mathbf{c} &= \sqrt{\Gamma} |g\rangle\langle e| \\ &\equiv \sqrt{\Gamma} \sigma \\ &\leftrightarrow \sqrt{\Gamma} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\end{aligned}$$

where Γ is a real scalar parameter and σ is known as the atomic lowering operator (for obvious reasons). Then

$$\begin{aligned}\text{Pr}(1) &= \langle \mathbf{c}^\dagger \mathbf{c} \rangle dt \\ &= \langle |e\rangle\langle g| |g\rangle\langle e| \rangle \Gamma dt \\ &= \langle \Pi_e \rangle \Gamma dt,\end{aligned}$$

so that the 'measurement' being performed is of the instantaneous probability to find the atom in its excited state. The parameter Γ thus appears as a 'strength' of the measurement, such that if $\Gamma \rightarrow (dt)^{-1}$ we would be making instantaneous projective measurements. However let's assume in the following that Γ is finite.

The selective evolution rule is simple. In any given time increment dt , there is probability $\langle \Pi_e \rangle \Gamma dt$ that the state evolves according to

$$\begin{aligned}\rho &\mapsto \frac{\sigma \rho \sigma^\dagger}{\text{Tr}[\sigma \rho \sigma^\dagger]} \\ &\propto |g\rangle\langle e| \rho |e\rangle\langle g| \\ &= |g\rangle\langle g|,\end{aligned}$$

assuming $\langle e | \rho | e \rangle > 0$. If this type of collapse does not occur, then

$$\begin{aligned}\rho(t+dt) &\propto \rho(t) - \frac{i}{\hbar} \left[\mathbf{H} - i\hbar \frac{\Gamma}{2} \Pi_e, \rho(t) \right] dt \\ &= \rho(t) - \frac{i}{\hbar} \left[\left(\hbar\omega_0 - i\hbar \frac{\Gamma}{2} \right) \Pi_e, \rho(t) \right] dt.\end{aligned}$$

If we focus our attention on the pure-state case, this is equivalent to the Schrödinger Equation

$$i\hbar \frac{d}{dt} |\Psi\rangle \propto \left(\hbar\omega_0 - i\hbar \frac{\Gamma}{2} \right) \Pi_e |\Psi\rangle,$$

which we know leads to the evolution

$$\begin{aligned}|\Psi(t)\rangle &\propto c_g(0) |g\rangle + c_e(0) \exp\left(-i\omega_0 t - \frac{\Gamma}{2} t\right) |e\rangle, \\ c_g(0) &\equiv \langle g | \Psi(0) \rangle, \quad c_e(0) \equiv \langle e | \Psi(0) \rangle.\end{aligned}$$

After normalization, we note that this produces a 'smooth decay' of $c_e(t)$ relative to $c_g(t)$ during periods of time in which no 'jumps' (outcome 1's) occur. Once a jump does occur, the system is left in the ground state and remains there forevermore since the probability of subsequently obtaining a 1 goes to zero.

Finally let's consider the non-selective evolution. In general for the type of operations we

are considering,

$$\begin{aligned}
\rho(t+dt) &= \Omega_0 \rho(t) \Omega_0^\dagger + \Omega_1 \rho(t) \Omega_1^\dagger \\
&= \rho(t) + \left(-\frac{i}{\hbar} \mathbf{H} \rho(t) + \frac{i}{\hbar} \rho(t) \mathbf{H} \right) dt \\
&\quad - \frac{1}{2} \left(\mathbf{c}^\dagger \mathbf{c} \rho(t) + \rho(t) \mathbf{c}^\dagger \mathbf{c} \right) dt + \mathbf{c} \rho(t) \mathbf{c}^\dagger dt, \\
\rho(t+dt) - \rho(t) &= \frac{-i}{\hbar} [\mathbf{H}, \rho(t)] dt + \frac{1}{2} \left(2\mathbf{c} \rho(t) \mathbf{c}^\dagger - \mathbf{c}^\dagger \mathbf{c} \rho(t) - \rho(t) \mathbf{c}^\dagger \mathbf{c} \right) dt, \\
\frac{d}{dt} \rho(t) &= \frac{-i}{\hbar} [\mathbf{H}, \rho(t)] + \frac{1}{2} \left(2\mathbf{c} \rho(t) \mathbf{c}^\dagger - \mathbf{c}^\dagger \mathbf{c} \rho(t) - \rho(t) \mathbf{c}^\dagger \mathbf{c} \right).
\end{aligned}$$

Substituting $\mathbf{H} = \hbar\omega_0 \Pi_e$ and $\mathbf{c} = \sqrt{\Gamma} \sigma$ we find

$$\frac{d}{dt} \rho(t) = \frac{-i}{\hbar} [\hbar\omega_0 \Pi_e, \rho(t)] + \frac{\Gamma}{2} \left(2\sigma \rho(t) \sigma^\dagger - \sigma^\dagger \sigma \rho(t) - \rho(t) \sigma^\dagger \sigma \right),$$

which may be recognized as the Master Equation for a two-level atom undergoing decay via spontaneous emission at rate Γ .

The interpretation of this is that in atomic decay, we could think of the set of electromagnetic vacuum modes as an ‘ancilla’ mediating an indirect measurement of whether the atom is in its ground or excited state. If the atom is in its excited state, it has some probability per unit time ($\Gamma \Delta t$) to emit a photon into the EM field. If we are ‘watching’ all of the relevant field modes with perfect photodetectors (this is not really possible in practice), we can be sure to catch this photoemission event when and if it occurs. If we do detect such an event, we know the atom must have decayed and that it will therefore be left in its ground state. However, if we watch the atom for a long period of time and detect no photon, the likelihood for it to be in the excited state becomes smaller and smaller – hence the smooth decay term in the effective Hamiltonian. Spontaneous emission without continuous observation can be thought of as the non-selective version of this evolution, in which we don’t explicitly monitor the field modes to try to detect the decay photon.

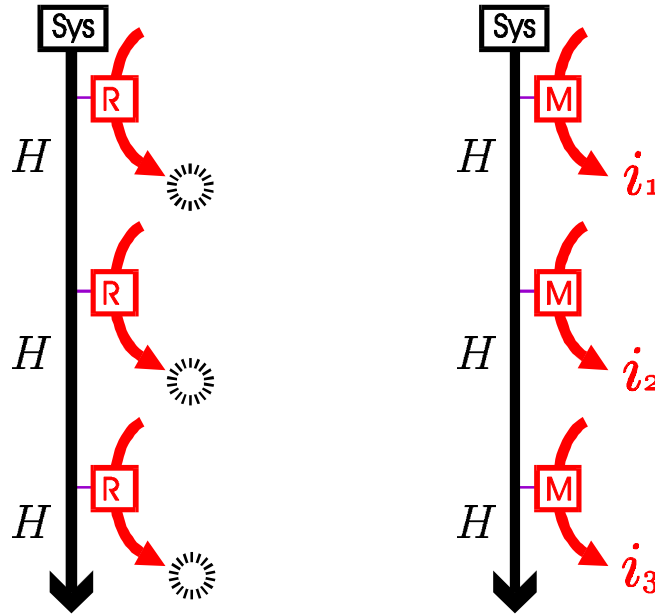
Unconditional evolution and the Master Equation

Above we saw that the *selective* evolution of a quantum system under continuous indirect measurement $\{\Omega_0, \Omega_1\}$ can be described using a Stochastic Schrödinger Equation (SSE). We also showed that the *non-selective* evolution

$$\rho(t+dt) = \Omega_0 \rho(t) \Omega_0^\dagger + \Omega_1 \rho(t) \Omega_1^\dagger$$

corresponds to a Master Equation. As we discussed previously, non-selective evolution for a measured system can be thought of as the result of performing an indirect measurement procedure but ignoring the result.

Way back in first term, we noted that if two quantum systems are allowed to interact but then one of them is ‘removed’ or ‘thrown away,’ an efficient description of the remaining system’s state can be obtained by taking the *partial trace* over the discarded degrees of freedom. It can in fact be shown that this is equivalent to the non-selective evolution rule above, which leads to the following illustration of selective (right) and non-selective (left) evolution under continuous measurement:



In the selective case, the system evolution is conditioned upon the sequence of measurement results $\{i\}$.

There is one very important thing to note about the class of generalized measurements we have been considering, with

$$\begin{aligned}\Omega_1 &= \sqrt{dt} \mathbf{c}, \\ \Omega_0 &= \mathbf{1} - \left(i\mathbf{H} + \frac{1}{2} \mathbf{c}^\dagger \mathbf{c} \right) dt.\end{aligned}$$

The 'transition' probability

$$\begin{aligned}\text{Pr}(1) &= \text{Tr} \left[\Omega_1^\dagger \Omega_1 \rho(t) \right] \\ &= \left(\frac{dt}{2} \right) \text{Tr} \left[\mathbf{c}^\dagger \mathbf{c} \rho(t) \right]\end{aligned}$$

is seen to be proportional to the time interval dt , indicating that this generalized measurement *must* be the result of coupling the system of interest to a continuum of final states in the joint space of system + ancilla. For this reason, the ancilla in such a scenario is referred to as a 'reservoir' or 'bath.' In the usual formulation of quantum statistical mechanics, the role of reservoir will typically be played by a continuum of modes of some field, such as the electromagnetic field (photons) or the field of vibrations in a crystal lattice (phonons). It is then not so easy to imagine that a 'new copy' of the ancilla (reservoir) is brought up and coupled to the system of interest in each and every time-interval dt . However a formal treatment via the above approach can often be justified (especially in the case of quantum optics), and the concomitant set of approximations are collectively known as the *quantum Markov approximation*. This approximation is often not valid in low-temperature condensed matter physics, so much more complicated theories are required to describe decoherence in, e.g., single-electron transistors.

Comparing the selective and non-selective evolution rules for our quantum system, it should be noted that the SSE maps pure states to pure states while the Master Equation generally leads to mixed states. In fact, the density operator $\rho(t)$ obtained by integrating the

Master Equation to a given time t can be thought of as the density operator representing the ensemble of pure states that would be generated up to time t by multiple ‘runs’ of the SSE, with independently-generated random numbers to determine the measurement outcomes in each time interval:

$$\rho(t) = \sum |\psi(t)\rangle\langle\psi(t)|.$$

For this reason SSE methods are often used for Monte Carlo integration of the Master Equation, as the SSE has the numerical advantage of requiring storage (in computer memory) of pure state vectors of size N , rather than density operator matrices of size N^2 (for a Hilbert space of dimension N).

So far we have been consider the class of Master Equations associated with our measurement $\{\Omega_0, \Omega_1\}$ described above,

$$\frac{d}{dt}\rho(t) = \frac{-i}{\hbar}[\mathbf{H}, \rho(t)] + \frac{1}{2} (2\mathbf{c}\rho(t)\mathbf{c}^\dagger - \mathbf{c}^\dagger\mathbf{c}\rho(t) - \rho(t)\mathbf{c}^\dagger\mathbf{c}).$$

Although this may seem like a fairly restricted class, it should be noted that *any* Markovian Master Equation can be written in the ‘Lindblad’ form

$$\frac{d}{dt}\rho(t) = \frac{-i}{\hbar}[\mathbf{H}, \rho(t)] + \sum_j \frac{1}{2} (2\mathbf{c}_j\rho(t)\mathbf{c}_j^\dagger - \mathbf{c}_j^\dagger\mathbf{c}_j\rho(t) - \rho(t)\mathbf{c}_j^\dagger\mathbf{c}_j),$$

where the $\{\mathbf{c}_j\}$ are now a set of positive jump operators acting on the system Hilbert space. One often writes the abstract notation

$$\begin{aligned} \frac{d}{dt}\rho(t) &= \frac{-i}{\hbar}[\mathbf{H}, \rho(t)] + \sum_j \frac{1}{2} (2\mathbf{c}_j\rho(t)\mathbf{c}_j^\dagger - \mathbf{c}_j^\dagger\mathbf{c}_j\rho(t) - \rho(t)\mathbf{c}_j^\dagger\mathbf{c}_j) \\ &\equiv \mathbf{L}\{\rho(t)\}, \end{aligned}$$

where \mathbf{L} thus defined is known as the *Liouville superoperator* or *Liouvillian*. Note that even when the system is finite-dimensional and $\rho(t)$ can thus be thought of as a simple matrix, \mathbf{L} cannot itself be represented simply by a matrix. The basic reason for this is that operators such as \mathbf{H} and \mathbf{c}_j and $\mathbf{c}_j^\dagger\mathbf{c}_j$ have to act on $\rho(t)$ from both sides, in the Master Equation. But the relationship between ρ and $\dot{\rho}$ in a Master Equation of Lindblad form is simply linear, so one often uses the trick of re-writing ρ as a vector (of length N^2) rather than a matrix (e.g. by just stacking its columns on top of one another), in which case it is possible to find an $N^2 \times N^2$ matrix representation for \mathbf{L} .

Either way, note that it is formally possible to define a generalization of the time-development operator, for Master Equations:

$$\begin{aligned} \rho(t_2) &= \exp[\mathbf{L}(t_2 - t_1)]\rho(t_1) \\ &\equiv \mathbf{T}(t_2, t_1)\rho(t_1). \end{aligned}$$

Likewise, one can – under certain conditions – use $\mathbf{T}(t_2, t_1)$ to define ‘rotating frames’ that are useful in the computation of operator correlation functions (see below).

It is often the case that the Master Equation admits steady-state solutions ρ_0 , such that

$$\frac{d}{dt}\rho_0 = \mathbf{L}\rho_0 = 0.$$

If the system somehow comes to be in state ρ_0 at time t , non-selective evolution via the Master Equation will not change the state thereafter. It is usually the case that such a steady-state is ‘attracting,’ in the sense that arbitrary initial states for the system will evolve into ρ_0 and then stay there. For example, in our previous example of a two-level atom

undergoing spontaneous emission (without a driving term), the atomic ground state was a steady state of the Master Equation. To see a less trivial example, consider

$$\frac{d}{dt}\rho(t) = \frac{-i}{\hbar} [\hbar\omega_0\Pi_e + \hbar\Omega_R \sin(\omega_0 t)(\sigma + \sigma^\dagger), \rho(t)] + \frac{\Gamma}{2} (2\sigma\rho(t)\sigma^\dagger - \sigma^\dagger\sigma\rho(t) - \rho(t)\sigma^\dagger\sigma),$$

where we have added a driving term $\hbar\Omega_R(\sigma + \sigma^\dagger)$ to the atomic Hamiltonian. As you saw (if you can remember!) way back in first term, the effect of this type of term is to induce Rabi oscillations between the atomic states:

$$|\Psi(t)\rangle \sim \cos(\Omega_R t)|g\rangle + \sin(\Omega_R t)|e\rangle.$$

The steady state of this Master Equation in the limit $\Omega_R \gg \Gamma$ is given by

$$\rho_0 \simeq \frac{1}{2} [|g\rangle\langle g| + |e\rangle\langle e|],$$

the even incoherent mixture of ground and excited state. Thinking about this evolution in the SSE picture, we see that the average population of $|e\rangle$ across an ensemble of quantum trajectories should be $1/2$, but the phase coherence between them is destroyed by stochastic interruption of the Rabi oscillation by spontaneous emission events.

Correlation functions

Suppose we want to answer the following question. If I get a count (measurement outcome = 1) at time t , what is the probability that I also get a count at time $t + \tau$?

Using our measurement formalism, we know that if the system state just before time t is given by $\rho(t)$, the system state just after obtaining the count will be

$$\rho(t+dt) = \frac{\Omega_1 \rho(t) \Omega_1^\dagger}{\text{Tr}[\Omega_1 \rho(t) \Omega_1^\dagger]}.$$

I can then propagate this state under non-selective evolution via the Master Equation,

$$\rho(t+\tau) = \exp(\mathbf{L}\tau)\{\rho(t+dt)\},$$

then compute the probability for a count at the final time:

$$\text{Tr}[\Omega_1^\dagger \Omega_1 \rho(t+\tau)] = \frac{\text{Tr}[\Omega_1^\dagger \Omega_1 \exp(\mathbf{L}\tau)\{\Omega_1 \rho(t) \Omega_1^\dagger\}]}{\text{Tr}[\Omega_1 \rho(t) \Omega_1^\dagger]}.$$

It is common to assume take $\rho(t)$ equal to the steady-state solution of the Master Equation, in which we obtain a stationary quantity (depending only on the time *difference* τ). The notation with curly-braces $\{\}$ after the time-development superoperator is meant to remind us that $\exp(\mathbf{L}\tau)$ really acts on its argument from both sides.

Generalizing, consider any normal operator on the system Hilbert space,

$$\mathbf{O} = \sum_j \lambda_j \Pi_j,$$

where λ_j is the j^{th} eigenvalue of \mathbf{O} and Π_j is a projector onto the corresponding eigenstate. We can then define a direct measurement in the eigenbasis of \mathbf{O} . Say we perform this measurement at some time t when the system is known to be in steady state ρ_0 . If we obtain outcome j ,

$$j : \quad \rho(t+dt) = \frac{\Pi_j \rho_0 \Pi_j^\dagger}{\Pr(j)}.$$

If we make a second measurement at time $t + \tau$, with non-selective evolution in between, the joint probabilities for the outcomes k will then be

$$\begin{aligned} \Pr(k,j) &= \Pr(k|j) \Pr(j) = \text{Tr} \left[\Pi_k^\dagger \Pi_k \exp(\mathbf{L}\tau) \left\{ \frac{\Pi_j \rho_0 \Pi_j^\dagger}{\Pr(j)} \right\} \right] \Pr(j) \\ &= \text{Tr} [\Pi_k \exp(\mathbf{L}\tau) \{ \Pi_j \rho_0 \Pi_j \}]. \end{aligned}$$

Note that since $\exp(\mathbf{L}\tau)$ is a superoperator, we cannot simply invoke cyclic property of Trace to obtain a δ_{jk} . If we compute now the average of the product of the eigenvalues associated with the first and second measurements, we find

$$\begin{aligned} \sum_{k,j} \lambda_k \lambda_j \Pr(k,j) &= \sum_{k,j} \lambda_k \lambda_j \text{Tr} [\Pi_k \exp(\mathbf{L}\tau) \{ \Pi_j \rho_0 \Pi_j \}] \\ &= \sum_{k,j} \text{Tr} [\lambda_k \Pi_k \exp(\mathbf{L}\tau) \{ \lambda_j \Pi_j \rho_0 \Pi_j \}] \\ &= \text{Tr} \left[\mathbf{O} \exp(\mathbf{L}\tau) \left\{ \sum_j \lambda_j \Pi_j \rho_0 \Pi_j \right\} \right] \\ &= \text{Tr} [\mathbf{O} \exp(\mathbf{L}\tau) \{ \mathbf{O} \rho_0 \}] ? \\ &\equiv \langle \mathbf{O}(t+\tau) \mathbf{O}(t) \rangle \\ &= \langle \mathbf{O}(\tau) \mathbf{O} \rangle, \end{aligned}$$

where we have used the spectral decomposition of \mathbf{O} and completeness of its eigenbasis, as well as linearity of evolution under the Master Equation. This final quantity, when viewed as a function of τ , is known as the stationary correlation function of the operator \mathbf{O} .