1 Problem 1.(A)

1.1 a. 

\[ \vec{L} = \vec{r} \times \vec{p} \] 

In components:

\[ L_x = y p_z - z p_y \quad L_y = z p_x - x p_z \quad L_z = x p_y - y p_x \] 

Express \( x_i \) and \( p_i \) in terms of creation and annihilation operators.

\[ x = \sqrt{\frac{\hbar}{2m\omega}}(a_x + a_x^\dagger) \quad p_x = \sqrt{\frac{m\hbar^2}{2} - \frac{a_x^\dagger}{i}} \] 

Similar expressions hold for the other coordinates and momenta. Substituting (3) into (2) we get

\[ L_x = -i\hbar(a_x^\dagger a_y^\dagger - a_x a_y) \] 

\[ L_y = -i\hbar(a_y^\dagger a_x^\dagger - a_y a_x) \] 

\[ L_z = -i\hbar(a_z^\dagger a_y^\dagger - a_z a_y) \] 

We can summarize the above relations as \( L_i = -i\hbar \epsilon_{ijk} a_j^\dagger a_k \).

Now consider the operator \( N = a_x^\dagger a_x + a_y^\dagger a_y + a_z^\dagger a_z \). We want to find the commutator \([N, L^2]\). Let’s first calculate the commutators of \( N \) with different components of \( \vec{L} \). We will use the standard commutation relations between creation and annihilation operators:

\[ [a_i, a_j] = [a_i^\dagger, a_j] = 0 \quad [a_i, a_j^\dagger] = \delta_{ij} \] 

Then we get

\[ [L_x, a_x^\dagger a_x] = -i\hbar[a_y^\dagger a_z - a_z^\dagger a_y, a_x^\dagger a_x] = 0 \]
\[ [L_x, a_y^\dagger a_y] = -i\hbar[a_y^\dagger a_y - a_y a_y^\dagger] = -i\hbar[a_y^\dagger a_y^\dagger a_y a_y - a_y^\dagger a_y a_y^\dagger a_y] = i\hbar(a_y^\dagger a_y + a_y a_y^\dagger) \tag{9} \]
\[ [L_x, a_z^\dagger a_z] = -i\hbar[a_y^\dagger a_z^\dagger a_y a_z - a_y^\dagger a_z a_y^\dagger a_z] = -i\hbar(a_y^\dagger a_z a_y^\dagger a_z - a_y^\dagger a_z a_y^\dagger a_z) = -i\hbar(a_y^\dagger a_z + a_z a_y^\dagger) \tag{10} \]

Thus
\[ [L_x, N] = [L_x, a_x^\dagger a_x + a_y^\dagger a_y + a_z^\dagger a_z] = 0 \tag{13} \]

The similar calculation gives that
\[ [L_y, N] = [L_z, N] = 0 \tag{14} \]

From (13) and (14) it follows immediately that
\[ [L^2, N] = 0 \tag{15} \]

1.2 b.

We know from the study of one-dimensional linear harmonic oscillator that the spectrum of the operator \( a_x^\dagger a_x \) is all integer non-negative numbers:
\[ a_x^\dagger a_x |n_x\rangle = n_x |n_x\rangle , n_x = 0, 1, 2, \ldots \tag{16} \]

The same is true for \( a_y^\dagger a_y \) and \( a_z^\dagger a_z \). The spectrum of \( N \) is just the sum
\[ N|n_x n_y n_z\rangle = n|n_x n_y n_z\rangle \quad n = n_x + n_y + n_z \tag{17} \]

Obviously \( n \) can be any non-negative integer and if \( n > 0 \) it is degenerate (since many different values of \( (n_x, n_y, n_z) \) give the same \( n \)). For example, \( n = 1 \) level is three times degenerate, it has the eigenvectors \( |n_x = 1, n_y = 0, n_z = 0\rangle, |n_x = 0, n_y = 1, n_z = 0\rangle, |n_x = 0, n_y = 0, n_z = 1\rangle \).

Generally the degree of degeneracy of \( n \) is equal to the number of different ways we can write \( n \) as a sum of three integer non-negative numbers.
\[ \text{deg}(n) = \frac{(n+1)(n+2)}{2} \tag{18} \]

1.3 c.

Write the hamiltonian in terms of creation and annihilation operators. Since \( H \) is the sum of three one-dimensional oscillators we get
\[ H = \hbar \omega (a_x^\dagger a_x + a_y^\dagger a_y + a_z^\dagger a_z + \frac{3}{2}) = \hbar \omega (N + \frac{3}{2}) \tag{19} \]

From (14) and (15) we infer that
\[ [H, L^2] = [H, L_z] = 0 \tag{20} \]

\[ 2 \]
We see that $H, \mathbf{L}^2$ and $L_z$ commute with each other and thus can be simultaneously diagonalized. The energy eigenstates can be written as $|nlm\rangle$ where

$$L^2|nlm\rangle = l(l+1)|nlm\rangle, \quad L_z|nlm\rangle = m|nlm\rangle, \quad H|nlm\rangle = E_{nlm}|nlm\rangle \quad (21)$$

The energy eigenvalue $E_{nlm}$ can depend on $l, m$ and on additional quantum number $n$. (We can choose $n$ to be the same number as in part b or not - it’s a matter of convention.) In fact it’s easy to show that $E_{nlm}$ does not depend on $m$. We use the fact that $H$ commutes with $L_x, L_y$ and thus with $L_+$. We know that

$$L_+|nlm\rangle = \sqrt{(l-m)(l+m+1)}|nlm+1\rangle \quad (22)$$

Acting by $H$ on both sides of the last eqn. we get

$$LHS = HL_+|nlm\rangle = L_+H|nlm\rangle = E_{nlm}L_+|nlm\rangle \quad (23)$$

$$RHS = \sqrt{(l-m)(l+m+1)}E_{nlm+1}E_{nlm+1}|nlm+1\rangle = E_{nlm+1}L_+|nlm\rangle \quad (24)$$

Compare LHS with RHS:

$$E_{nlm+1} = E_{nlm} \quad (25)$$

Then starting with $l = -m$ we get

$$E_{nl,-l} = E_{nl,-l+1} = \cdots = E_{nl,l} \quad (26)$$

that is indeed $E = E_{nl}$ doesn’t depend on $m$. The spectrum of $H$ is degenerate in accordance with the result of part b.

Note. We could expect from the analysis above that the degree of degeneracy of $E_{nl}$ be $(2l+1)$ (the dimension of the multiplet with angular momentum $l$). This contradicts the formula (18). The reason is that there is an additional degeneracy between energy levels with different $l$'s.

1.4 d.

Consider coherent state $|\alpha_x\alpha_y\alpha_z\rangle$. Assume that it is also the eigenstate of $L_z$, i.e.

$$L_z|\alpha_x\alpha_y\alpha_z\rangle = m|\alpha_x\alpha_y\alpha_z\rangle \quad (27)$$

Then it’s easy to show that the commutators of $L_z$ with annihilation operators $a_i$, ($i = x, y, z$) must annihilate the coherent state. Indeed

$$[L_z, a_i]|\alpha_x\alpha_y\alpha_z\rangle = (L_z a_i - a_i L_z)|\alpha_x\alpha_y\alpha_z\rangle = (ma_i - a_i m)|\alpha_x\alpha_y\alpha_z\rangle = 0 \quad (28)$$

Calculate the commutators using the results of part a.

$$[L_z, a_x] = -i\hbar[a_x^\dagger a_y - a_y^\dagger a_x, a_x] = -i\hbar a_y[a_x^\dagger, a_x] = i\hbar a_y \quad (29)$$

$$[L_z, a_y] = -i\hbar[a_x^\dagger a_y - a_y^\dagger a_x, a_y] = i\hbar a_x[a_y^\dagger, a_y] = -i\hbar a_x \quad (30)$$

$$[L_z, a_z] = -i\hbar[a_x^\dagger a_y - a_y^\dagger a_x, a_z] = 0 \quad (31)$$

3
Then the following eqns. must be satisfied:

\[
\begin{align*}
[L_z, a_x] = i\hbar \alpha_x |\alpha_x\alpha_y\alpha_z\rangle &= 0 \\
[L_z, a_y] = -i\hbar \alpha_y |\alpha_x\alpha_y\alpha_z\rangle &= 0
\end{align*}
\]

\Rightarrow \alpha_x = \alpha_y = 0 \text{ are the necessary conditions} (34)

These conditions are also sufficient since

\[
L_z |00\alpha_z\rangle = -i\hbar(a_x^\dagger a_y - a_y^\dagger a_x)|00\alpha_z\rangle = 0 (35)
\]

We conclude that the coherent states with \(\alpha_x = \alpha_y = 0\) are the eigenstates of \(L_z\) with the eigenvalue \(m = 0\) and these are the only simultaneous eigenstates of annihilation operators and \(L_z\). \(\alpha_z\) can be an arbitrary complex number.

2 Problem 2.(A)

We have the Schrödinger Eqn.:

\[
-\frac{\hbar^2}{2m} \frac{d^2}{dx^2}\Psi(x) + g\delta(x)\Psi(x) = E\Psi(x)
\]

(36)

The solution with appropriate asymptotic behaviour is:

\[
\Psi(x) = \begin{cases} 
e^{ikx} + re^{-ikx}, & x < 0 \text{ (incident plus reflected waves)} \\
t e^{ikx}, & x > 0 \text{ (transmitted wave)} 
\end{cases}
\]

(37)

where \(k = \sqrt{2mE/\hbar}\). The amplitude of the incident wave can be arbitrary, we’ve chosen it to be just unity.

The solution must satisfy the boundary conditions at \(x = 0\).

\[
\Psi(-0) = \Psi(+0) \Rightarrow 1 + r = t \text{ (continuity)}
\]

(38)

\[
\Psi'(+0) - \Psi'(-0) = 2\frac{mg}{\hbar^2}\Psi(0) \Rightarrow ikt - ik(1 - r) = \frac{2mg}{\hbar^2}t
\]

(39)

( the derivative of \(\Psi(x)\) is discontinuous due to the presence of the delta-function potential).

After some simple algebra we get the solution to the system of eqns. (38), (39).

\[
t = \frac{1}{1 + \frac{2mg}{\hbar^2k}} \quad r = -\frac{1}{1 - \frac{2mg}{\hbar^2k}}
\]

(40)

The transmission coefficient is

\[
T = |t|^2 = \frac{1}{1 + \frac{\omega^2g^2}{\hbar^2k}} = \frac{2\hbar^2E}{mg^2 + 2\hbar^2E}
\]

(41)

while the reflection coefficient is

\[
R = |r|^2 = \frac{1}{1 + \frac{k^2\hbar^2g^2}{m^2\hbar}} = \frac{mg^2}{mg^2 + 2\hbar^2E}
\]

(42)

Note that \(T + R = 1\) as is required by flux conservation.
3 Problem 3.(G)

We expect that when \( V << \frac{p^2}{2m} \) for any states, the energy vs. momentum dispersion curve will be approximately parabolic for all values of momentum. In this perturbative limit, the eigenstates are still \( |\mathbf{k}\rangle \), where \( p = \hbar k \). The perturbative energy corrections

\[
\Delta E_k = \langle \mathbf{k}|V(x)|\mathbf{k}\rangle + \sum_{k' \neq k} \frac{|\langle \mathbf{k}'|V(x)|\mathbf{k}\rangle|^2}{E_k' - E_k}.
\]

As in the lecture notes on 1/30, the first term only leads to a uniform shifting of the parabola curve. And the second term only is significant for \( k = n\pi/\xi \). So if and only if even for \( k = n\pi/\xi, \frac{p^2}{2m} = (\hbar k)^2 / 2m \gg V_0 \), the dispersion curve will be almost parabolic for all values of momentum. That leads to \( (\hbar\pi)^2 / 2m\xi^2 \gg V_0 \).

An alternative way: Still the right condition is that the kinetic energy potential energy for all values of \( k \). The kinetic energy can be calculated by \( \frac{1}{2m} (-i\hbar \frac{\partial}{\partial x} + \hbar k)^2 u(x) \), where \( (x) \) is periodic: \( u(x) = u(x - \xi) \). Then the derivative term can be estimated as \( \frac{d}{dx} u \sim \frac{1}{\xi} u \). If \( k \) is large ( \( k\xi > 1 \) ) kinetic energy is about \( (\hbar k)^2 / 2m \), but if \( k \) is small ( \( k\xi < 1 \) ) the derivative term dominates and kinetic energy is \( \hbar^2 / 2m\xi^2 \). In both cases potential energy is small if \( (\hbar\pi)^2 / 2m\xi^2 \gg V_0 \).

(note that \( (\hbar\pi)^2 / 2m\xi^2 \gg V_0 \) ~ \( \hbar^2 / 2m\xi^2 \gg V_0 \))

4 Problem 4.(G)

4.1

Dimension of space \( H = H^A \otimes H^B \) is \((2j^A + 1)(2j^B + 1) = 4 + 1 = 3\), so the possible values of total \( j \) are 0 and 1.

4.2

Denote \( C(j^A j^B; m^A m^B | j; m) \) as \( (m^A m^B | jm) \). Since \( m = m^A + m^B \), only \((\pm \frac{1}{2}, \pm \frac{1}{2}|0, 0), (\pm \frac{1}{2}, \mp \frac{1}{2}|1, 0), (\frac{1}{2}, \frac{1}{2}|1, 1), (\frac{1}{2}, -\frac{1}{2}|1, -1)\) are non-zero. And since \((\frac{1}{2}, \frac{1}{2}|1, 1)\) is the only non-zero C-G coefficient for \( |1,1\), \((\frac{1}{2}, \frac{1}{2}|1, 1) = 1\). As the same, \((-\frac{1}{2}, -\frac{1}{2}|1, -1) = 1\).

4.3

\[
J^2 = \hbar^2 \begin{pmatrix}
2 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 2
\end{pmatrix}
\]
in the uncoupled basis. \( J^2 |0, 0\rangle = 0 \), so \( |0, 0\rangle = \frac{1}{\sqrt{2}} |\frac{1}{2}, -\frac{1}{2}\rangle - \frac{1}{\sqrt{2}} |\frac{1}{2}, \frac{1}{2}\rangle \).
so $(\pm \frac{1}{2}, \mp \frac{1}{2} | 0, 0) = \pm \frac{1}{\sqrt{2}}$.

4.4

$|1, 0\rangle = (\frac{1}{2}, -\frac{1}{2}|1, 0\rangle |\frac{1}{2}, -\frac{1}{2}) + (-\frac{1}{2}, \frac{1}{2}|1, 0\rangle |\frac{1}{2}, \frac{1}{2})$.

$\langle 0, 0 | 1, 0\rangle = 0$, so $(\pm \frac{1}{2}, \mp \frac{1}{2}|1, 0) = \frac{1}{\sqrt{2}}$.

All the six non-zero C-G coefficients are determined.

State $|1, 0\rangle$ and $|0, 0\rangle$ are entangled states Since these two states cannot be expressed as the direct products of some states in Hilbert-space $H^A$ and $H^B$. (For example $|1, 1\rangle$ is not entangled since $|1, 1\rangle = |\frac{1}{2}, \frac{1}{2}\rangle^A \otimes |\frac{1}{2}, \frac{1}{2}\rangle^B$.)