ON A GEOMETRIC INTERPRETATION OF THE PENTAGON RELATION

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Abstract. In this article we use the Betti-de Rham comparison for the unipotent fundamental groupoid of certain configuration spaces (and their Kummer coverings) to give a geometric interpretation of the pentagon relation among (cyclotomic) multiple zeta values.

Contents

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1. Introduction

The pentagon relation appears in [6] as an identity which should be satisfied by the twisting factor of the co-associativity for the coproduct of a quasi-Hopf algebra. In [7], Drinfeld shows that for a deformation quasi-triangular quasi-Hopf algebra over $\mathbb{C}$ such a twisting factor is given, uniquely up to gauge transformation, as $G_2^{-1}G_1$, where G_1 and G_2 are solutions of the KZ-equation with certain asymptotic behavior at singular points. More precisely, if one puts $\phi_{KZ}(A,B) = G_2^{-1}G_1$, where G_1 and G_2 are solutions (with certain asymptotic behavior at 0 and 1) for the differential equation

$$G'(x) = \frac{1}{2\pi i}\left(\frac{A}{x} + \frac{B}{x-1}\right)G(x), \tag{1}$$

with G a formal power series in noncommutative indeterminates A and B with coefficients that are analytic functions in x, then ϕ_{KZ} satisfies the so called pentagon relation:

$$\begin{aligned}\phi_{KZ}(X^{12}, X^{23}+X^{24})\phi_{KZ}(X^{13}+X^{23}, X^{34}) = \\ = \phi_{KZ}(X^{23}, X^{34})\phi_{KZ}(X^{12}+X^{13}, X^{24}+X^{34})\phi_{KZ}(X^{12}, X^{23}).\end{aligned} \tag{2}$$

We should also mention that there are two hexagon relations that are satisfied by ϕ_{KZ} which guarantee the compatibility of the co-commutativity and co-associativity of the quasi-triangular quasi-Hopf algebra.

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On the other hand, multiple zeta values appear as coefficients in the expansion of ϕ_{KZ} (for example see [10, Proposition 3.2.3]) and therefore the pentagon relation (2) gives rise to a family of relations among multiple zeta values. Indeed, pentagon relation plays a central role in the theory of multiple zeta values. In order to explain this more precisely, let us recall some basic definitions.

Definition 1.1. *Let $(s_1, s_2, \dots, s_d)$ be a d-uple of natural numbers with $s_1 \geq 2$ and let $(z_1, z_2, \dots, z_d)$ be a d-uple of N-th roots of unity, for some $N \in \mathbb{N}$. Then the cyclotomic multiple zeta value $\zeta(s_1, \dots, s_d; z_1, \dots, z_d)$ is defined as:*

$$\zeta(s_1, \dots, s_d; z_1, \dots, z_d) := \sum_{n_1 > n_2 > \cdots > n_d > 0} \frac{z_1^{n_1} \dots z_d^{n_d}}{n_1^{s_1} \dots n_d^{s_d}}.$$

In the case of $N = 1$, we omit the z_i from the notation and the number $\zeta(s_1, \dots, s_d)$ will be called a multiple zeta value.

Remark 1.2. Note that the value defined by the above series makes sense as long as $(s_1, z_1) \neq (1, 1)$, but the condition $s_1 \geq 2$ makes the defining series to be absolutely convergent.

Now it is a straightforward calculation to show that the product of any two (cyclotomic) multiple zeta values can be expressed as a linear combination with integer coefficients of (cyclotomic) multiple zeta values (relations of this form are called shuffle relations). Therefore, the $\mathbb{Q}$-vector space spanned by (cyclotomic) multiple zeta values is a subalgebra of $\mathbb{C}$ over $\mathbb{Q}$. Studying the structure of this subalgebra, which involves understanding the relations between (cyclotomic) multiple zeta values, is an interesting deep problem which has been considered by many mathematicians.

Now, when we said that the pentagon relation plays a central role in the theory of multiple zeta values what we meant is the following. In [11], using combinatorial properties of the cell decomposition of $\bar{\mathcal{M}}_{0,5}$, Furusho shows that the class of hexagon relations among multiple zeta values is a consequence of the pentagon relations, and in [12] he shows that the generalized shuffle relations can also be derived from the pentagon relation.

Also, in the case of cyclotomic multiple zeta values, Broadhurst in [1] for $N = 2$ and Okuda in [16] for $N = 4$ show that exceptional symmetries of $\mathbb{P}^1 \setminus \{0, \infty, \mu_N\}$ give rise to families of relations among cyclotomic multiple zeta values. The analog of Furusho's work for obtaining hexagon relations from the pentagon relation in the case of cyclotomic multiple zeta values for $N = 2$ has been developed in [9].

In all the above mentioned works, the geometry and symmetries of the spaces $\mathcal{M}_{0,4} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$, $\mathcal{M}_{0,5}$, and certain coverings of these spaces play significant role. Our goal in this article is to explore these roles and give a purely geometric interpretation of the pentagon relation by studying the geometry and symmetries of $\mathcal{M}_{0,4}$ and $\mathcal{M}_{0,5}$. This, in view of Furusho's works, gives also a geometric realization for the hexagon and generalized shuffle relations for multiple zeta values. We also outline how one might extend this machinery to Kummer coverings of $\mathcal{M}_{0,4}$ and $\mathcal{M}_{0,5}$ and get a geometric interpretation for similar identities among cyclotomic multiple zeta values.

This article is organized as follows. In section 2 we review three different constructions for the unipotent hull of an abstract group and apply these constructions to the case of the fundamental group of a space which gives rise to the notion of the Betti unipotent fundamental group. We also discuss the notion of de Rham unipotent fundamental group of a variety and the Betti-de Rham comparison isomorphism. Finally, we outline the theory of base points at infinity in Betti and de Rham versions. Section 3 is devoted to an explicit calculation of the de Rham unipotent fundamental group of varieties which admit a projectivization with trivial Picard variety. This is applied to the case of the moduli spaces of 4 and 5 distinct points on $\mathbb{P}^1$ and their Kummer coverings. Finally, in section 4, we show how the geometry and symmetries of the moduli spaces of 4 and 5 distinct points on $\mathbb{P}^1$ and their Kummer coverings together with the Betti-de Rham comparison isomorphism between their unipotent fundamental groups leads to the pentagon and other relations among (cyclotomic) multiple zeta values.

2. Unipotent completion and fundamental groups

Our geometric interpretation of the pentagon relation is based on the comparison between the Betti (or topological) and the de Rham unipotent fundamental groups associated to certain varieties and the symmetries of this system coming from geometry. Therefore, in this section, we develop the necessary tools concerning the Betti and the de Rham unipotent fundamental groups.

Let us begin by recalling the Malčev completion, which is a functorial way of associating a unipotent algebraic group scheme over $\mathbb{Q}$ to any finitely generated nilpotent torsion free abstract group. For any abstract group G, let $Z^\bullet G$ denote the descending central series of G. That is, Z^1G is defined to be G itself and $Z^{i+1}G := [G, Z^iG]$, for any $i \geq 1$. Recall that a group G is called unipotent of class n if $Z^{n+1}G = 0$.

Lemma 2.1. *Let G be a finitely generated nilpotent torsion free group and let $x, y \in G$ be such that $x^m = y^m$ for some $m \in \mathbb{Z}$. Then $x = y$.*

Proof. The proof is by induction on the nilpotent class of G. Note that the assertion is evident if G is abelian. So assume that G is of nilpotent class at least 2 and let $\mathfrak{Z}(G)$ denote the center of G. Then $G/\mathfrak{Z}(G)$ is torsion free and its nilpotent class is strictly less than G. Thus, by induction hypothesis, we have $x = zy$ for some element z in $\mathfrak{Z}(G)$. But in this case, $y^m = x^m = z^m y^m$ and hence $z^m = 1$. This, together with the fact that $\mathfrak{Z}(G)$ is torsion free, implies that $z = 1$ and thus $x = y$. □

Remark 2.2. Note that if G is a torsion free group, then the subquotients of its ascending central series are torsion free as well, but the same is not true for the descending central series (see Example 2.3).

Now let G be a finitely generated group and for any $n \geq 1$ put

$$\Gamma^n := \left(G/Z^{n+1}G\right)/\text{Torsion},$$

which is a finitely generated torsion free nilpotent group of nilpotent class n. For every $1 \leq i \leq n$, let $F_i := Z^i\Gamma^n/Z^{i+1}\Gamma^n$, which is a finitely generated abelian group (not necessarily torsion free). Then we have:

$$F_i \cong \mathbb{Z}^{r_i} \oplus T_i,$$

where T_i is a finite group and r_i is the rank of F_i as a $\mathbb{Z}$-module. For any i, fix a basis $e_1^{(i)}, \ldots, e_{r_i}^{(i)}$ for the free part of F_i and lift this basis to elements $e_1^i, \ldots, e_{r_i}^i$ in Γ^n. Then, every element $\gamma \in \Gamma^n$ admits a unique representation of the form

$$\gamma = (e_1^1)^{q_1^1} \ldots (e_{r_n}^n)^{q_{r_n}^n}, \tag{3}$$

with rational exponents (note that raising to rational exponents is well defined by Lemma 2.1). Moreover, the group operation of Γ^n, after expressing everything in the form (3), is given by polynomial formulas in terms of the exponents. The Malčev completion $\Gamma^n_{\mathbb{Q}}$ of Γ^n is then defined to be the group $\oplus_{i=1}^n \oplus_{j=1}^{r_i} \mathbb{Q}.e_j^i$, where the multiplication is given by the same polynomial formulas. Note that $\Gamma^n_{\mathbb{Q}}$ is an algebraic group scheme over $\mathbb{Q}$ (evidently, one has to verify the independence of the construction, up to isomorphism, from the choices). Finally, we define the Malčev or the unipotent completion $G_{\mathbb{Q}}$ of G to be:

$$G_{\mathbb{Q}} := \varprojlim_n \Gamma^n_{\mathbb{Q}},$$

which is a pro-unipotent group scheme over $\mathbb{Q}$.

Furthermore, note that there is a natural increasing filtration $F^\bullet$ on the coordinate ring $\mathbb{Q}[G_{\mathbb{Q}}]$ of the Malčev completion of G, which is induced by the descending central series of G. Namely, take F^0

to be spanned by constants, F^1 to be spanned by F^0 and linear combinations of variables associated to a basis of the free part of Z^1G/Z^2G, F^2 to be spanned by F^1, products of two elements in F^1, and linear combinations of variables coming from Z^2G/Z^3G, and so on.

Example 2.3. *Consider the group G defined as:*

$$G := \langle x, y, c | c \in \mathfrak{Z}(G), yx = xyc^2 \rangle.$$

Then G is of nilpotent class 2 and its descending central series has the form

$$G = Z^1G \supset \ \langle c^2 \rangle \ \supset 1.$$

That implies that $F_1 \cong \mathbb{Z}.x \oplus \mathbb{Z}.y \oplus \mathbb{Z}/2\mathbb{Z}$ and $F_2 \cong \mathbb{Z}.c^2$. Now every element $g \in G$ can be uniquely written in the form $g = x^p y^q (c^2)^r$ with $p, q \in \mathbb{Z}$ and $r \in \frac{1}{2}.\mathbb{Z}$ and the group multiplication is given by

$$(p, q, r).(p', q', r') = (p + p', q + q', r + r' + qp').$$

Therefore, the unipotent completion $G_{\mathbb{Q}}$ of G is isomorphic to the subgroup:

$$\{\begin{pmatrix} 1 & 0 & 0 \\ p & 1 & 0 \\ r & q & 1 \end{pmatrix} : p, q, r \in \mathbb{Q}\}$$

of the matrix group GL_3 over $\mathbb{Q}$.

One can easily check that the canonical filtration induced by the descending central series on the coordinate ring of $G_{\mathbb{Q}}$ is given by considering p and q of degree 1 and r of degree 2.

Before applying the unipotent completion to the special case of fundamental groups, let us briefly mention two other approaches toward unipotent completion of abstract groups.

For any finitely generated group G let $\mathbb{Q}[G]$ denote the group algebra of G over $\mathbb{Q}$ and let $I \subset \mathbb{Q}[G]$ be the augmentation ideal, that is I is the kernel of the augmentation map $\epsilon : \mathbb{Q}[G] \to \mathbb{Q}$. Then I is preserved by the Hopf algebra structure of $\mathbb{Q}[G]$, induced by the group structure of G, and therefore, for all $n \geq 1$, the quotient $\mathbb{Q}[G]/I^n$ inherits a Hopf algebra structure. Quillen's description of the unipotent completion of G is as follows:

$$G_{\mathbb{Q}} \cong \operatorname{Spec}\left(\varinjlim_n (\mathbb{Q}[G]/I^n)^\vee\right).$$

In this interpretation, the filtration $F^\bullet$ on the coordinate algebra $\varinjlim_n (\mathbb{Q}[G]/I^n)^\vee$ is the one induced by powers of I.

Finally, for any finitely generated group G, let $\operatorname{Rep}_{\mathbb{Q}}(G)$ be the category of finite dimensional representations of G over $\mathbb{Q}$ and let $\operatorname{Rep}_{\mathbb{Q}}^{un}(G)$ be the full subcategory of unipotent representations. Then the category $\operatorname{Rep}_{\mathbb{Q}}^{un}(G)$, equipped with the forgetful functor as a fibre functor, forms a neutral Tannakian category over $\mathbb{Q}$ and the unipotent completion $G_{\mathbb{Q}}$ of G can be defined as the corresponding group scheme over $\mathbb{Q}$.

All the above constructions for unipotent completion work for an arbitrary finitely generated group. But we are particularly interested in the case where $G = \pi_1(X, x)$ is the fundamental group of a topological space X with a base point x. In this situation, in addition to the unipotent completion of $\pi_1(X, x)$, which is a pro-unipotent group scheme over $\mathbb{Q}$ denoted by $\pi_1^B(X, x)$, for an arbitrary point y in X, we are interested in the "unipotent completion" of the space of homotopy classes of paths from y to x, which is going to be a torsor over $\pi_1^B(X, x)$.

Let X be a topological space of a complex algebraic variety. For any point $x \in X$, let Λ_x denote the group algebra $\mathbb{Q}[\pi_1(X, x)]$ and let I_x denote the augmentation ideal of Λ_x. Then, it is known that there is a functorial correspondence between Λ_x-modules and local systems on X in a way that if $\tilde{V}$ denotes the local system on X associated to a Λ_x-module V, then the fibre $\tilde{V}[x]$ of $\tilde{V}$ at x is canonically identified with V. Therefore, for any $n \geq 1$, the local system $\widetilde{(\Lambda_x/I_x^n)}$ together with the element $1 \in (\Lambda_x/I_x^n) = \widetilde{(\Lambda_x/I_x^n)}[x]$ is the universal unipotent local system of unipotent class n equipped with a distinguished element in its fibre at x. That is, for any unipotent local system L of unipotent class n with a fixed element $l \in L[x]$, there exits a unique morphism of local systems from $\widetilde{(\Lambda_x/I_x^n)}$ to L which sends 1 to l. This simply follows from the fact that for any Λ_x-module V with $I_x^n V = 0$ and any element $v \in V$, there exists a unique Λ_x-linear map from Λ_x/I_x^n to V which maps 1 to v.

Now fix two points $x, y \in X$ and for any $0 \leq j \leq n$ consider the subset $Y_j := \{t_j = t_{j+1}\} \subset X^n$, where t_0 and t_{n+1} are taken to be y and x respectively and for any i between 1 and n, t_i denotes the i-th coordinate of a point in X^n. For any $J \subset \Delta_n := \{0, \ldots, n\}$, put $Y_J := \cap_{j \in J} Y_j$ and let $\mathbb{Q}_J = i_*(\mathbb{Q})$, where $i : Y_J \hookrightarrow X^n$ is the embedding of Y_J in X^n and $\mathbb{Q}$ is the constant sheaf with value $\mathbb{Q}$ on Y_J. Then, for any $J \subset K$ there is a restriction map $\mathbb{Q}_J \to \mathbb{Q}_K$ and the alternating sum of these restriction maps form differential maps of a complex of sheaves on X^n of the form:

$$\mathbb{Q} \to \bigoplus_{0 \leq j \leq n} \mathbb{Q}_{\{j\}} \to \ldots \to \bigoplus_{J \subset \Delta_n,\ |J| = p} \mathbb{Q}_J \to \ldots \to \mathbb{Q}_{\Delta_n}. \tag{4}$$

Note that $\mathbb{Q}_{\Delta_n} = 0$ unless $n = 0$ or $x = y$, where it is the skyscraper sheaf on the point $t_1 = \cdots = t_n = x$ with fibre $\mathbb{Q}$. Let ${}_y\mathfrak{K}_x^n$ denote the truncation of the complex (4) obtained by removing the last term and putting the other terms in degrees 0 to n. Then we have the following:

Proposition 2.4. [5, Proposition 3.4] *Using the above notations, we have:*

(1) *For any $i < n$, $\mathbb{H}^i(X^n,\ {}_y\mathfrak{K}_x^n) = 0$.*
(2) *For varying y, the local system $\mathbb{H}^n(X^n,\ {}_y\mathfrak{K}_x^n)$ on X, equipped with the linear map $\mathbb{H}^n(X^n,\ {}_x\mathfrak{K}_x^n) \to \mathbb{Q}$ induced by the map of complexes ${}_x\mathfrak{K}_x^n \to \mathbb{Q}_{\{t_1 = \cdots = t_n = x\}}[-n]$, is the dual of the local system $\widetilde{(\Lambda_x/I_x^n)}$ equipped with the linear map $\mathbb{Q} \to \Lambda_x/I_x^n$ induced by $1 \in \lambda_x/I_x^n$.*

This fundamental proposition shows that the unipotent completion of the fundamental group and path torsors of a topological space is governed by cohomology of certain spaces and gives an explicit description of the quotients of the dual of the coordinate ring of the unipotent fundamental group by powers of its augmentation ideal in terms of cohomology of these spaces.

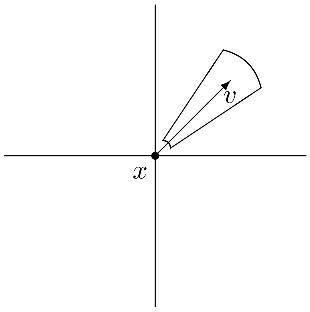
Now let us turn our attention to the de Rham fundamental group and path torsors. Assume that X is an algebraic variety defined over a subfield k of the field of complex numbers and fix a base point $x \in X(k)$. To define the de Rham counterpart of the Betti unipotent fundamental group, consider the category $\mathcal{U}^{dR}$ of vector bundles with flat connection over X which are iterated extensions of the trivial vector bundle with connection $(\mathcal{O}_X, d)$. Then $\mathcal{U}^{dR}$ equipped with the fibre functor $\mathcal{F}_x : (\mathcal{V}, \nabla) \mapsto \mathcal{V}_x$ forms a neutral Tannakian category over k, and thus we may define the de Rham unipotent fundamental group $\pi_1^{dR}(X, x)$ as the pro-unipotent k-group scheme associated to $(\mathcal{U}^{dR}, \mathcal{F}_x)$. Similarly, for an arbitrary point $y \in X(k)$, we have a fibre functor $\mathcal{F}_y : (\mathcal{V}, \nabla) \mapsto \mathcal{V}_y$, and by Tannakian formalism the space of natural transformations from $\mathcal{F}_y$ to $\mathcal{F}_x$ is representable by a torsor over the de Rham unipotent fundamental group of (X, x) (this torsor is called the torsor of homotopy classes of de Rham paths from y to x and is denoted by $\pi_1^{dR}(X; x, y)$). Very important to us is that the Betti and the de Rham unipotent fundamental groups and path torsors are closely related. More precisely, by Riemann-Hilbert correspondence, the category of complex local systems over $X(\mathbb{C})$ is equivalent to the category of vector bundles with flat connection on $X_{\mathbb{C}}$. One can easily check that this correspondence respects being unipotent and thus establishes an isomorphism of the form:

$$\pi_1^B(X, x) \otimes_{\mathbb{Q}} \mathbb{C} \cong \pi_1^{dR}(X, x) \otimes_k \mathbb{C}. \tag{5}$$

One also has a similar isomorphism between Betti and de Rham path torsors.

Remark 2.5. We would like to mention that the above theory of unipotent fundamental groups and path torsors, which is discussed in the Betti and de Rham versions, has analogues in the étale, crystalline, and algebraic cycle (motivic) worlds. For more details in those directions and applications we refer the reader to [4], [5], [14], and [15]. See also [17] where Ünver uses the crystalline-de Rham comparison isomorphism in order to derive Drinfeld-Ihara relations for p-adic multiple zeta values.

Let us conclude this section with a brief discussion of the notion of base point at infinity (following [4, Section 15]). For the remainder of the section, assume that k is a subfield of the field of complex numbers, let $\bar{X}$ be a smooth projective variety over k, let D be a strict normal crossing divisor on $\bar{X}$, and put $X := \bar{X} \setminus D$. For any point $x \in X(k)$, we have already seen how to define the Betti and the de Rham fundamental group of X based at x. A useful fact is that one can make sense of these notions even if the base point x is a "point at infinity", that is x belongs to the divisor D, as soon as one fixes a nonzero tangent vector at x which points towards X. Here, what we mean by "points towards X" is the following. Fix a point $x \in D$ and locally at x let $D = \bigcup_{i\in I} D_i$ be the decomposition of D into its irreducible components (i.e. $\{D_i\}_{i\in I}$ is the set of irreducible components of the restriction of D to the local ring of x in $\bar{X}$). Then for each $i \in I$, the tangent space T_x of $\bar{X}$ at x contains a hyperplane $T_{x,i}$ which is the tangent space of D_i at x (note that D_i is assumed to be a smooth divisor in $\bar{X}$). We say that a tangent vector v at x points towards X if it belongs to $T_x \setminus \bigcup_{i\in I} T_{x,i}$. Now, using this notations, for a point $x \in D$, any tangent vector pointing towards X can serve as a base point in the theory of fundamental groups. The idea in the Betti version can be visualized as follows. First note that in the theory of topological fundamental groups, a contractible subset of X can play the role of a base point, and then note that, for a fixed point x in D, a tangent vector which points towards X determines a contractible subset of X (see the following picture).



In the case of de Rham theory, the situation is more technical. So let us briefly go through the main idea in the one dimensional case and refer the reader to [4, Section 15] for higher dimensional cases. In the one dimensional case, $\bar{X}$ is a smooth projective curve over k, D is a finite set of k-points of $\bar{X}$, and X is the complement of D in $\bar{X}$. Let x be one of the finitely many points in D and let v be a nonzero tangent vector at x (note that in the one dimensional case any nonzero tangent vector points towards X). Then in order to define the unipotent de Rham fundamental group $\pi_1^{dR}(X, v)$ with base point at v, using the Tannakian formalism, we need to construct a fibre functor $\mathcal{F}_v$ on the category $\mathcal{U}^{dR}$ of unipotent vector bundles with integrable connection on X. After restricting a unipotent vector bundle with flat connection to the local neighborhood around x, we may assume that $\bar{X} = \mathrm{Spec}(k[[t]])$, that x is the closed point of $\bar{X}$, and that $X = \bar{X} \setminus \{s\} = \mathrm{Spec}(K)$ where K is the fraction field of $k[[t]]$. The following proposition is the key technical tool for defining $\mathcal{F}_v$.

Proposition 2.6. *Using the above notations, let T_x° be the tangent space of x punctured at origin. Then there is an exact tensor functor from the category of unipotent vector bundles with integrable connection on X to the same category on T_x°.*

Proof. (Sketch) First of all, note that for any object $(\mathcal{V}, \nabla)$ in $\mathcal{U}^{dR}$, the connection ∇ has regular singularities at x in the sense of [2]. On the other hand, since $X = \operatorname{Spec}(K)$, a vector bundle $\mathcal{V}$ on X is nothing but a vector space V over K and, by definition, the connection ∇ has regular singularities at x if there exists a $k[[t]]$-lattice L in V with the following property:

- There exists a basis B of the lattice L with respect to which the connection ∇ has the form:

$$\nabla(v) = d(v) + \Gamma(v),$$

where $\Gamma \in \Omega^1(K) \otimes \operatorname{End}(V)$ has simple poles.

Now, by changing the basis from B to B', if A denotes the change of basis matrix, the matrix of 1-forms Γ would change according to the classic formula:

$$\Gamma' = A^{-1}dA + A^{-1}\Gamma A. \tag{6}$$

Therefore, the residue at x of the matrix Γ, as a linear transformation of the k-vector space $L \otimes_{k[[t]]} k$, is independent of the choice of the basis. Let us denote this linear transformation by $\operatorname{res}(\nabla)$. Finally, the vector bundle with connection $(\mathcal{V}_x, \nabla_x)$ on T_x° associated to $(\mathcal{V}, \nabla)$ can be constructed as follows. The vector bundle $\mathcal{V}_x$ is defined to be the trivial vector bundle with fibre $L \otimes_{k[[t]]} k$ and the connection ∇_x is defined via the formula:

$$\nabla_x := d + \operatorname{res}(\nabla)\frac{da}{a},$$

where a is any linear form on T_x (note that the logarithmic differential da/a is independent of the choice of the linear form a). □

Now, since any nonzero vector v in the tangent space of x is a point of T_x°, the construction of the fibre functor $\mathcal{F}_v$ on $\mathcal{U}^{dR}$ is evident. Take any object $(\mathcal{V}, \nabla)$ in $\mathcal{U}^{dR}$, transform it using the above proposition into a vector bundle with connection $(\mathcal{V}_x, \nabla_x)$ on T_x°, and apply the usual fibre functor associated to v as a point in T_x°.

Remark 2.7. Note that in the Betti-de Rham comparison isomorphism (5), we used the Riemann-Hilbert correspondence between the category of local systems of finite dimensional complex vector spaces and the category of vector bundles with integrable connection. Therefore, in order to carry over this comparison isomorphism to the case of base points at infinity, one needs to have the analogue of Proposition 2.6 for local systems. We would like to mention that such a construction for local systems exists, and the Betti-de Rham comparison isomorphism extends to the case of base points at infinity (see [4, Section 15]).

3. Explicit de Rham in an special case

In this section, following [4], we want to present an explicit description of the Lie algebra of the de Rham fundamental group of a certain class of varieties. Let k be a subfield of $\mathbb{C}$, assume that X is a smooth variety over k, that X admits a smooth projectivization $X \subset \bar{X}$ such that the divisor at infinity $D := \bar{X} \setminus X$ is a simple normal crossing divisor, and that $\operatorname{Pic}^0(\bar{X}) = 0$ (note that, thanks to the resolution of singularities, the existence of a smooth projectivization with a nice divisor at infinity is automatic and the only restricting assumption is the one on the Picard variety). Now, by [2], any regular singular connection $(\mathcal{V}, \nabla)$ with unipotent monodromy along D has a canonical extension $(\bar{\mathcal{V}}, \bar{\nabla})$ to $\bar{X}$ where $\bar{\nabla}$ has logarithmic poles along D. Moreover, if $(\mathcal{V}, \nabla)$ is unipotent, i.e. is iterated extension of $(\mathcal{O}_X, d)$, so is $(\bar{\mathcal{V}}, \bar{\nabla})$ (by uniqueness of the canonical extension). But since $H^1(\bar{X}, \mathcal{O}_{\bar{X}})$, being the Lie algebra of the Picard variety, vanishes under our assumption, we have:

$$\operatorname{Ext}^1_{\bar{X}}(\mathcal{O}^n_{\bar{X}}, \mathcal{O}^m_{\bar{X}}) = \operatorname{Ext}^1_{\bar{X}}(\mathcal{O}_{\bar{X}}, \mathcal{O}_{\bar{X}})^{mn} = H^1(\bar{X}, \mathcal{O}_{\bar{X}})^{mn} = 0.$$

This implies that $\bar{\mathcal{V}}$, being an iterated extension of $\mathcal{O}_{\bar{X}}$, is isomorphic to $\mathcal{O}^r_{\bar{X}}$ as a vector bundle, where $r = \operatorname{Rank}(\bar{\mathcal{V}}) = \operatorname{Rank}(\mathcal{V})$. Therefore, the following lemma can be applied to $\bar{\mathcal{V}}$.

Lemma 3.1. *Every connection $\bar{\nabla}$ with logarithmic poles along D on the trivial vector bundle $\mathcal{O}_{\bar{X}}^r \cong k^r \otimes_k \mathcal{O}_{\bar{X}}$ has the form $d + \Omega$ for a unique $\Omega \in H^0(\bar{X}, \Omega^1(D)) \otimes \mathrm{End}(k^r)$.*

Proof. In general, by definition, a connection with logarithmic poles along D on a vector bundle $\mathcal{V}$ on $\bar{X}$ is a k-linear map from $\mathcal{V}$ to $\mathcal{V} \otimes_{\mathcal{O}_{\bar{X}}} \Omega^1(D)$ which satisfies the Leibnitz rule. A straightforward computation shows that the difference of any two such connections is $\mathcal{O}_{\bar{X}}$-linear and thus the space of all connections with logarithmic poles on $\mathcal{V}$ forms a torsor over $\mathrm{Hom}_{\mathcal{O}_{\bar{X}}}(\mathcal{V}, \mathcal{V} \otimes_{\mathcal{O}_{\bar{X}}} \Omega^1(D))$. In the special case of $\mathcal{V} = \mathcal{O}_{\bar{X}}^r$, this torsor has a canonical element induced by $d : \mathcal{O}_{\bar{X}}^r \to \Omega^1(D)^r$ and thus an arbitrary element can be uniquely written in the desired form. □

Using this lemma and the preceding discussion, we can prove the following proposition, which gives an explicit description of the Lie algebra of the de Rham unipotent fundamental group $\pi_1^{dR}(X, x)$.

Proposition 3.2. *Let $\bar{X}$ be a smooth projective variety over k with trivial Picard variety, let D be a simple normal crossing divisor on $\bar{X}$, and put $X := \bar{X} \setminus D$. Then for any point $x \in X(k)$, the Lie algebra $\mathcal{L}$ of the de Rham unipotent fundamental group $\pi_1^{dR}(X, x)$ has the form:*

$$\mathcal{L} = \langle H_1^{dR}(X) \mid \mathrm{Im}(H_2^{dR}(X) \to \wedge^2 H_1^{dR}(X)) \rangle,$$

where the map from $H_2^{dR}(X)$ to $\wedge^2 H_1^{dR}(X)$ is the dual of the cup product map. That is, $\mathcal{L}$ is generated as a Lie algebra over k by the vector space $H_1^{dR}(X)$ and the relations are given by the image of the dual of the cup product.

Proof. By definition, the de Rham unipotent fundamental group of X is the pro-unipotent group scheme over k whose category of finite dimensional representations is equivalent to the category $\mathcal{U}^{dR}$ of unipotent vector bundles with flat connection over X. On the other hand, by Lemma 3.1 and the discussion before it, we obtain a functorial correspondence between objects $(\mathcal{V}, \nabla)$ of $\mathcal{U}^{dR}$ and pairs (V, Ω) consisting of a finite dimensional vector space V over k and a 1-form Ω with logarithmic poles at D with values in $\mathrm{End}(V)$ (i.e. $\Omega \in H^0(\bar{X}, \Omega^1(D)) \otimes \mathrm{End}(V)$) which satisfies the Maurer-Cartan equation $d\Omega + \Omega \wedge \Omega = 0$ (the Maurer-Cartan equation for Ω is equivalent to the integrability of the connection ∇). Now, from general facts in Hodge theory, we know that the spectral sequence:

$$E_1^{pq} := H^q(\bar{X}, \Omega^p(D)) \Rightarrow H_{dR}^{p+q}(X)$$

degenerates at the first page (see [3, Corollaire 3.2.13 (ii)]), which, in particular, has the following consequences:

- $H^0(\bar{X}, \Omega^1(D)) = E_1^{10} = E_\infty^{10} = H_{dR}^1(X)$ (note that $E_1^{01} = H^1(\bar{X}, \mathcal{O}_{\bar{X}}) = 0$ by our assumption).
- Ω is closed, that is $d\Omega = 0$ in $H^0(\bar{X}, \Omega^2(D))$, since $\Omega \in E_1^{10} = E_\infty^{10}$.
- $H^0(\bar{X}, \Omega^2(D)) \subset H_{dR}^2(X)$.

Therefore, if $c : H_{dR}^1(X) \otimes H_{dR}^1(X) \to H_{dR}^2(X)$ denotes the cup product map, the element $\Omega \in H_{dR}^1(X) \otimes \mathrm{End}(V)$ gives rise to a sequence:

$$V \to V \otimes H_{dR}^1(X) \to V \otimes \wedge^2(H_{dR}^1(X)) \xrightarrow{c} V \otimes H_{dR}^2(X), \tag{7}$$

where the composition of all the three maps is the zero map. By taking dual, we get the sequence:

$$H_2^{dR}(X) \otimes V^\vee \to \wedge^2(H_1^{dR}(X)) \otimes V^\vee \to H_1^{dR}(X) \otimes V^\vee \to V^\vee, \tag{8}$$

where the composition of all three maps is zero (this accounts for integrability of the connection ∇). On the other hand, this data is evidently equivalent to the data of a finite dimensional nilpotent representation of the Lie algebra:

$$\mathcal{L} = \langle H_1^{dR}(X) \mid \mathrm{Im}(H_2^{dR}(X) \to \wedge^2 H_1^{dR}(X) \rangle.$$

The upshot is an equivalence between the category of finite dimensional nilpotent representations of $\mathcal{L}$ and the category of finite dimensional unipotent representations of $\pi_1^{dR}(X, x)$, and hence we are done. □

Remark 3.3. Under our assumption on the Picard variety of $\bar{X}$, which in particular implies that the extension $\bar{\mathcal{V}}$ of $\mathcal{V}$ can be trivialized on $\bar{X}$, for any two points $x, y \in X(k)$ the corresponding fibre functors $\mathcal{F}_x$ and $\mathcal{F}_y$ on $\mathcal{U}^{dR}$ are canonically isomorphic. In other words, there are canonical de Rham paths between points of X which are compatible with concatenation. This means that, under the hypothesis $\mathrm{Pic}^0(\bar{X}) = 0$, the de Rham fundamental groupoid can be trivialized (which is not true in general). Therefore, under the assumption $\mathrm{Pic}^0(\bar{X}) = 0$, we can drop the base point from our notation and speak of the de Rham fundamental group $\pi_1^{dR}(X)$.

Let us conclude this section with two important examples where the above proposition can be applied.

Example 3.4. *Let $S \subset \mathbb{P}^1(k)$ be a finite set of rational points on the projective line and put $X := \mathbb{P}^1 \backslash S$. Then for any point $s \in S$, we have a residue map:*

$$\mathrm{res}_s : H^1_{dR}(X) = H^0(\mathbb{P}^1, \Omega^1(S)) \to k.$$

This residue map can be viewed as an element $e_s \in H_1^{dR}(X)$ and by the residue theorem we know that $\sum_{s \in S} e_s = 0$. Therefore,

$$H_1^{dR}(X) = \langle \{e_s\}_{s\in S} | \sum_{s\in S} e_s = 0 \rangle.$$

Now if $S \neq \emptyset$, $H_2^{dR}(X) = 0$ and thus the Lie algebra of $\pi_1^{dR}(X)$ is the free Lie algebra generated by $H_1^{dR}(X)$.

As a particular case, let ξ_N be a primitive N-th root of unity and consider $X_N := \mathbb{P}^1 \setminus \{0, \infty, \mu_N\}$ over $k = \mathbb{Q}(\xi_N)$, where $\mu_N = \{\xi_N^i\}_{i=1}^N$ is the set of all N-th roots of unity. Then, if e_0 and $\{e_{\xi_N^i}\}_{1\le i\le N}$ denote the residue maps at zero and N-th roots of unity respectively, $H_1^{dR}(X_N)$ is freely generated by e_0 and $\{e_{\xi_N^i}\}_{1\le i\le N}$. This implies that the Lie algebra $\mathcal{L}_N$ of $\pi_1^{dR}(X_N)$ is the free Lie algebra on $N+1$ generators. So the universal enveloping algebra $\mathcal{U}_N$ of $\mathcal{L}_N$ is the free associative k-algebra on $N+1$ variables, i.e.

$$\mathcal{U}_N = k\langle W_0, \{W_{\xi_N^i}\}_{1\le i\le N}\rangle.$$

On the other hand, it is known that the dual of the affine coordinate ring of $\pi_1^{dR}(X_N)$ is the I-adic completion $\hat{\mathcal{U}}_N$ of $\mathcal{U}_N$, where $I \subset \mathcal{U}_N$ is the augmentation ideal (see [5, Proposition A.6]*). This in particular implies that for any field extension k' of k, the group of k'-rational points of $\pi_1^{dR}(X_N)$ can be embedded into $k'\langle\langle W_0, \{W_{\xi_N^i}\}_{1\le i\le N}\rangle\rangle$ as the multiplicative subgroup of group-like elements. Recall that a group-like element is an element $e \in k'\langle\langle W_0, \{W_{\xi_N^i}\}_{1\le i\le N}\rangle\rangle$ such that $\epsilon(e) = 1$ and $\Delta(e) = e \otimes e$, where Δ is the coproduct of $k'\langle\langle W_0, \{W_{\xi_N^i}\}_{1\le i\le N}\rangle\rangle$. The case $N = 1$ of this example, where $X_1 = \mathcal{M}_{0,4} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ is the moduli space of Riemann spheres with 4 marked points, will be of particular importance to us.*

Example 3.5. *Let $X = M_{0,n}$ be the moduli space of $n \ge 3$ distinct marked points on $\mathbb{P}^1$ and let $\bar{X} = \bar{M}_{0,n}$ be the Deligne-Mumford compactification of $M_{0,n}$ (note that $\bar{M}_{0,n}$ is a rational variety and hence has a trivial Picard variety). For any $1 \le i \ne j \le n$, there is a divisor D_{ij} at infinity corresponding to the collision of the i-th and the j-th marked points. Let $e_{ij} \in H_1^{dR}(X)$ be the element associated to the residue map along D_{ij}, and for convenience put $e_{ii} = 0$ for all $1 \le i \le n$ (note that $e_{ij} = e_{ji}$). Then $H_1^{dR}(X)$ is generated by $\{e_{ij}\}_{1\le i,j\le n}$ and there are two types of relations among these residue maps, namely:*

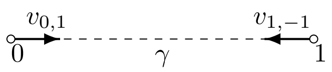
- *If $\{i, j\} \cap \{k, l\} = \emptyset$, then e_{ij} and e_{kl} commute, that is $[e_{ij}, e_{kl}] = 0$.*
- *For any fixed value of i, the residue theorem implies that $\sum_{j=1}^n e_{ij} = 0$.*

Combining the above relations, one easily verifies that, for distinct i, j, and k, $[e_{ij}, e_{ik} + e_{jk}] = 0$ in $H_1^{dR}(X)$.

4. Straight path, (cyclotomic) multiple zeta values, and the pentagon relation

In this final section, we want to show how the geometry and symmetry of the moduli space $M_{0,5}$ and its Kummer coverings lead to the pentagon relation among (cyclotomic) multiple zeta values. Recall from Example 3.4 that for $X := M_{0,4} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$, $H_1^{dR}(X)$ is generated by e_0 and e_1, which correspond to the residue maps at 0 and 1 respectively. Moreover, as $H_2^{dR}(X) = (0)$, the Lie algebra of the de Rham fundamental group is the free Lie algebra generated by $H_1^{dR}(X)$ and therefore, its enveloping algebra is the free associative algebra over $\mathbb{Q}$ with two variables. This implies that the group of $\mathbb{Q}$-points of $\pi_1^{dR}(X)$ can be identified with the group of group-like elements in $\mathbb{Q}\langle\langle W_0, W_1\rangle\rangle$ (here $\mathbb{Q}\langle\langle W_0, W_1\rangle\rangle$ is the free associative formal power series algebra over $\mathbb{Q}$ in two indeterminates W_0 and W_1 and an element $g \in \mathbb{Q}\langle\langle W_0, W_1\rangle\rangle$ is called group-like if it satisfies $\epsilon(g) = 1$ and $\Delta(g) = g \otimes g$, where $\Delta : \mathbb{Q}\langle\langle W_0, W_1\rangle\rangle \to \mathbb{Q}\langle\langle W_0, W_1\rangle\rangle \otimes \mathbb{Q}\langle\langle W_0, W_1\rangle\rangle$ is the coproduct).

Now let $v_{0,1}$ (resp., $v_{1,-1}$) denote the tangent vector 1 at 0 (resp., the tangent vector -1 at 1) in X and consider the straight path $\gamma \in \pi_1^B(X; v_{0,1}, v_{1,-1})$ (see the following figure):



Using the Betti-de Rham comparison isomorphism (5) we can transform the Betti path γ to an element p_γ in $\pi_1^{dR}(X; v_{0,1}, v_{1,-1}) \otimes \mathbb{C}$. On the other hand, since $\mathrm{Pic}^0(\mathbb{P}^1) = (0)$, the de Rham path torsor $\pi_1^{dR}(X; v_{0,1}, v_{1,-1})$ admits a canonical trivialization (see Remark 3.3) and thus p_γ gives rise to a group-like element:

$$f_\gamma(W_0, W_1) \in \mathbb{C}\langle\langle W_0, W_1\rangle\rangle. \tag{9}$$

The relation between this geometric constructions and multiple zeta values is established by the following proposition, which in particular shows that the power series f_γ coincides with Drinfeld's associator ϕ_{KZ} that appeared in the introduction (see [10, Proposition 3.2.3]).

Proposition 4.1. [5, Proposition 5.17] *If the coefficient of a word w in the letters W_0 and W_1 in the expansion of $f_\gamma(W_0, W_1)$ is denoted by c_w, then for any sequence $(s_1, s_2, \ldots, s_d) \in \mathbb{N}^d$ with $s_1 \geq 2$, we have:*

$$c_{W_0^{s_1-1}W_1W_0^{s_2-1}W_1\ldots W_0^{s_d-1}W_1} = (-1)^d\zeta(s_1, s_2, \ldots, s_d).$$

Remark 4.2. Actually, the above proposition is a special case of [5, Proposition 5.17] and more generally one has an analogue of the above proposition for cyclotomic multiple zeta values, which is as follows. Suppose that, for any $N \geq 2$, we consider the straight path γ in the space:

$$X_N := \mathbb{P}^1 \setminus \{0, 1, \mu_N\},$$

where μ_N is the set of all N-th roots of unity. In this situation, $H_1^{dR}(X_N)$ is generated by the residue maps e_0 and $\{e_\xi\}_{\xi\in\mu_N}$ and thus γ generates a formal power series in $\mathbb{C}\langle\langle W_0, \{W_\xi\}_{\xi\in\mu_N}\rangle\rangle$. Then, [5, Proposition 5.17] states that certain coefficients in this power series can be compared to cyclotomic multiple zeta values.

Remark 4.3. The appearance of (cyclotomic) multiple zeta values in Proposition 4.1 can be explained by Chen's theory of iterated integrals. Recall that for any smooth path $\gamma : [0, 1] \to M$ in a manifold M and any collection $\{\omega_1, \ldots, \omega_k\}$ of 1-forms on M, the iterated integral $\int_\gamma \omega_1 \circ \cdots \circ \omega_k$ can be defined as follows. γ induces a map $\gamma^{[k]} : \Delta^k \to M^k$, where $\Delta^k = \{(t_1, \ldots, t_k) : 0 \leq t_1 \leq \cdots \leq t_k \leq 1\}$ is the

k-dimensional simplex, via the rule:

$$\gamma^{[k]}(t_1,\ldots,t_k) := (\gamma(t_1),\ldots,\gamma(t_k))\,.$$

Then the desired iterated integral is defined as:

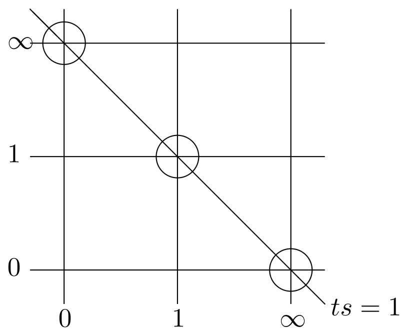
$$\int_\gamma \omega_1\circ\cdots\circ\omega_k := \int_{0<t_1<\cdots<t_k<1}\gamma^{[k]*}(\omega_1\wedge\cdots\wedge\omega_k).$$

Now, the reason that one gets (cyclotomic) multiple zeta values in Proposition 4.1, is the iterated integral interpretation of (cyclotomic) multiple zeta values.

In order to use this relation in giving a geometric interpretation for the pentagon relation, we are going to consider the space $M_{0,5}$. Recall that $M_{0,5}$ parametrizes the configurations of 5 distinct marked points on $\mathbb{P}^1$. After identifying 3 of these 5 points with 0, 1, and ∞ (using the GL_2 action on $\mathbb{P}^1$), $M_{0,5}$ can be identified with the space:

$$M_{0,5} = \{(0,1,x,y,\infty) : x,y\in\mathbb{P}^1\setminus\{0,1,\infty\}, x\neq y\}.$$

If we use the variables $t = xy^{-1}$ and $s = y$, $M_{0,5}$ can be visualized geometrically as follows: Let $\bar{M}_{0,5}$ denote the blow-up of $\mathbb{P}^1\times\mathbb{P}^1$ at points $(0,\infty)$, $(1,1)$, and $(\infty,0)$ (this is the Deligne-Mumford compactification of $M_{0,5}$). Then $M_{0,5} = \bar{M}_{0,5}\setminus D$, where the divisor at infinity D is the union of the lines $t = 0,1,\infty$, $s = 0,1,\infty$, $ts = 1$, and the three exceptional divisors resulted by the blow-ups (see the following figure).

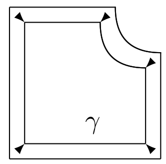


Recall from Example 3.5 that we have the following set of generators and relations for the Lie algebra of the de Rham unipotent fundamental group of $M_{0,5}$:

$$Lie\left(\pi_1^{dR}(M_{0,5})\right) = \langle\{W_{ij}\}_{1\le i,j\le 5}\ :$$

$$W_{ij} = W_{ji}, W_{ii} = 0, \sum_{j=1}^{5} W_{ij} = 0, [W_{ij},W_{kl}] = 0 \text{ if } \{i,j\}\cap\{k,l\} = \emptyset\rangle.$$

Also, note that the symmetric group on 5 letters S_5 acts canonically on $M_{0,5}$, by permuting the marked points, and the induced action on the above Lie algebra is via the action of S_5 on the indices of $W_{i,j}$'s. Now let us amplify the lower left corner of the above picture of $M_{0,5}$, which indeed looks like a pentagon.



Since the lower edge of this pentagon is the straight path γ (see Remark 4.5) and all the other edges are obtained from the lower edge after applying a suitable element of S_5, the contractibility of this pentagon in $M_{0,5}$ implies the relation:

$$f_\gamma(W_{12}, W_{23})f_\gamma(W_{34}, W_{45})f_\gamma(W_{51}, W_{12})f_\gamma(W_{23}, W_{34})f_\gamma(W_{45}, W_{51}) = 1. \tag{10}$$

Together with Proposition 4.1, the above relation implies the pentagon relation among multiple zeta values.

Remark 4.4. Note that the identities (10) and and (2) are equivalent after a suitable change of coordinates (see [11, Lemma 5]).

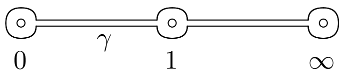
Remark 4.5. Note that the edges of the above pentagon are parts of the divisor at infinity D in $\bar{\mathcal{M}}_{0,5}$ and do not belong to $\mathcal{M}_{0,5}$. But each irreducible component of D is isomorphic to $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ and its normal bundle admits a nowhere vanishing section. Therefore, if $\mathcal{N}^*$ denotes the normal bundle of an irreducible component of D with the zero section removed, $\mathcal{N}^*$ contains a copy of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ and hence there is a map α from the unipotent fundamental groupoid of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ to the unipotent fundamental groupoid of $\mathcal{N}^*$. On the other hand, using the theory of base points at infinity for unipotent fundamental groupoids, there is a map β from unipotent fundamental groupoid of $\mathcal{N}^*$ to the unipotent fundamental groupoid of $\mathcal{M}_{0,5}$. Now, when we say that the lower edge of the above pentagon is the straight path γ, we mean that it is the image of the straight path γ under $\beta \circ \alpha$ from the unipotent fundamental groupoid of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ to the one of $\mathcal{M}_{0,5}$.

Remark 4.6. In the fundamental paper [12], Furusho shows that the identity (10) implies all shuffle relations which express the product of two multiple zeta values as a linear sum of multiple zeta values (see [12, Theorem 1.1]). For example, relations of the form:

$$\zeta(p)\zeta(q) = \zeta(p, q) + \zeta(q, p) + \zeta(p + q).$$

It is also shown in [11] that the hexagon relations are consequences of the pentagon relation. We would like to mention at this point that, in view of Grothendieck's general yoga explained in "Esquisse d'un Programme" ([13]), it is not surprising that "all" relations among multiple zeta values can be explained via geometry of $M_{0,5}$. Indeed, it is expected in Grothendieck's program that the whole Grothendieck-Teichmüller tower in genus 0 is generated by the fundamental groupoid of $M_{0,4}$ and that all the relations in this tower come from the fundamental groupoid of $M_{0,5}$. On the other hand, multiple zeta values are periods of the fundamental group of $M_{0,4}$ and so the central role of $M_{0,5}$ in the theory of multiple zeta values is perfectly compatible with Grothendieck-Teichmüller theory.

Remark 4.7. One can also study the symmetries of the space $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ together with Proposition 4.1 to get geometric interpretation for other relations among multiple zeta values. For example consider the following contractible path in $\mathbb{P}^1 \setminus \{0, 1, \infty\}$:



where the lower left straight segment is the straight path γ and all the other straight segments are obtained from it by the S_3 action (here S_3 acts on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ by permuting the points 0, 1, and ∞). Then the contractibility of the above path in $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, using the Betti-de Rham comparison isomorphism, is equivalent to an identity involving $f_\gamma(W_0, W_1)$ in $\mathbb{C}\langle\langle W_0, W_1\rangle\rangle$, which, according to Proposition 4.1, has a potential of producing identities among multiple zeta values. Actually, one can show that (see [4, Section 18] for example) this way one recovers identities of the form:

$$\zeta(2n) = (-1)^{n-1} \frac{B_{2n}}{2.(2n)!} (2\pi)^{2n}, \tag{11}$$

which were first discovered by Euler (here B_* are the Bernoulli numbers). Note that the above identities are identities involving the values of the zeta function at even integers and powers of the number π^2. We have seen that values of the zeta function appear as periods of the straight path γ. On the other hand, note that the number $2\pi i$ is the period of a circle going around zero in $\mathbb{G}_m$ and all this goes perfectly well along with the geometry (note the small circles going around 0, 1, and ∞ in the above contractible path).

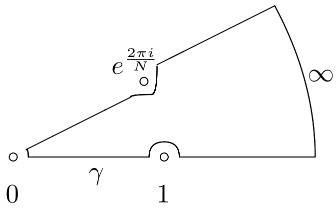
Remark 4.8. In Remark 4.2 we saw that if one considers the straight path γ in $X_N = \mathbb{P}^1 \setminus \{0, \infty, \mu_N\}$ then one recovers the cyclotomic multiple zeta values as coefficients of the power series in $\mathbb{C}\langle\langle W_0, \{W_\xi\}_{\xi\in\mu_N}\rangle\rangle$ associated to γ. Evidently, a natural expectation would be that symmetries of X_N should lead to relations among cyclotomic multiple zeta values. In this remark, as a continuation to Remark 4.2, we sketch a geometric structure that accounts for the analogue of the pentagon relation for cyclotomic multiple zeta values.

First of all, note that X_N can be realized as a degree N covering of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ given by the map $z \mapsto z^N$. On the other hand, recall that $M_{0,5}$ is isomorphic to the complement of 10 lines in the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at 3 points; the lines $t = 0, 1, \infty$, $s = 0, 1, \infty$, $ts = 1$, and the three exceptional divisors (here t and s are the coordinates of the $\mathbb{P}^1$ as the first and second axis respectively). Each of these 10 lines at infinity is isomorphic to a copy of $\mathbb{P}^1$ which intersects the other lines in 3 different points. In particular, the projections p_t and p_s from $\bar{M}_{0,5}$ to the first and second coordinate induce projections from $M_{0,5}$ to $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ (this projections are nothing but the projections induced from $M_{0,5}$ to $M_{0,4}$ induced by forgetting one of the marked points not in $\{0, 1, \infty\}$). Now let $\tilde{M}_{0,5}$ be the degree N^2 covering of $M_{0,5}$ induced by joining the N-th roots of t and s to the coordinate ring, and consider the following diagram:

$$\begin{array}{ccc} \tilde{M}_{0,5} & \longrightarrow & M_{0,5} \\ \big\downarrow{\scriptstyle \tilde{p}} & & \big\downarrow{\scriptstyle p} \\ X_N & \xrightarrow{z\mapsto z^N} & \mathbb{P}^1 \setminus \{0, 1, \infty\} \end{array}$$

where p stands for either of the projections p_t or p_s (p extends to a projection $\tilde{p}$ because the pullback of the coordinate function of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ via p is either t or s and both t and s admit N-th root in the coordinate ring of $\tilde{M}_{0,5}$).

Now this geometric objects have symmetries and the contractible paths in $M_{0,5}$ and $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ we used to obtain the pentagon relation (10) and Euler's identities (11) lift to the coverings $\tilde{M}_{0,5}$ and X_N. For example, the contractible path in $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ of the previous remark lifts to the following contractible path in X_N:



and the contractible pentagon in $M_{0,5}$ that we used in obtaining the pentagon relation (10) lifts to a contractible path in $\tilde{M}_{0,5}$. Contractibility of these lifted paths together with the fact that different segments in these paths are related to the straight path γ by the symmetries of the ambient spaces gives rise to relations among cyclotomic multiple zeta values analogue to the pentagon relation and Euler's identities (see also [8]).

Acknowledgments

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