Lectures on Étale Fundamental Groups

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Contents

Chapter 1

Étale Fundamental Groups

1.1 General Philosophy

In order to motivate the theory of étale fundamental groups, let us start by a brief recall of the theory of classical fundamental groups and the analogies between this theory and Galois theory.

Recall that, in topology, for a connected pointed space (X, b) the fundamental group $\pi_1(X, b)$ of X with base point b is defined to be the group consisting of homotopy classes of continuous pointed maps from $(S^1, *)$ to (X, b), where the group operation is induced by concatenation of loops. On the other hand, a covering of the space X is defined to be a continuous map $X' \xrightarrow{\pi} X$ such that for every point $x \in X$ there exists an open neighborhood U of x with the property that:

$$\pi^{-1}(U) = \coprod U_i$$

with U_i being π-homeomorphic to U for all i.

Now, for a "reasonable" space X, there is a very nice connection between coverings and the fundamental group of X. Namely, there is an equivalence between categories $Cov(X)$ of coverings of X and $\pi_1(X, b)$-sets. First, observe that for any covering $X' \xrightarrow{\pi} X$ of X, $\pi_1(X, b)$ acts on the set $\pi^{-1}(b)$ as follows. For any point $i \in \pi^{-1}(b)$ and any loop γ in X based at b, one can lift γ to a path γ' in X' with the initial point i and define i^γ to be the end point of γ'. This defines an action of $\pi_1(X, b)$ on $\pi^{-1}(b)$ and a functor from $Cov(X)$ to $\pi_1(X, b)$-sets. In order to define the reverse functor, we need the notion of the universal covering. Roughly speaking, the universal covering $\tilde{X}$ of (X, b) is the space of homotopy classes of paths in X with initial point b where the homotopies are relative to the end points of paths. There is a canonical map $\tilde{X} \to X$ which sends a path $\gamma : [0, 1] \to X$ to $\gamma(1) \in X$. It can be checked that $\tilde{X}$ is a simply connected covering of X which admits a "nice" action by $\pi_1(X, b)$. Now let S be a $\pi_1(X, b)$-set and let $S = \coprod_\alpha S_\alpha$ be the decomposition of S into its orbits. For any α, the choice of an element in S_α gives rise to a stabilizer subgroup $H_\alpha \subset \pi_1(X, x)$ such that $S_\alpha \sim_{\pi_1} \pi_1/H_\alpha$. Then the quotient space $\tilde{X}/H_\alpha$ forms a covering of X and $X' = \coprod_\alpha \tilde{X}/H_\alpha$ is the covering space associated to the $\pi_1(X,b)$-set S. It can be checked that the above functors establish the desired equivalence between $Cov(X)$ and $\pi_1(X,b)$-sets (note that under this equivalence, homogenous $\pi_1(X,b)$-sets correspond to connected covers). Therefore, if one wants to describe the fundamental group $\pi_1(X,b)$ in terms of coverings, the question to consider is whether the group $\pi_1(X,b)$ can be recovered from the category $\pi_1(X,b)$-sets. The following result gives an affirmative answer to this question.

Theorem 1.1.1. *Let G be a group and $\mathcal{F}$ be the forgetful functor from the category of G-sets to the category of sets. Then $G \cong Aut(\mathcal{F})$, where by an automorphism α of $\mathcal{F}$ we mean a family of automorphisms $\alpha_S : \mathcal{F}(S) \to \mathcal{F}(S)$ such that for any morphism $w : S_1 \to S_2$ of G-sets, the following diagram commutes*

$$\begin{array}{ccc} \mathcal{F}(S_1) & \xrightarrow{\mathcal{F}(w)} & \mathcal{F}(S_2) \\ \downarrow{\scriptstyle \alpha_{S_1}} & & \downarrow{\scriptstyle \alpha_{S_2}} \\ \mathcal{F}(S_1) & \xrightarrow{\mathcal{F}(w)} & \mathcal{F}(S_2). \end{array}$$

Proof. First lets define a homomorphism $\phi : G \to Aut(\mathcal{F})$. To do this, note that for any $g \in G$ and any G-set S the action of g on the underlying set $\mathcal{F}(S)$ of S defines a functorial automorphism of $\mathcal{F}(S)$, which we consider as $\phi(g) \in Aut(\mathcal{F})$. Obviously this gives a group homomorphism from G to $Aut(\mathcal{F})$, which can be checked to be injective by considering the paticular G-set G with left multiplication. To prove the surjectivity take an arbitrary element $\alpha \in Aut(\mathcal{F})$. Then α_G acts on the underlying set $\mathcal{F}(G)$ of G. Let $g = \alpha_G(1)$. Since α_G must commute with all automorphisms of G as a G-set with left multiplication, and one can show that this automorphism group is G^{op} which acts on G by right multiplication, then it is easy to show that α_G must be multiplication by g from left. For any subgroup H of G, using the surjective G-morphism from G to G/H, one can show again that $\alpha_{G/H}$ must also coinside with left multiplication by g. Finally for a general G-set X one can decompose it to its orbits and use the previous results to show that α_X acts the same on $\mathcal{F}(X)$ as g does. This shows that $\alpha = \phi(g)$ and hence ϕ is surjective. □

Remark 1.1.2. *We would like to mention that the above theorem has very interesting variants. Namely, if one considers an specific subcategory of the category of G-sets, instead of the full category, the automorphism group of the forgetful functor gives interesting invariants associated to G. For example, by considering the category of finite dimensional G-representations or the category of finite sets on which G acts, one obtains the algebraic hull and the profinite completion of the group G respectively.*

In algebraic geometry it is not straightforward to define the analogues of path and homotopy, but there is a good notion of a covering space, namely étale coverings. The purpose of this course is to use this and the above ideas to define algebraic fundamental groups in the context of algebraic geometry. As a warm up, let us consider the Galois situation.

Let k be a field and k^{sep} be a separable closure of k (in analogy with topological situation, if we think of k as a "space", the choice of a separable closure k^{sep} is like choosing a base point). Then the absolute Galois group $Gal(k)$ of k is defined to be the automorphism group of the extension k^{sep}/k. Now, for any finite separable extension k' of k (separable extensions should be thought of as connected coverings), the group $Gal(k)$ acts on "the fibre of k' above the base point k^{sep}". More precisely, if we reinterpret k' as $k[x]/P(x)$, with $P(x)$ an irreducible polynomial, then one has:

$$k^{sep} \otimes_k k' \cong k^{sep} \otimes_k k[x]/P(x) \cong \prod_{\alpha} k_{\alpha}^{sep},$$

where α runs through roots of $P(x)$. Now, $Gal(k)$ acts on $k^{sep} \otimes_k k'$ via acting on roots of $P(x)$. Conversely, for every finite homogenous $Gal(k)$-set S with an stabilizer subgroup H, the fixed field $k' = Fix(H)$ in k^{sep} is a finite separable extension of k that we associate to S. This establishes an equivalence between finite separable extensions of k and finite homogenous $Gal(k)$-sets. Therefore, $Gal(k)$ can be thought of as a profinite fundamental group of k with finite separable extensions play the role of finite connected covering spaces. Our first goal in this course is to generalize these constructions when k is replaced by an arbitrary scheme.

1.2 Étale Maps

Definition 1.2.1. *A morphism $A \to B$ between commutative rings with unit is called formally smooth (resp., formally unramified, resp., formally etale), provided that for each commutative diagram*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & \swarrow & \downarrow \\ C & \longrightarrow & C/I \end{array} \tag{1.1}$$

with $I^2 = 0$ there exists at least one lift (resp., at most one lift, resp., exactly one lift) from B to C which makes the above diagram commutative. If in addition $A \to B$ is of finite presentation, i.e. $B = A[T_1, \ldots, T_m]/(f_1, \ldots, f_n)$, then we drop the adjective formally.

Lemma 1.2.2. *A map $A \to B$ is formally unramified if and only if $\Omega_{B/A} = 0$.*

Proof. First assume that $\Omega_{B/A} = 0$. By contradiction suppose that there are two lifts $\phi_1, \phi_2 : B \to C$ which make diagram (1.1) commutative. Consider the map $d := \phi_1 - \phi_2 : B \to C$. Since ϕ_1 and ϕ_2 become the same after composing with the projection $C \to C/I$, the image of d lies in I, and if we endow I with a B-module structure using either ϕ_1 or ϕ_2 one can check that $d : B \to I$ is an A-derivation. Hence d should factor through $\Omega_{B/A}$ which is 0, and hence $\phi_1 = \phi_2$.

Now assume that $A \to B$ is formally unramified. Put $C := B \oplus \Omega_{B/A}$ and consider it as a ring with multiplication $(b_1, w_1).(b_2, w_2) = (b_1 b_2, b_1 w_2 + b_2 w_1)$. Then if we consider the ideal $I := \Omega_{B/A}$, one has $I^2 = 0$, $C/I \cong B$, and the following diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \cong \\ C & \longrightarrow & C/I \end{array}$$

We have at least two lifts $\phi_1, \phi_2 : B \to C$, namely

$$\phi_1(b) = (b, 0)\ , \quad \phi_2(b) = (b, d_{B/A}(b))$$

By assumption ϕ_1 is equal to ϕ_2, hence $d_{B/A} : B \to \Omega_{B/A}$ is zero. We are done since $d_{B/A}$ is known to be surjective. □

Now assume that $A \to B = A[T_1, \ldots, T_m]/(f_1, \ldots, f_n)$ is a morphism of finite presentation and put $C = A[T_1, \ldots, T_m]$ and $I = (f_1, \ldots, f_n)$. Then we have the following criterion for smoothness

Lemma 1.2.3. (Jacobian criterion for smoothness) *With the above notations we have the following exact sequence*

$$I/I^2 \xrightarrow{\alpha} \Omega_{C/A} \otimes_C B \cong \oplus_{i=1}^m B.dT_i \to \Omega_{B/A} \to 0.$$

Then $A \to B$ is smooth if and only if α has a left inverse (i.e. there exists a $\beta : \Omega_{C/A} \otimes_C B \to I/I^2$ such that $\beta \circ \alpha = id_{I/I^2}$).

Proof. ($\Rightarrow$) Since $A \to B = C/I$ is assumed to be smooth, there exists at least one lift $w : C/I \to C/I^2$ which makes the following diagram commutative

$$\begin{array}{ccc} A & \longrightarrow & B = C/I \\ \downarrow & \overset{w}{\swarrow} & \downarrow = \\ C/I^2 & \longrightarrow & C/I \end{array}$$

Now we have two homomorphisms from C to C/I^2, one is the natural projection, and the other one is the composition $C \to C/I \xrightarrow{w} C/I^2$. The difference of these two homomorphisms gives an A-derivation from $C \to I/I^2$ which corresponds to a C-linear map from $\Omega_{C/A}$ to I/I^2. Since I/I^2 has a B-module structure, we can extend the scalars and obtain $\beta : \Omega_{C/A} \otimes_C B \to I/I^2$. Now we must show that $\beta \circ \alpha = id_{I/I^2}$, but this is easy to check simply because the composition of w with natural projection from $C \to C/I$ sends I to zero, and the natural projection $C \to C/I^2$ restricted to I/I^2 is identity.

($\Leftarrow$) Using the existence of β we show the smoothness of $A \to B = C/I$. For this consider as usual a diagram of the form

$$\begin{array}{ccc} A & \longrightarrow & B = C/I \\ \downarrow & & \downarrow \\ D & \longrightarrow & D/J \end{array}$$

with $J^2 = 0$. $A \to C$ is obviously smooth and thus there exists a lift $\phi : C \to D$ (simply lift the images of T_i's). Now we use β to correct it so that it vanishes on I and gives a lift $C/I \to D$. Since $\phi(I) \subset J$, ϕ induces a linear map $I/I^2 \to J/J^2$. Composing it with β we obtain a linear map $\Omega_{C/A} \otimes_C B \to J/J^2$ which corresponds to a derivation from $C \to J/J^2$. If we subtract this derivation from ϕ, we obtain the desired lift. □

Combining the above two lemmas we obtain the following important

Corollary 1.2.4. *Using the above notations, $A \to B = C/I$ is etale if and only if $I/I^2 \cong \Omega_{C/A} \otimes_C B$.*

As an application of the above results we show that notions of smooth, unramified, and etale have good behavior under filtered inductive limits. Assume $A = \varinjlim A_\alpha$ (e.g. $A = \cup$ finitely generated $\mathbb{Z}$-algebras). Then any finitely presented A-algebra B is of the form $B = B_\alpha \otimes_{A_\alpha} A$ for some finitely presented A_α-algebra B_α. For each $\beta \geq \alpha$ put $B_\beta := B_\alpha \otimes_{A_\alpha} A_\beta$. Then one has the following

Lemma 1.2.5. *B/A is smooth (resp. unramified, resp. etale) if and only if B_β/A_β is smooth (resp. unramified, resp. etale) for big enough β.*

Proof. First note that since being smooth (resp. unramified, resp. etale) is obviously preserved under extention of scalars, ($\Leftarrow$) is trivial. Hence suffices to prove the ($\Rightarrow$) part. Using same notations as before one has $B_\alpha = A_\alpha[T_1, \ldots, T_m]/(f_1, \ldots, f_n)$, and so $B_\beta = A_\beta[T_1, \ldots, T_m]/(f_1, \ldots, f_n)$. First lets do the unramified case. Consider the following linear map

$$\phi = \left(\frac{\partial f_i}{\partial T_j}\right) : B^n \to B^m$$

then one knows that

$$\Omega_{B/A} = \left(\bigoplus_{i=1}^{m} B dT_i\right) / Im(\phi)$$

This means that B being unramified over A is the same as surjectivity of ϕ which in turn is the same as the existence of vectors $\vec{b_i} = (b_{i,\mu}) \in B^n$ mapping to dT_i for all i. These vectors lie in some B_β^n, $\beta \geq \alpha$, hence the images of $\vec{b_i}$'s also lie in B_β^m and map to dT_i in B^m. Now the diferences of the images of $\vec{b_i}$'s and dT_i's can be written as a combination of finitely many polynomials, which involves finitely many coefficients and hence lies in B_γ for some $\gamma \geq \beta$. Now the map ϕ restricted to B_γ is surjective and hence B_γ/A_γ is unramified as well.

Now suppose that B/A is smooth. With notations of Lemma 1.2.3, this means that there exists a section $\beta : \Omega_{C/A} \otimes_C B = \oplus B dT_i \to I/I^2$. Giving the section β is the same as giving $\beta(dT_i) = \sum_j b_{i,j} f_j \in I/I^2$ for all i. Again one can find a $\mu \geq \alpha$ such that all $b_{i,j}$ lie in B_μ. We have a map β over B_γ for all $\gamma \geq \mu$. To check that β is the desired section, it is enough to check that $\beta(\partial f_i/\partial T_i) = f_i$ in I_γ/I_γ^2 for all i. Known that this is true in I/I^2, differences lie in I^2 and again involve only finitely many coefficients, and hence everything is fine for big enough γ.

The etale case is an immediate consequence of the above cases. □

To see some more applications of what we have done so far, let $A \to B$ be of finite presentation and denote by I the kernel of the multiplication map $B \otimes_A B \to B$. Note that this map corresponds to the diagonal map $\Delta : Spec(B) \to Spec(B) \times_{Spec(A)} Spec(B)$, which is a closed immersion since we have affine schemes. The other important thing to mention is that the ideal I is generated by the elements of the form $b \otimes 1 - 1 \otimes b$ when b runs over a generating set of B as an A-algebra, and hence is finitely generated in this case. Now we have

Claim 1.2.6. *With above assumptions, $A \to B$ is unramified if and only if the diagonal map Δ is also an open immersion.*

Proof. Since we have $\Omega_{B/A} = I/I^2$, by Lemma 1.2.2, $A \to B$ is unramified if and only if $I = I^2$. Now for ($\Rightarrow$), first we show that the ideal I is generated by an idempotent element. Since $I = I^2$ and I is finitely generated, there exists an element $f \in 1 + I$ such that $f.I = 0$. Then $e := 1 - f \in I$ and $f.e = (1 - e).e = 0$ and hence e is an idempotent. Now put $C := B \otimes_A B$ and write $C = e.C \oplus (1 - e).C$. Note that $e.C \subset I$ and $(1 - e).I = 0$, we deduce that $I = e.C$. Now $Im(\Delta)$ is the same as $Spec(C/I)$, which in turn is the same as $Spec((1 - e).C)$ and hence is both open and close in $Spec(C)$.

The other direction is easy, because when $Im(\Delta) = Spec(C/I)$ is both open and close, one can decompose $C = I \oplus (C/I)$, but then obviously I is generated by an idempotent and hence satisfies $I = I^2$. □

One other important application is the fact that smoothness implies flatness. So we formulate this fact as

Claim 1.2.7. *Suppose that $A \to B$ is a smooth map, then B is a flat A-algebra.*

Proof. We separate the proof into parts:

Step 1) First note that by Lemma 1.2.5, using the same notations, we can assume that A and B are Noetherian or even finitely generated $\mathbb{Z}$-algebras. Because if we assume B is smooth over A, B_α will be smooth and hence flat over

A_α, and B will be flat over A since flatness is preserved by extension of scalars.

Step 2) Let $\mathfrak{q}$ be a prime ideal in B and $\mathfrak{p} \subset A$ be the preimage of $\mathfrak{q}$. It follows from definition that when B is smooth over A, $B_\mathfrak{q}$ is also smooth over $A_\mathfrak{p}$. Since flatness is a local property, we are reduced to prove the claim when A and B are local Noetherian rings and $A \to B$ is a local morphism.

Step 3) If we assume that $B = C/I$ where $C = A[T_1, \ldots, T_m]$, then the prime ideal $\mathfrak{q}$ corresponds to a prime ideal in C, which we denote by the same symbol $\mathfrak{q}$, and one has $B_\mathfrak{q} \cong C_\mathfrak{q}/I_\mathfrak{q}$. By Jacobian criterion $B_\mathfrak{q}$ is smooth over $A_\mathfrak{p}$ if and only if $I_\mathfrak{q}/I_\mathfrak{q}^2 \to \Omega_{C_\mathfrak{q}/A_\mathfrak{p}} \otimes_{C_\mathfrak{q}} B_\mathfrak{q}$ has a left inverse. Note that we have $\Omega_{C/A} \otimes_C - = \Omega_{C_\mathfrak{q}/A} \otimes_{C_\mathfrak{q}} -$ (universal property of Käler differentials), and since localizations are formally etale (follows from definition), we have $\Omega_{A_\mathfrak{p}/A} = 0$. Then by considering the exact sequence $\Omega_{A_\mathfrak{p}/A} \otimes_{A_\mathfrak{p}} C_\mathfrak{q} \to \Omega_{C_\mathfrak{q}/A} \to \Omega_{C_\mathfrak{q}/A_\mathfrak{p}} \to 0$, which comes from $A \to A_\mathfrak{p} \to C_\mathfrak{q}$, one obtaines that $\Omega_{C_\mathfrak{q}/A} \cong \Omega_{C_\mathfrak{q}/A_\mathfrak{p}}$. So all this shows that $B_\mathfrak{q}$ is smooth over $A_\mathfrak{p}$ if and only if $I_\mathfrak{q}/I_\mathfrak{q}^2$ is a free $B_\mathfrak{q}$-module (note that projective modules over local rings are free) which is a direct summand of $\Omega_{C/A} \otimes_C B_\mathfrak{q}$, and this in turn is equivalent to the existence of polynomials $f_{j_1}, \ldots, f_{j_r}$ which generate $I_\mathfrak{q}/I_\mathfrak{q}^2$, and variables $T_{i_1}, \ldots, T_{i_r}$ such that $\left(\partial f_{j_\mu}/\partial T_{i_\nu}\right)_{r \times r}$ is invertible.

Step 4) Now since $f_{j_1}, \ldots, f_{j_r}$ generate $I_\mathfrak{q}/I_\mathfrak{q}^2$ over $C_\mathfrak{q}$ and $I \subset \mathfrak{q}$, they also generate $I_\mathfrak{q}/\mathfrak{q}I_\mathfrak{q}$, and by Nakayama's lemma, $I_\mathfrak{q}$ itself over $C_\mathfrak{q}$. So far we have shown that when B is smooth over A, locally at a prime ideal $\mathfrak{q}$ we have

$$B_\mathfrak{q} = \frac{(A_\mathfrak{p}[T_1, \ldots, T_m])_\mathfrak{q}}{(f_1, \ldots, f_r)}, \text{ with } \left(\frac{\partial f_i}{\partial T_j}\right)_{1 \leq i,j \leq r} \text{ invertible.}$$

and hence it suffices to prove that such an $A_\mathfrak{p}$-algebra is flat.

Step 5) In this step we assume that $A_\mathfrak{p} = k$ is a field. Then flatness is obvious, but we prove more. We actually prove that $f_1, \ldots, f_n$ form a regular system of parameters and hence the quotient $B_\mathfrak{q}$ is a regular local ring. Consider the following diagram, whose first exact row comes from the surjection $C_\mathfrak{q} \twoheadrightarrow C_\mathfrak{q}/\mathfrak{q}_\mathfrak{q}$

$$\begin{array}{ccccccc}
(\mathfrak{q}/\mathfrak{q}^2)_\mathfrak{q} & \longrightarrow & \Omega_{C_\mathfrak{q}/k} \otimes_{C_\mathfrak{q}} C_\mathfrak{q}/\mathfrak{q}_\mathfrak{q} & \longrightarrow & \Omega_{(C_\mathfrak{q}/\mathfrak{q}_\mathfrak{q})/k} & \longrightarrow & 0 \\
\uparrow \psi & \nearrow \phi & & & & & \\
(f_i) = I_\mathfrak{q}/\mathfrak{q}I_\mathfrak{q} & & & & & &
\end{array}$$

Since by assumption the Jacobian matrix of f_i's has full rank, the map ϕ and hence the map ψ in above diagram are injections. The injectivity of ψ is equivalent to the fact that f_i form a regualr sequence and hence the quotient $B_\mathfrak{q}$ is a regular local ring (See [2, Theorem 14.2.]).

Step 6) In order to proceed to the general setting, we need to prove the following

Lemma 1.2.8. *Let $(A, m) \to (B, n)$ be a local homomorphism of Noetherian local rings, $k = A/m$, and M be a finitely generated B-module. Then M is flat over A if and only if $Tor_1^A(M, k) = 0$.*

Proof. ($\Rightarrow$) is clear.

For ($\Leftarrow$) one must show that for all A-modules N, $Tor_1^A(M, N) = 0$. First note that for any short exact sequence $0 \to N_1 \to N \to N_2 \to 0$, if the assertion is true for N_1 and N_2, then it is also true for N. On the other hand since Tor commutes with direct limits, it is enough to prove the assertion for finitely generated A-modules. By induction on the number of generators, one reduces to the cyclic case, i.e. $N = A/\mathfrak{a}$ for an ideal $\mathfrak{a}$ in A. Assume the contrary and choose a maximal ideal $\mathfrak{a}$ for which the assertion fails. For any $x \notin \mathfrak{a}$ consider the following short exact sequence

$$0 \to A/(\mathfrak{a} : x) \xrightarrow{.x} A/\mathfrak{a} \to A/(\mathfrak{a} + xA) \to 0$$

Since $\mathfrak{a}$ was a maximal ideal for which assertion fails, we obtain that for all $x \notin \mathfrak{a}$, $(\mathfrak{a} : x) = \mathfrak{a}$, which means that $\mathfrak{a}$ is a prime ideal. Choose an element $x \in m \setminus \mathfrak{a}$ and consider the exact sequence $0 \to A/\mathfrak{a} \xrightarrow{.x} A/\mathfrak{a} \to A/(\mathfrak{a} + xA) \to 0$ and the following part of the associated long exact sequence

$$Tor_1^A(M, A/\mathfrak{a}) \xrightarrow{.x} Tor_1^A(M, A/\mathfrak{a}) \to Tor_1^A(M, A/(\mathfrak{a} + xA)) = 0$$

Now since these Tor groups are finitely generated B-modules, x is in m, and multiplication by x is surjective, by Nakayama's lemma one deduces that $Tor_1^A(M, A/\mathfrak{a}) = 0$. $\square$

Step7) So far we have reduced the problem to showing that $Tor_1^{A_\mathfrak{p}}(C_\mathfrak{q}/I_\mathfrak{q}, A_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}) = 0$. Consider the surjection $C_\mathfrak{q}^n \xrightarrow{e_i \to f_i} I_\mathfrak{q} \to 0$, and let R be the submodule in $C_\mathfrak{q}^n$ which is generated by trivial relations (i.e. for all $i < j$ the vector $f_j.e_i - f_i.e_j$). Now consider the exact sequence $0 \to I_\mathfrak{q} \to C_\mathfrak{q} \to C_\mathfrak{q}/I_\mathfrak{q} \to 0$, and the associated long Tor-exact sequence, which looks like:

$$Tor_1^{A_\mathfrak{p}}(C_\mathfrak{q}/I_\mathfrak{q}, \kappa(\mathfrak{p})) = Ker\{I_\mathfrak{q} \otimes_{A_\mathfrak{p}} \kappa(\mathfrak{p}) \to C_\mathfrak{q} \otimes_{A_\mathfrak{p}} \kappa(\mathfrak{p})\}$$

We already know that in $C_\mathfrak{q} \otimes \kappa(\mathfrak{p})$, f_i form a regular sequence and hence we have the exact sequence $(C_\mathfrak{q} \otimes \kappa(\mathfrak{p}))^{\frac{n(n-1)}{2}} \to (C_\mathfrak{q} \otimes \kappa(\mathfrak{p}))^n \xrightarrow{e_i \to f_i} C_\mathfrak{q} \otimes \kappa(\mathfrak{p})$, which fits to the following commutative diagram

$$\begin{array}{ccccc}
(C_\mathfrak{q} \otimes \kappa(\mathfrak{p}))^{(\frac{n(n-1)}{2})} & \longrightarrow & (C_\mathfrak{q} \otimes \kappa(\mathfrak{p}))^n & \xrightarrow{e_i \to f_i} & C_\mathfrak{q} \otimes \kappa(\mathfrak{p}) \\
\downarrow \psi & & \downarrow \phi: e_i \to f_i & & \downarrow = \\
Tor_1^{A_\mathfrak{p}}(C_\mathfrak{q}/I_\mathfrak{q}, \kappa(\mathfrak{p})) & \longrightarrow & I_\mathfrak{q} \otimes \kappa(\mathfrak{p}) & \longrightarrow & C_\mathfrak{q} \otimes \kappa(\mathfrak{p})
\end{array}$$

Now since the map ϕ is surjective, a simple diagram chassing shows that the map ψ is surjective as well. As a result the Tor group under study is generated by trivial relations, hence is zero (Note that ϕ is zero on trivial relations). This finally completes the proof of our claim and also shows that f_i form a regular sequence. □

Note that in the last step of the above proof, we used the following lemma from commutative algebra. The proof can be done by a simple induction argument, or from the Koszul complex.

Lemma 1.2.9. *Let A be a commutative ring with unit. A sequence $f_1, \dots, f_n \in A$ forms a regular sequence if and only if the kernel of the map $A^n \to A$ which sends the vector e_i to f_i is generated by trivial relations. i.e. the kernel be the ideal generated by the $\binom{n}{2}$ elements $f_j.e_i - f_i.e_j, \quad 1 \le j < i \le n$.*

We finish this section by prooving the following very important theorem which roughly says that if a morphism is etale outside a subset of codimention 2, then it is everywhere etale.

Theorem 1.2.10. (Zariski-Nagata Purity in dim. 2) *Let A be a Noetherian regular ring, and $A \subset B$ a finite, torsion-free, integral, normal A-algebra such that $B_\mathfrak{p}/A_\mathfrak{p}$ is etale for all prime ideal $\mathfrak{p}$ with $ht(\mathfrak{p}) \le 1$. Then B is an etale A-algebra.*

Proof. We may assume that A is a local ring of dimension 2. Then B, being finite over A, is a semi-local ring of dimension 2. On the other hand, since B is normal, it satisfies the condition S_2 (see the following lemma or [2, Theorem 23.8.]), and therefor $depth_A B = 2$. By a general result in commutative algebra, we know that $depth_A(B) + proj.\ dim_A(B) = 2$ (see [2, Theorem 19.1.]). So B is a projective and hence a free A-module. This allows us to define a trace $tr_{B/A} : B \to A$. Now we claim that the bilinear form $tr_{B/A}(b_1.b_2)$ is non-degenerate, i.e. its determinant is invertible. If not, the determinant must be contained in a prime ideal $\mathfrak{p} \subset A$, with $ht(\mathfrak{p}) = 1$ (Note that by Krull's principal ideal theorem, every minimal prime ideal containing an element has hight 1). So $tr_{B_\mathfrak{p}/A_\mathfrak{p}}$ degenerates too, and the same for $B \otimes \kappa(\mathfrak{p})/\kappa(\mathfrak{p})$. But we know by assumption that $B \otimes \kappa(\mathfrak{p})$ is etale over $\kappa(\mathfrak{p})$, and we show that this prevents the degeneracy of the trace. Being étale over the field $\kappa(\mathfrak{p})$, $B \otimes \kappa(\mathfrak{p})$ must be a finite product of finite seperable field extensions of $\kappa(\mathfrak{p})$, and in this case it is well known that the trace form is non-degenerate. This proves our claim and hence B/A has a non-degenerate trace form. In particular, for any $\mathfrak{p} \in Spec(A)$, $B \otimes \kappa(\mathfrak{p})$ has a non-degenerate trace form, hence is reduced (simply because nilpotent elements make the trace form to generate!), and therefore is isomorphic to a finite product of finite separable field extensions of $\kappa(\mathfrak{p})$, which is etale over $\kappa(\mathfrak{p})$. Since being unramified can be checked at the level of fibers, B is unramified over A. Finally, note that B is free and hence flat over A, and thus in order to check that B is étale over A, it suffices to check that its fibers are so. □

For the sake of completeness, let us give a proof of that part of Serre's theorem that we used in the above proof. Recall that for every natural number

$i \geq 0$, one says that a commutative ring A satisfies the condition S_i if for every prime ideal $\mathfrak{p}$ in A, $depth(A_{\mathfrak{p}}) \geq \min(ht(\mathfrak{p}), i)$.

Lemma 1.2.11. *Every normal Noetherian domain satisfies the condition S_2.*

Proof. Obviously, we may assume that A is local. First note that for an integral domain, the condition S_2 is equivalent to the property that every associated prime of a nonzero principal ideal has height one. Now suppose that $0 \neq a \in A$ be an element and that $\mathfrak{p}$ is an associated prime of (a). Then there is an element $b \in A$ such that $(a : b)_A = \mathfrak{p}$, and hence $(a : b)_{A_{\mathfrak{p}}} = \mathfrak{m}$, where $\mathfrak{m}$ denotes the maximal ideal of $A_{\mathfrak{p}}$. But this means that $ba^{-1} \in \mathfrak{m}^{-1} \setminus A_{\mathfrak{p}}$. If $ba^{-1}\mathfrak{m} \subset \mathfrak{m}$, by the determinant trick in the proof of Nakayama's lemma, ba^{-1} would be integral over $A_{\mathfrak{p}}$, which contradicts the normality assumption. Therefore, $ab^{-1}\mathfrak{m} = A_{\mathfrak{p}}$ and thus $\mathfrak{m}$ is an invertible fractional ideal. But an invertible fractional ideal over a local ring is principal, and thus $ht(\mathfrak{p}) = ht(\mathfrak{m}) = 1$. □

1.3 Definition

Let S be a connected scheme and fix a geometric point $\bar{s} \in S$, i.e. $Spec(K = \bar{K}) \xrightarrow{\bar{s}} S$. Consider the category $\mathcal{C}$ of finite etale maps $T \xrightarrow{\pi} S$ to S. It can be easily checked that the morphisms in this category are automatically finite and etale. Consider the fibre functor associated to the base point $\bar{s}$:

$$\mathcal{F} : \mathcal{C} \to \mathfrak{Sets} \ , \quad \mathcal{F}(T \xrightarrow{\pi} S) = \pi^{-1}(\bar{s})$$

We denote by $deg(T/S) = Card(\mathcal{F}(T)) = \#\{\text{lifts of } \bar{s} \text{ to } \bar{t} \in T(K)\}$.

Definition 1.3.1. *Let $T \to S$ be an object in $\mathcal{C}$. We say that T is a Galois covering of S with group G, G a finite group which is part of data, when G acts on T as a covering of S and the map $(g, t) \mapsto (g.t, t)$ gives an isomorphism $G \times T \xrightarrow{\sim} T \times_S T$. In particular if you look at the fiber above $\bar{s}$ of this isomorphism, you see that $\pi^{-1}(\bar{s})$ is a principal homogeneous space over G.*

Now we study a number of important properties of the category $\mathcal{C}$.

- Every object T in $\mathcal{C}$ admits a decomposition $T = \coprod T_i$ as a disjoint union of its connected components. Simply note that since the inclusion of a connected component is both open and closed immersion, it is both étale and finite!

- $\mathcal{C}$ admits fiber products. To see this let T_1, T_2, and T be objects of $\mathcal{C}$ and $\phi_i \in Hom_{\mathcal{C}}(T_i, T)$ for $i = 1, 2$. Since the morphisms ϕ_i are finite etale, the projections from $T_1 \times_T T_2$ to T_1 and T_2 are also finite etale, and hence $T_1 \times_T T_2$ belongs to $\mathcal{C}$

- For any object T in $\mathcal{C}$, the diagonal map $T \hookrightarrow T \times_S T$ is both open and close immersion and hence the diagonal is a union of connected components.

- The functor $\mathcal{F}$ is faithful, conservative (i.e. if $\mathcal{F}(f)$ is an isomorphism for a morphism $T \xrightarrow{f} X$ in $\mathcal{C}$, then f itself is an isomorphism), and exact (i.e. it commutes with fiber products).
- Every object T of $\mathcal{C}$ can be dominated by a Galois cover. In fact T can be dominated by an S_n-cover, where $n = deg(T/S)$. To prove this, first assume $\mathcal{F}(T) = \pi^{-1}(\bar{s}) = \{\bar{t_1}, \ldots, \bar{t_n}\}$. Consider T^n, and for each pair $i \neq j \in \{1, \ldots, n\}$ denote by Δ_{ij} the subset $\{x_i = x_j\}$ which is the preimage in T^n of the diagonal $T \hookrightarrow T \times_S T$ under (i,j)-projection. It is a union of connected components and does not contain the point $(\bar{t_1}, \ldots, \bar{t_n}) \in T^n(K)$. Consider $T' := T^n \setminus \bigcup_{i \neq j} \Delta_{ij}$, which is a union of connected components. T' admits an action of S_n, and the claim is that it is actually a Galois S_n-covering. Note that $\mathcal{F}(T') =$ all elements of $\{\bar{t_1}, \ldots, \bar{t_n}\}^n$ with no repetitions, and S_n acts on this fiber via permutations. Thus we have a morphism $S_n \times T' \to T' \times_S T'$ which becomes an isomorphism after applying the functor $\mathcal{F}$, and since $\mathcal{F}$ is conservative, $S_n \times T' \xrightarrow{\sim} T' \times_S T'$.
- Now let $X \to S$ be a connected G-covering, then one has $Aut_S(X) = G$. In order to see this, fix an element $g \in Aut_S(X)$ and an element $x_0 \in \mathcal{F}(X)$. Then $g.x_0 = x_1 \in \mathcal{F}(X)$. Now by definition there exist a unique element $g' \in G$ such that $g'.x_0 = x_1$. Hence $g = g'$ on the connected component of X containing x_0, and we are done since X is connected.

Finally we are prepared to give the definition of the etale fundamental group.

Definition 1.3.2. *With above notations, we define*

$$\pi_1(S, \bar{s}) := \varprojlim Aut_S(X)$$

where the the inverse limit is taken over all connected pointed Galois-coverings X of S, up to isomorphism. We consider them up to isomorphism in order to have a set for indexes of the projective limit.

In order to obtain information about étale fundamental groups, we need to study Galois coverings more closely. We do this in a series of lemmas.

Lemma 1.3.3. *Let $X_1 \xrightarrow{f} X_2$ be a map of connected Galois coverings, with Galois groups G_1 and G_2 respectively. Then there exists a unique surjective group homomorphism $\tilde{f} : G_1 \to G_2$ with respect to which f is equivariant.*

Proof. First note that since f is finite and étale, $f(X_1)$ is closed and open in X_2 and hence f is surjective. Now for any $x_1 \in \mathcal{F}(X_1)$ and any $g_1 \in G_1$, there exists a unique $g_2 \in G_2$ such that $f(x_1.g_1) = f(x_1).g_2$. This must hold on a union of connected components of X_1, and hence everywhere. Let $\tilde{f} : G_1 \to G_2$ be defined via the rule $\tilde{f}(g_1) = g_2$. $\tilde{f}$ is evidently a group homomorphism and it's surjectivity follows from surjectivity of f. □

Lemma 1.3.4. *Pointed connected Galois coverings form a filtered category.*

Proof. Let $(X_i, x_i) \to S$ be pointed connected Galois coverings $(i = 1, 2)$. Denote by X the connected component of $X_1 \times_S X_2$ containenig the point (x_1, x_2), and by G the stabilizer of this connected component in the action of $Aut_S(X_1 \times_S X_2) = G_1 \times G_2$. One can easily check, by compairing the degree of X over S and the order of G that X is a G-covering of S which dominates both X_1 and X_2. □

Lemma 1.3.5. *For any pointed connected Galois covering* (X, x)*, the canonical map* $\pi_1(S, \bar{s}) \to Aut(X)$ *is surjective.*

Proof. For any element $g \in Aut(X)$ and any finitely many pointed connected Galois coverings (X_i, x_i) of S, one can choose a pointed connected Galois covering (Y, y) of S which dominates X and X_i's at the same time. Since $Aut(Y) \twoheadrightarrow Aut(X)$, we can lift g to $\tilde{g} \in Aut(Y)$ and then map $\tilde{g}$ to $g_i \in Aut(X_i)$. This way, for any finite subcategory of pointed connected Galois coverings of S one obtains a compatible system of automorphisms which are compatible with $g \in Aut(X)$ as well. Now these form a filtered inverse system of non-empty closed subsets in $\prod Aut(T)$, which by Tichonoff's theorem gives rise to a lift of g in $\pi_1(S, \bar{s})$. □

Now we are ready to prove the following fundamental result, which establishes a connection between finite $\pi_1(S, \bar{s})$-sets and finite étale coverings of S. Compare this result with the classical situation where coverings of a topological space are related to sets equipped with an action of the fundamental group (see Section 1.1).

Theorem 1.3.6. *There is an equivalence between the category of finite étale coverings of* S *and the category of finite continiuous* $\pi_1(S, \bar{s})$*-sets.*

Proof. Consider a finite étale covering $X \to S$, and let $X = \coprod X_i$ be its decomposition into connected components. For any i choose a pointed Galois covering $(Y_i, y_i) \xrightarrow{f_i} X_i$ which dominates X_i. Put $G_i := Aut_S(Y_i)$ and define $H_i := \{g_i \in G_i : f_i(y_i.g_i) = f_i(y_i)\}$. H_i is a subgroup of G_i and the coset space $H_i \backslash G_i$ is in bijection with $\mathcal{F}(X_i)$ by sending $H_i.g_i$ to $f_i(y_i.g_i)$. $H_i \backslash G_i$ can be furnished with a π_1-action via the epimorphism $\pi_1 \twoheadrightarrow G_i$, and using the fact that pointed Galois coverings form a filtered category one can easily check that these π_1-sets are independent of the choice of Y_i's. This association certainly commutes with fiber products, sends connected coverings to homogeneous π_1-sets, and respects the degree. Note that for any element $x_1 \in \mathcal{F}(X_1)$ and $x_2 \in \mathcal{F}(X_2)$, there exists a π_1-map from $H_1 \backslash G_1 \to H_2 \backslash G_2$ which sends x_1 to x_2 if and only if the stabilizer of x_1 is contained in the stabilizer of x_2, and this happens if and only if the connected component of $X_1 \times_S X_2$ containing (x_1, x_2) maps isomorphically to X_1 under the first projection. Hence $Hom_{\pi_1}(\mathcal{F}(X_1), \mathcal{F}(X_2))$ as well as $Hom_S(X_1, X_2)$ is in bijection with the set of connected components of $X_1 \times_S X_2$ which are isomorphic with X_1 via first projection. This shows that the functor we have constructed from the category of finite étale coverings of S to finite continuous π_1-sets is fully faithful. To prove the equivalence, it remains to show that this functor is essentially surjective. To do this, one must show that the category of finite étale coverings of S is closed under taking quotient by group actions. More precisely we have:

Claim 1.3.7. *For any connected Galois covering Y of S with automorphism group G, and any subgroup H of G, the quotient Y/H exists in the category of finite étale coverings of S, i.e. there exists a finite étale covering Y/H of S with a faithfully flat morphism $Y \twoheadrightarrow Y/H$, such that Y is an H-torsor over Y/H, and any H-invariant map $Y \to Z$ factors uniquely through Y/H. Moreover one has $\mathcal{F}(Y/H) = G/H$.*

Proof. (*of Claim*) Y gives a finite flat coherent $\mathcal{O}_S$-algebra $\mathcal{A} := \mathcal{O}_Y$ on which G acts. The aim is to show that $\mathcal{B} := \mathcal{A}^H$ defines the finite étale covering Y/H with desired properties. Since the question is local over S, we can assume S is affine. Therefore, assume $S = Spec(A)$, $Y = Spec(B)$, where B is a finite étale faithfully flat A-algebra, G operates on B as an A-algebra, and finally $B \otimes_A B \xrightarrow{\sim} \oplus_{g \in G} B_g$. Now we are going to prove that B^H is an étale A-algebra as well, and hence $Y/H = Spec(B^H)$ plays the role of the desired quotient. Using descent theory we know that an A-algebra C is equivalent to the following data. A B-algebra C_B together with an isomorphism $(B \otimes_A B) \otimes_{i_1,B} C_B \cong (B \otimes_A B) \otimes_{i_2,B} C_B$ in such a way that over $B \otimes_A B \otimes_A B$, three possible pushforwards are compatibly isomorphic (see Section A.2). So instead of A-algebras we can work with B-algebras provided the descent datum is respected. Over B, $B \otimes_A B \cong \oplus_{g \in G} B_g$ is the trivial G-covering, and hence $(B \otimes_A B)^H = \oplus_{g \in (H \backslash G)} B_g$ is also finite étale. Descent datum is respected under taking H-invariants, hence one deduces that B^H is an étale A-algebra. □

Now consider any finite π_1-set S. By decomposing it into its orbit spaces, we can assume that it is of the form $H \backslash G$, where G is a finite quotient of π_1 and hence by definition G is the automorphism group of a pointed connected Galois covering Y of S. Obviously Y/H maps to the π_1-set S under the functor we have constructed and we are done (note that we did the proof of above Claim only in the Galois case, but that was enough to finish the proof of our Theorem, and now the theorem implies the claim in full generality). □

We finish this section by the following example which shows that Galois theory is a very particular case of this theory.

Example 1.3.8. *Consider $S = Spec(k)$, where k is a field. For the geometric point $\bar{s}$ fix an embeding $k \hookrightarrow \Omega$ of k into an algebraically closed field. Any étale k-algebra is of the form $\prod k_i$, where k_i's are finite seperable extensions of k. Hence pointed connected coverings of k are intermediate finite seperable extensions $k \hookrightarrow k_i \hookrightarrow \Omega$, and such a covering is Galois if and only if k_i/k is a Galois extension. Puting these together, one obtains that if we denote by k^{sep} the seperable algebraic closure of k in Ω, then $\pi_1(S, \bar{s}) = Aut_k(k^{sep})$ is the absolute Galois group of k.*

1.4 Functoriality

In this section we study some basic and important properties of the étale fundamental group.

Lemma 1.4.1. *Let $\bar{s}_1 \in S(\Omega_1)$ be a geometric point, and for any extension of algebraically closed fields $\Omega_1 \subset \Omega_2$ consider the induced geometric point $\bar{s}_2 : Spec(\Omega_2) \to Spec(\Omega_1) \xrightarrow{\bar{s}_1} S$. Then $\pi_1(S, \bar{s}_1) = \pi_2(S, \bar{s}_2)$.*

Proof. Simply note that for any finite étale covering $X \xrightarrow{f} S$, $f^{-1}(\bar{s}_1) = f^{-1}(\bar{s}_2)$. □

In order to prove the following important properties of π_1, we mention a general procedure. Let P_i be an arbitrary profinite group for $i = 1, 2$, and $\mathcal{C}_i$ denote the category of continuous finite P_i-sets. Any continuous homomorphism $\varphi : P_2 \to P_1$ induces an exact functor $\Phi : \mathcal{C}_1 \to \mathcal{C}_2$ which preserves cardinality. Note that if we conjugate φ by an element in P_1, the functor Φ doesn't change. The following proposition says that this correspondence is actually a bijection.

Proposition 1.4.2. *With above notations, any exact functor $\Phi : \mathcal{C}_1 \to \mathcal{C}_2$ which preserves cardinality is induced by a continiuous homomorphism $\varphi : P_2 \to P_1$ which is determined uniquely upto conjugation by elements in P_1.*

Proof. For any finite quotient G of P_1, the canonical surjection $P_1 \twoheadrightarrow G$ makes the underlying set of G into a continuous P_1-set. Because of the transitivity of the action, the automorphism group $Aut_{\mathcal{C}_1}(G)$ is isomorphic to G which acts on itself by right multiplication. Applying the functor Φ we obtain $\Phi(G)$ as a continuous P_2-set on which G acts on the right and these two action commute. After choosing a base point $g \in G$ this gives us a continuous homomorphism from P_2 to G. By Tichonoff we can choose compatible system of base points for all finite quotients of P_1, and obtain compatible continuous homomorphisms from P_2 to all of them and hence to P_1 (note that choosing compatible system of base points in all finite quotients of P_1 is the same as choosing an element in P_1, and changing this element changes the resulting homomorphism by conjugation). If we call this homomorphism $\varphi : P_2 \to P_1$, it is easy to check that Φ is induced by φ in the sence explained before the proposition. □

Theorem 1.4.3. (Functoriality) *For any pointed map $f : (S, \bar{s}) \to (T, \bar{t})$, where $f : S \to T$ is a map of schemes, $\bar{s} \in S(\Omega)$, and $\bar{t} := f \circ \bar{s} \in T(\Omega)$, there exists a natural map $\pi_1(f) : \pi_1(S, \bar{s}) \to \pi_1(T, \bar{t})$.*

Proof. If we denote by $\mathcal{E}t(S)$ (resp. $\mathcal{E}t(T)$) the category of finite étale coverings of S (resp. T), then the algebraic map $f : S \to T$ induces an exact functor $f^* : \mathcal{E}t(T) \to \mathcal{E}t(S)$ which sends a finite étale covering X of T to $X \times_T S$. Note that since f is pointed by assumption, the number of points above $\bar{t}$ in X is equal to the number of points above $\bar{s}$ in $f^*(X)$. On the other hand, by Theorem 1.3.6 we know that the category $\mathcal{E}t(S)$ (resp. $\mathcal{E}t(T)$) is equivalent to the category of continuous $\pi_1(S, \bar{s})$-sets (resp. $\pi_1(T, \bar{t})$-sets). Whence the functor f^* induces an exact functor Φ from the category of continuous $\pi_1(T,\bar{t})$-sets to the category of continuous $\pi_1(S,\bar{s})$-sets, which preserve cardinality. By Proposition 1.4.2 this gives a continuous group homomorphism $\varphi : \pi_1(S,\bar{s}) \to \pi_1(T,\bar{t})$ which is uniquely determined up to conjugation by elements in $\pi_1(T,\bar{t})$. To fix this indeterminacy, note that it was a result of ambiguity in choosing base points. Here we can overcome this difficulty by noticing that finite quotients of $\pi_1(T,\bar{t})$ are the same as finite pointed connected Galois coverings of T. Let (X,x) be such a covering, then one can choose the natural point $(x,\bar{s})$ in $X \times_T S$. This points give rise to a compatible system of points in the course of proof of Proposition 1.4.2, and hence we get a canonical homomorphism $\pi_1(f) : \pi_1(S,\bar{s}) \to \pi_1(T,\bar{t})$. □

Finally we prove the independence of the étale fundamental group from choosing the base point. Note that here, like in topology, the independence is upto inner automorphisms.

Theorem 1.4.4. *Given two base points $\bar{s}_1, \bar{s}_2 \in S(\Omega)$ one has $\pi_1(S,\bar{s}_1) \cong \pi_1(S,\bar{s}_2)$. Moreover this isomorphism is unique upto conjugation by an element.*

Proof. Just note that by Theorem 1.3.6, both categories of finite continuous $\pi_1(S,\bar{s}_1)$-sets and finite continuous $\pi_1(S,\bar{s}_2)$-sets are equivalent to the category of finite étale coverings of S, and use Proposition 1.4.2. □

1.5 Topological Invariance

In this section we are going to prove some invariance theorems for the étale fundamental group under topological assumptions. We start with the following trivial one.

Lemma 1.5.1. *Let I be a nilpotent ideal in A and put $\bar{A} := A/I$. Then any étale $\bar{A}$-algebra $\bar{B}$ lifts to an étale algebra B over A.*

Proof. As we have seen in the proof of Claim 1.2.7, $\bar{B}$ is locally of the form $\bar{A}[T_1,\dots,T_n]/(\bar{f}_1,\dots,\bar{f}_n)$, where $det(\partial\bar{f}_i/\partial T_j)_{ij}$ is invertible. Let f_i be an arbitrary lift of $\bar{f}_i$ to $A[T_1,\dots,T_n]$, and put $B := A[T_1,\dots,T_n]/(f_1,\dots,f_n)$. Note that since I is nilpotent any lift of an invertible element in $\bar{A}$ to A in invertible. In particular $det(\partial f_i/\partial T_j)_{ij}$ is invertible in A and hence B is an étale A-algebra lifting $\bar{B}$. □

Remark 1.5.2. *Note that, more generally, in the above result, we may assume that I is a nil ideal. That is, assume that $A' = A/I$ where I is an ideal consisting of nilpotent elements. Then one can write $A = \cup A_\alpha$ where A_α's are finitely generated $\mathbb{Z}$-subalgebras, and note that $I_\alpha := I \cap A_\alpha$ is nilpotent, and one has $A' = \cup(A_\alpha/I_\alpha)$. In this situation C'/A' comes from C'_α over A_α/I_α for some α. Now by the above lemma, C'_α descends to C_α over A_α and we are done.*

Corollary 1.5.3. *Let S be a scheme and $T \hookrightarrow S$ be a closed subscheme with the same underlying Zariski topological set. Then there is an equivalence between categories of finite étale coverings of S and T. In particular, the induced morphism from the étale fundamental group of T to that of S is an isomorphism.*

Definition 1.5.4. *A map $f : S_1 \to S_2$ of schemes is called a universal homeomorphism (resp. universal bijection), if for any $S_2' \to S_2$ the induced map $S_1 \times_{S_2} S_2' \to S_2'$ is a homeomorphism (resp. bijective) on the underlying topological spaces.*

Proposition 1.5.5. *$f : S_1 \to S_2$ is a universal bijection if and only if for each point $s_2 \in S_2$, $f^{-1}(\{s_2\}) = \{s_1\}$ consists only of one point, and $\kappa(s_1)/\kappa(s_2)$ is a purely inseperable algebraic extension.*

Proof. By definition, being universally bijective is equivalent to the property that for any field extension $\kappa(s_2')/\kappa(s_2)$, $Spec(\kappa(s_1) \otimes_{\kappa(s_2)} \kappa(s_2'))$ has cardinality 1. This is now a simple exercise in field theory to show that this is in turn equivalent to $\kappa(s_1)$ being purely inseparable and algebraic over $\kappa(s_2)$. □

Lemma 1.5.6. *Let S be a scheme, and $X, Y \in \mathcal{E}t(S)$ be two finite étale coverings. Then $Hom_S(X, Y)$ is in bijection with open and closed subsets Z in $X \times_S Y$ which map universally bijectively to X under the first projection.*

Proof. Given a morphism $f : X \to Y$, take Z to be the graph of f in $X \times_S Y$. Z is open and closed in $X \times_S Y$ and maps isomorphically, and hence universal bijectively, to X. For the other direction, assume such a subset $Z \subset X \times_S Y$ is given. The map $Z \to X$ induced by first projection is finite étale, and by hypothesis, universally bijective. We are going to prove that this map is an isomorphism. Since the question is local on X and $Z \to X$ is affine, we may assume X and Z are affine and show that the induced map $\mathcal{O}_X \to \mathcal{O}_Z$ is an isomorphism. Note that for any point $x \in X$ the fiber of $Z \to X$ over x is a finite disjoint union of spectrum of finite seperable field extensions over $\kappa(x)$. On the other hand, by Proposition 1.5.5, there must be only one point in the fiber with purely inseparable residue field. This implies that the fiber is isomorphic to $\{x\}$ itself or equivalently $\mathcal{O}_{X,x}/\mathfrak{m} \cong \mathcal{O}_{Z,z}/\mathfrak{m}.\mathcal{O}_{Z,z}$. Using Nakayama's lemma, one obtains that the map $\mathcal{O}_{X,x} \to \mathcal{O}_{Z,z}$ is surjective. But this map being flat and local, is faithfully flat and hence injective. This shows that the map $\mathcal{O}_X \to \mathcal{O}_Z$ is locally, and hence globally as well, an isomorphism. □

Corollary 1.5.7. *Let $f : S_1 \to S_2$ be a universal homeomorphism. Then $f^* : \mathcal{E}t(S_2) \to \mathcal{E}t(S_1)$ is a fully faithful functor.*

Proof. For any two finite étale covering X_2 and Y_2 of S_2, since $S_1 \to S_2$ is a universal homeomorphism, $f^*(X_2) \times_{S_1} f^*(Y_2)$ is homeomorphic to $X_2 \times_{S_2} Y_2$. Hence the correspondence $Z_2 \mapsto Z_1 := S_1 \times_{S_2} Z_2$ is a bijection between the open and closed subsets of $X_2 \times_{S_2} Y_2$ which map universally bijectively to X_2 and the open and closed subsets of $f^*(X_2) \times_{S_1} f^*(Y_2)$ which map universal bijectively to $f^*(X_2)$. We conclude applying the above lemma. □

This corollary sugests an equivalence between categories of finite étale coverings when we have a universal homeomorphism. In order to have such a result, we need to add some extra assumptions, namely we have

Theorem 1.5.8. *Let $f : S' \to S$ be a universal homeomorphism of finite type, and assume S is Noetherian. Then f^* is an equivalence between the categories of finite étale coverings over S and S'.*

Proof. We have seen in the above corollary that f^* is full and faithful. So it remains to prove it's essential surjectivity, i.e. we claim that any finite étale covering T' of S' is the pullback of such a cover of S. Since S is Noetherian, there exists a maximal open subset U of S on which the above claim is true. Let s be a point in $S \setminus U$ and $s' := f^{-1}(s)$. It suffices to show that the process of descending finite étale covers can be extended to a neighborhood of the point s.

Using faithfully flat descent, we may replace S (resp., S') by $Spec(A)$ where $A := \widehat{\mathcal{O}_{S,s}}$ is a complete local ring with maximal ideal $\mathfrak{m}$ (resp., $Spec(B)$ where $B := \mathcal{O}_{S',s'} \otimes_{\mathcal{O}_{S,s}} \widehat{\mathcal{O}_{S,s}}$ has a unique maximal ideal above $\mathfrak{m}$).

Fix a prime ideal $\mathfrak{p}$ of A and denote its residue field by κ. Then $\kappa \otimes_A B$ is an Artinian ring of finite type over κ and its residue field is a purely inseperable algebraic extension of κ (recall that $Spec(B) \to Spec(A)$ is a universal homeomorphism and apriory a universal bijection). Hence $\kappa \otimes_A B$ is finite over κ and since $\mathfrak{p}$ was arbitrary, $A \to B$ is quasi-finite. Now one can apply Zariski's main theorem (see EGA, Chapter III), to obtain $B = B_1 \times B_2$ where B_1 is finite over A, and $B_2/\mathfrak{m}.B_2 = 0$. But $Spec(B)$ is connected and hence $B_2 = (0)$ and B is finite over A.

Now let $Spec(C')$ be the pullback of T' to $Spec(B)$. Hence C' is finite étale over B and we are going to show that it descends to a finite étale A-algebra. Since $B/\mathfrak{m}.B$ is purely inseperable algebraic over $A/\mathfrak{m}$, they have the same étale fundamental groups, and hence $C'/\mathfrak{m}.C'$ over $B/\mathfrak{m}.B$ descends to $A/\mathfrak{m}$. By Lemma 1.5.1, $C'/\mathfrak{m}^n.C'$ over $B/\mathfrak{m}^n.B$ descends to C_n over $A/\mathfrak{m}^n$ for any $n \geq 1$. Put $C := \varprojlim C_n$ and check easily that C is finite étale over A which satisfies $C \otimes_A B = \varprojlim (C'/\mathfrak{m}^n.C') = C'$. □

Remark 1.5.9. *Note that the 'finite type' condition can be dropped in the following case. When k is a field, k'/k is a purely inseparable extension, A is a k-algebra, and $A' = A \otimes_k k'$. In this situation we have a finite étale A'-algebra C' and going to show that it has a descent datum for $A \to A'$ (note that A' is faithfully flat over A). i.e. we are going to prove that two base extensions to $A'' := A' \otimes_A A' = A \otimes_k (k' \otimes_k k')$ are isomorphic, and this isomorphisms are compatibly transitive on $A''' = A \otimes_k (k' \otimes_k k' \otimes_k k')$. Here the key ingredient is that the canonical surjection $k' \otimes_k k' \twoheadrightarrow k'$ has a nilpotent kernel.*

1.6 An Exact Sequence

In this section we consider the following general question. Let S be a connected scheme and $\bar{s} \in S(\Omega)$ be a geometric point. Let $f : X \to S$ be a proper smooth (in fact flat with geometrically reduced fibers is enough!) map, and fix a geometric point $\bar{x} \in X(\Omega)$ lying above $\bar{s}$. We have the following sequence

$$\pi_1(X_{\bar{s}}, \bar{x}) \to \pi_1(X, \bar{x}) \to \pi_1(S, \bar{s}).$$

Note that by the following cartesian diagram, the composition factors through the étale fundamental group of $Spec(\Omega)$, and hence is zero.

$$\begin{array}{ccc} X_{\bar{s}} & \longrightarrow & Spec(\Omega) \\ \downarrow & & \downarrow \bar{s} \\ X & \longrightarrow & S \end{array}$$

The general question now is if the above sequence is exact, and under which assumptions do we actually get a short exact sequence? Note that by our intuition coming from topology, the natural thing to add on the right of the above sequence is $\pi_0(X_{\bar{s}})$. This suggests that in order to have an epimorphism on the right, we must assume the fiber $X_{\bar{s}}$ is connected. We come back to this later.

Recall that since f is a proper map, we have the following Stein factorization (See [1, III.11.5.] for projective case)

$$X \xrightarrow{\varphi} Spec(f_*\mathcal{O}_X) \xrightarrow{\psi} S,$$

where φ has geometrically connected fibers and ψ is finite. We claim moreover that in our situation, where $X \to S$ has geometrically reduced fibres, ψ is an étale map as well. When $S = \{s\} = Spec(k)$, where k is a field, $f_*\mathcal{O}_X$ is just $\Gamma(X, \mathcal{O}_X)$. This is, by assumption, a finite geometrically redued k-algebra, hence a finite product of finite separable field extensions. To prove the general case, we first recall the following 'flat base change theorem':

Theorem 1.6.1. (Flat Base Change) *Consider a proper morphism $f : X \to Y$, a flat morphism $q : Y_1 \to Y$, and the following cartesian diagram*

$$\begin{array}{ccc} X_1 := X \times_Y Y_1 & \xrightarrow{q_1} & X \\ \downarrow f_1 & & \downarrow f \\ Y_1 & \xrightarrow{q} & Y \end{array}$$

Then for any coherent $\mathcal{O}_X$-module $\mathcal{F}$, and any non-negative integer n, one has the following isomorphism

$$R^n f_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_1} \xrightarrow{\sim} R^n f_{1*}(q_1^*(\mathcal{F})).$$

Proof. EGA, Chapter III, Proposition 1.4.15. □

Using the above theorem, one can easily see that Stein factorization commutes with flat base change, i.e. if $X \to S' \to S$ is the Stein factorization of a proper map $X \to S$, then for any flat morphism $S_1 \to S$, $X \times_S S_1 \to S' \times_S S_1 \to S_1$ is the Stein factorization of $X \times_S S_1$.

Now consider the proper flat morphism $f : X \to S$ with geometrically reduced fibers. We are going to show that $S' := Spec(f_*\mathcal{O}_X)$ is étale over S. Since the claim is local over S, and by above observation concerning the compatibility of the Stein factorization with flat base change, we can replace S by local rings of its points. Furthermore, by faithfully flat descent, we can replace these local rings by their completions and hence reduce to the case of $S = Spec(A)$ where A is a complete local ring with maximal ideal $\mathfrak{m}$. As we have seen, $\Gamma(X \otimes_A A/\mathfrak{m}, \mathcal{O})$ is finite étale over $A/\mathfrak{m}$. Lift it to a finite étale A-algebra B (Hensel's lemma). For any $n \geq 0$, consider the schemes $X_n := X \otimes_A A/\mathfrak{m}^n$ and note that $X_0 = f^{-1}(\{\mathfrak{m}\})$. Because of the canonical map $B \twoheadrightarrow B \otimes_A A/\mathfrak{m} = \Gamma(X_0, \mathcal{O}_{X_0})$, we have the induced morphism $X_0 \to Spec(B)$ over A. Since B is étale over A, and for any $n \geq 0$, X_0 is a closed sub-scheme of X_n defined by a nilpotent ideal, there are unique extensions to morphisms $X_n \to Spec(B)$ over A. Puting these together one obtains a morphism $\hat{f} : \hat{X} \to Spec(B)$ over A, and hence $\hat{f}_*\mathcal{O}_{\hat{X}}$ is a B-algebra. By the theorem of formal functions $\hat{f}_*\mathcal{O}_{\hat{X}} \cong \widehat{f_*\mathcal{O}_X}$ (see EGA, Chapter III, Theorem 4.1.5.). But $f_*\mathcal{O}_X$ is already complete, being finite over A. Hence if we show that $\hat{f}$ is an isomorphism, we are done.

Note that in general for any A-module M which is annihilated by $\mathfrak{m}$, one has $M \cong_A M \otimes_A (A/\mathfrak{m})$, and moreover for any other A-module N which is also annihilated by $\mathfrak{m}$, we have $M \otimes_A N \cong_{A/\mathfrak{m}} M \otimes_{A/\mathfrak{m}} N$. If we apply these general facts to the A-modules $B \otimes_A (\mathfrak{m}^n/\mathfrak{m}^{n+1})$ and $\mathcal{O}_X \otimes_A (\mathfrak{m}^n/\mathfrak{m}^{n+1})$, and use the isomorphism ψ_0, we obtain the left vertical isomorphisms in the following diagram for all $n \geq 1$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B \otimes_A (\mathfrak{m}^n/\mathfrak{m}^{n+1}) & \longrightarrow & (B/\mathfrak{m}^{n+1}.B) & \longrightarrow & (B/\mathfrak{m}^n.B) & \longrightarrow & 0 \\
 & & \downarrow \cong & & \downarrow \psi_{n+1} & & \downarrow \psi_n & & \\
0 & \longrightarrow & \Gamma(X, \mathcal{O}_X \otimes_A (\mathfrak{m}^n/\mathfrak{m}^{n+1})) & \longrightarrow & \Gamma(X, (\mathcal{O}_X/\mathfrak{m}^{n+1}.\mathcal{O}_X)) & \longrightarrow & \Gamma(X, (\mathcal{O}_X/\mathfrak{m}^n.\mathcal{O}_X)) & &
\end{array}$$

Note that ψ_0 is an isomorphism by construction, and since the left vertical map in above diagram is always an isomorphism, we deduce by induction that ψ_n is an isomorphism for all $n \geq 0$, and hence we completed the proof of the following

Theorem 1.6.2. *Let S be a connected scheme, $f : X \to S$ be a proper flat map with geometrically reduced fibers, and let $X \to S' \to S$ be its Stein factorization. Then $S' \to S$ is a finite étale covering.*

Corollary 1.6.3. *With notations and hypothesis of the above theorem, we have a functor from $\mathcal{E}t(X)$ to $\mathcal{E}t(S)$, which sends $g : Y \to X$ to $Spec((f \circ g)_*\mathcal{O}_Y)$ over S.*

In order to attack the general question raised at the beginning of this section, we must study the above functor more closely. Once again consider the proper flat morphism $f : X \to S$ with geometrically reduced fibers, and this time assume that X is connected. After fixing compatible geometric points $\bar{s} \in S$ and $\bar{x} \in X$, we obtain the induced map between fundamental groups:

$$f_* : \pi_1(X, \bar{x}) \to \pi_1(S, \bar{s}).$$

Consider $S' := Spec(f_* \mathcal{O}_X)$ which appears in the Stein factorization $X \xrightarrow{g} S' \to S$ of f and recall that by Theorem 1.6.2, $S' \to S$ is a finite étale covering. Now the map $f_* : \pi_1(X, \bar{x}) \to \pi_1(S, \bar{s})$ factors through $\pi_1(S', \bar{s}')$, where $\bar{s}' := g(\bar{x})$, and thus f_* is not surjective unless $S' \xrightarrow{\sim} S$. On the other hand, we show that the induced map $g_* : \pi_1(X, \bar{x}) \to \pi_1(S', \bar{s}')$ is surjective. Indeed, since g has geometrically connected fibers, for any connected finite étale covering $T' \to S'$, the pullback $g^*(T')$ is also connected. Equivalently, for any $\pi_1(S', \bar{s}')$-set N with only one orbit, the induced $\pi_1(X, \bar{x})$-set has also one orbit. In particular for any finite quotient G of $\pi_1(S', \bar{s}')$, the composition morphism $\pi_1(X, \bar{x}) \to \pi_1(S', \bar{s}') \twoheadrightarrow G$ is surjective as well. But this means that the map $g_* : \pi_1(X, \bar{x}) \to \pi_1(S', \bar{s}')$ is surjective itself. By replacing S by S' in our study, we can and will assume that the map $f_* : \pi_1(X, \bar{x}) \to \pi_1(S, \bar{s})$ is surjective. Let L denote the kernel of the map f_*. Note that with this notations, for any $\pi_1(X, \bar{x})$-set M, and any $\pi_1(S, \bar{s})$-set N, one has $Hom_{\pi_1(X,\bar{x})}(M, N) = Hom_{\pi_1(S,\bar{s})}(L.M \backslash M, N)$, where $L.M \backslash M$ is the space of L-orbits of M. Now we have the following commutative diagram

$$\begin{array}{ccccccc} \pi_1(X_{\bar{s}}, \bar{x}) & \longrightarrow & \pi_1(X, \bar{x}) & \longrightarrow & \pi_1(S, \bar{s}) & \longrightarrow & 0 \\ & \searrow^{\psi} & \uparrow & & & & \\ & & L & & & & \end{array}$$

and our aim now is to prove surjectivity of ψ. First note that surjectivity of ψ is equivalent to the following:

Claim 1.6.4. *In any finite $\pi_1(X, \bar{x})$-set, L and $\pi_1(X_{\bar{s}}, \bar{x})$ have the same number of orbits.*

Surjectivity of ψ clearly implies the above claim. Conversely if $Im(\psi)$ is a proper subgroup of L, then there is a finite quotient G of $\pi_1(X, \bar{x})$ which sees this properness, i.e. $Im(\psi)$ and L map to different subgroups of G. But this contradicts the claim. Hence to prove surjectivity of ψ, we prove the claim as follows:

Proof. (of claim) Instead of finite $\pi_1(X, \bar{x})$-sets, we can consider fibers of finite étale coverings of X. Fix such a covering $\varphi : Y \to X$ and consider $T := Spec((f \circ \varphi)_* \mathcal{O}_Y)$ which by Theorem 1.6.2 is a finite étale covering of S. If we denote by M_Y the corresponding $\pi_1(X, \bar{x})$-set, the number of L-orbits in M_Y is the number of elements of $L.M_Y \backslash M_Y$, which is the number of elements of the corresponding $\pi_1(S, \bar{s})$-set, which in turn is nothing else than $deg_S(T)$. On the other hand number of $\pi_1(X_{\bar{s}}, \bar{x})$-orbits is the number of connected components of the induced finite étale covering $Y_{\bar{s}} \to X_{\bar{s}}$. Considering the following cartesian diagram

$$\begin{array}{ccc} Y_{\bar{s}} & \longrightarrow & X_{\bar{s}} \\ \downarrow{\scriptstyle \theta} & & \downarrow \\ T_{\bar{s}} & \longrightarrow & \{\bar{s}\} \end{array}$$

and noticing that θ has geometrically connected fibers, it is obvious that the number of connected components of $Y_{\bar{s}}$ is equal to the number of points of $T_{\bar{s}}$, which by definition is $deg_S(T)$. □

The upshot of all these is the following

Theorem 1.6.5. *Let $f : X \to S$ be a proper, flat morphism with geometrically reduced fibers between connected schemes. Assume moreover that $f_*\mathcal{O}_X = \mathcal{O}_S$. After fixing compatible geometric base points $\bar{x} \in X$ and $\bar{s} \in S$, one has the following exact sequence*

$$\pi_1(X_{\bar{s}}, \bar{x}) \to \pi_1(X, \bar{x}) \to \pi_1(S, \bar{s}) \to 0.$$

Now we want to study the injectivity of the first map in the above exact sequence. To prove some results in that direction, we need preliminaries. We start with the following:

Lemma 1.6.6. *Let S be a connected scheme and $f : X \to S$ be a proper flat map with geometrically reduced fibers. Fix a geometric point $\bar{s} \in S$ and consider finite étale covers Y_1 and Y_2 of X such that $Y_{1,\bar{s}} \cong Y_{2,\bar{s}}$. Then there is a finite étale cover S' of S such that Y_1 and Y_2 become isomorphic over $X_{S'}$.*

Proof. First note that $Isom_X(Y_1, Y_2)$ is in bijection with open and closed subsets $Z \subset Y_1 \times_X Y_2$ which are isomorphic to both Y_1 and Y_2 via the two projections. Since the projections from Z to Y_1 and Y_2 are finite étale maps, being isomorphic in this case is the same as having a 1-element geometric fiber. The isomorphism between $Y_{1,\bar{s}}$ and $Y_{2,\bar{s}}$ in hypothesis corresponds then to a subset $Z_{\bar{s}}$ of $(Y_1 \times_X Y_2)_{\bar{s}}$ with mentioned properties. Consider $T := Spec(f_*\mathcal{O}_{Y_1 \times_X Y_2})$, which by Theorem 1.6.2 is a finite étale covering of S. Since f is proper, $T_{\bar{s}}$ is in bijection with connected components of $(Y_1 \times_X Y_2)_{\bar{s}}$, and hence $Z_{\bar{s}}$ corresponds to a finite subset of it. On the other hand, note that any finite étale covering of S, can be trivialized by a base change to a finite étale covering, i.e. by replacing S by a connected finite étale cover one can assume that the finite étale cover is of the form $\coprod S$. After trivializing T, $Z_{\bar{s}}$ corresponds to a number of copies of S appearing in this disjoint union, whence it extends to an open and closed subset Z of $Y_1 \times_X Y_2$. Z projects to Y_1 and Y_2 with degree one over $\bar{s}$, hence the degree is one everywhere and hence Z gives the desired isomorphism in the statement. □

Using this Lemma we prove the following important proposition which says that the étale fundamental group of proper schemes over algebraically closed fields doesn't change if we base change to another algebraically closed field.

Proposition 1.6.7. *Let $k \hookrightarrow k'$ be an extension of algebraically closed fields, and let $X = X_k$ be a proper, reduced, and connected scheme over k. Then one has*

$$\pi_1(X_k, \bar{x}) \cong \pi_1(X_{k'}, \bar{x}).$$

Proof. We show that base extension induces an equivalence between the corresponding categories of finite étale covers.

To prove fully faithfulness, fix finite étale covers Y_1 and Y_2 of X_k and note as usual that $Hom_{X_k}(Y_1, Y_2)$ is in bijection with open and closed subsets of $Y_1 \times_{X_k} Y_2$ which map isomorphically to Y_1. But this set obviously doesn't change under base change $k \hookrightarrow k'$.

To prove essential surjectivity, consider an arbitrary finite étale covering Y' of $X_{k'}$. There exists a finitely generated k-subalgebra $A \subset k'$ such that Y' is defined over X_A, i.e. there is a finite étale cover $Y'_A \to X_A$ such that $Y' = Y'_A \otimes_A k'$. Since A is finitely generated over k, it admits a homomorphism $A \to k$, and hence from Y'_A one gets a finite étale cover Y over X_k such that Y'_A and Y_A are isomorphic after base change $A \to k$. Now one can apply the above lemma to show that Y'_A and Y_A become isomorphic after going to a finite étale axtension A' over A. Since k' is algebraically closed, it containes every finite étale extension of A, in particular A'. Hence Y' is isomorphic to $Y_{k'}$ and we are done. ◻

Finally we can prove the following very important and interesting result.

Theorem 1.6.8. *With all the notations and assumptions of Theorem 1.6.5, if one assume moreover that $S = Spec(k)$, where k is a field, then one has a short exact sequence of the form:*

$$0 \to \pi_1(X_{\bar{s}}, \bar{x}) \to \pi_1(X, \bar{x}) \to Gal(\bar{k}/k) \to 0.$$

Proof. Assume we can show that for any finite étale cover $Z \to X_{\bar{s}}$ there exists a finite étale cover $Y \to X$ such that $Y_{\bar{s}}$ surjects onto Z. In the language of fundamental groups, it means that for any finite index subgroup H of $\pi_1(X_{\bar{s}}, \bar{x})$ there exists a finite index subgroup H' of $\pi_1(X, \bar{x})$ such that the pre-image of H' in $\pi_1(X_{\bar{s}}, \bar{x})$ is contained in H. This in particular says that kernel of the map $\pi_1(X_{\bar{s}}, \bar{x}) \to \pi_1(X, \bar{x})$ is contained in every finite index subgroup and hence must be trivial because pro-finite groups are Housdorff. Furthermore, note that by Proposition 1.6.7, it suffices to work with $X_{\bar{k}}$ instead of $X_{\bar{s}}$.

Since $Spec(\bar{k}) \to Spec(k^{sep})$ is a universal homeomorphism, every finite étale cover $Z \to X_{\bar{k}}$ comes from a finite étale cover of $X_{k^{sep}}$, and so from a finite étale cover $Z' \to X_{k'}$ for some finite seperable extension k' of k. Now consider the following diagram

$$\begin{array}{ccccc}
Z' & \longrightarrow & X_{k'} & \longrightarrow & X \\
\uparrow & & \uparrow & & \uparrow \\
Z'_{\bar{k}} & \longrightarrow & X_{(k' \otimes_k \bar{k})} & \longrightarrow & X_{\bar{k}} \\
\downarrow {\scriptstyle Id} & & \downarrow {\scriptstyle Id} & & \\
\coprod (Z' \times_X X_{\bar{k}}) & & \coprod X_{\bar{k}} & &
\end{array}$$

where the coproducts in last line are over all embeddings of k' in $\bar{k}$. The identity embeding $k' \hookrightarrow \bar{k}$ in $Z'_{\bar{k}} = \coprod (Z' \times_X X_{\bar{k}})$ gives a copy of Z. Whence Z is a connected component of $Z'_{\bar{k}}$ and we are done. □

1.7 Specialization Map

In this section we will use the following theorem from EGA, which we state without proof.

Theorem 1.7.1. (Grothendieck's Algebraicity Theorem) *Let A be a Noetherian ring and I an ideal of A such that A is complete with respect to the I-adic topology. Put $Y := Spec(A)$ and consider a proper morphism $f : X \to Y$. For any $n \geq 1$, set $A_n := A/I^n$, $Y_n := Spec(A_n)$, and $X_n := X \times_Y Y_n$. Suppose, for every n, $\mathcal{F}_n$ is a coherent $\mathcal{O}_{X_n}$-module such that $\mathcal{F}_{n-1} \cong \mathcal{F}_n \otimes_{\mathcal{O}_{X_n}} \mathcal{O}_{X_{n-1}}$. Then there exists a coherent $\mathcal{O}_X$-module $\mathcal{F}$ such that $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_n} \xrightarrow{\sim} \mathcal{F}_n$, for every $n \geq 1$.*

Proof. EGA, Chapter III, Theorem 5.1.4. □

Now consider a complete discrete valuation ring V, with maximal ideal $\mathfrak{m}$ and algebraically closed residue field k. Put $S := Spec(V)$ and let s (resp. η) be the closed (resp. generic) point of S. Finally let $f : X \to S$ be a proper and flat map with geometrically reduced fibers such that $f_* \mathcal{O}_X = \mathcal{O}_S$. Our aim in this section is to compare the étale fundamental groups of X_s and $X_{\bar{\eta}}$, where $\bar{\eta}$ is a geometric point with topological image η. To start with, we prove the following:

Lemma 1.7.2. *There is an equivalence between the categories of finite étale covers of X and finite étale covers of X_s, and hence $\pi_1(X) \cong \pi_1(X_s)$.*

Proof. Since X_s is a closed sub-scheme of X, there is the pullback functor from $\mathcal{E}t(X)$ to $\mathcal{E}t(X_s)$. Conversely consider an arbitrary finite étale covering Y of X_s. By lemma 1.5.1, for any $n \geq 1$, it lifts to a finite étale cover of $X_n := X \otimes_V (V/\mathfrak{m}^n)$. This finite étale covers are given by coherent finite $\mathcal{O}_{X_n}$-algebras $\mathcal{F}_n$. Now one can apply Theorem 1.7.1 to obtain a coherent finite $\mathcal{O}_X$-algebra $\mathcal{F}$ which gives all $\mathcal{F}_n$'s. $\mathcal{F}$ gives a finite affine scheme over X which must be étale since it is so over the closed point. This gives a functor from $\mathcal{E}t(X_s)$ to $\mathcal{E}t(X)$, and obviously $\mathcal{E}t(X_s) \to \mathcal{E}t(X) \to \mathcal{E}t(X_s)$ is the identity functor. Hence the pullback functor is essentially surjective and full. It is plain to check its faithfulness and hence the claim. □

On the other hand, we have the generic fiber $X_\eta \hookrightarrow X$. Fixing any geometric point $\bar{\eta}$ gives a morphism $X_{\bar{\eta}} \to X_\eta$ and hence a map $\pi_1(X_{\bar{\eta}}) \to \pi_1(X)$. For any connected finite étale cover $g : Y \to X$, $T := Spec((f \circ g)_* \mathcal{O}_Y)$ is a finite étale cover of S. But $\pi_1(S) = \pi_1(\{s\}) = 0$ and thus $T = \coprod S$ is a trivial cover. On the other hand, Y is connected and hence $T = S$, which means that $f \circ g : Y \to S$ is it's own Stein factorization. But Stein factorization commutes with flat base change and $\bar{\eta} \to S$ is flat, hence $Y_{\bar{\eta}}$ is also connected. This means that the map $\pi_1(X_{\bar{\eta}}) \to \pi_1(X)$ is surjective. Combining this and Lemma 1.7.2 we obtain the following result.

Theorem 1.7.3. *Let V be a complete discrete valuation ring with algebraically closed residue field. Let $f : X \to S := Spec(V)$ be a proper flat map with geometrically reduced fibres such that $f_*(\mathcal{O}_X) = \mathcal{O}_S$, and let s and η denote the closed and the open point of S respectively. Then we have a surjective specialization map $\pi_1(X_{\bar{\eta}}) \twoheadrightarrow \pi_1(X_s)$, where $\bar{\eta}$ is a geometric point over η.*

Remark 1.7.4. *Here we make two important remarks concerning the above theorem, namely:*

1. *In the above arguments we assumed that S is the spectrum of a complete discrete valuation ring with algebraically closed residue field, but this was redundant! More precisely, any locally Noetherian base scheme S can be reduced to this case, simply because for any two points $s_1, s_2 \in S$ with $s_1 \in \overline{\{s_2\}}$ there exists a morphism from spectrum of such a DVR to S with image $\{s_1, s_2\}$ and we have the analogue of Theorem 1.7.3 in that generality. In particular this gives a semi-continuity theorem for fundamental groups of fibers of a proper fiberation.*

2. *The specialisation map is not always injective. For example consider a family of generically ordinary elliptic curves in positive characteristic which has a super-singular fiber. The corresponding specialisation map, goes from the fundamental group of an ordinary elliptic curve to the fundamental group of a super-singular one and hence can not be injective.*

By part 2 of the above remark, we know that the specialization map $\pi_1(X_{\bar{\eta}}) \to \pi_1(X_s)$ is not necessarily injective, but one can still study it more closely and obtain finer information about the kernel. In order to do that, we will use the following result.

Theorem 1.7.5. *With all the above notations and assumptions, let G be a finite group of order prime to $p = char(k)$ and assume moreover that $X \to S$ is smooth. Then any G-cover of $X_{\bar{\eta}}$ is induced by a G-cover of X.*

Proof. Let $Y_{\bar{\eta}} \to X_{\bar{\eta}}$ be a G-cover with field of definition K', which is a finite field extension of the fraction field K of V. For any finite extension K'' of K', if we denote by V'' the integral closure of V in K'', then V'' is a complete discrete valuation ring with residue field k. Hence X and $X_{V''}$ have isomorphic special fibers, hence isomorphic étale fundamental groups (by Lemma 1.7.2). Therefore, it suffices to show, after passing to a finite extension K'', that $Y_{K''}$ extends to a finite étale cover $Y_{V''}$ of $X_{V''}$. Also, replacing K by K', we may assume that $Y_{\bar{\eta}}$ is the pullback of a G-cover Y_η of X_η.

Take Y to be the normalization of X in Y_η, then $Y \to X$ is finite and is étale over X_η, i.e. it is étale over the complement of the special fiber $X_s \subset X$. Let $X_s = \coprod X_s^i$ be the decomposition of X_s into it's connected components. If X_s^i is one of these components with generic point z, then $W := \mathcal{O}_{X,z}$ is a discrete valuation ring. Now fix a point $z' \in Y$ above z and consider $W \subset W' := \mathcal{O}_{Y,z'}$. The group G acts on W' and let D (resp., I, resp., P) be the decomposition group (resp., inertia group, resp., wild inertia group) of z' over z (see section A.3). Since order of G is not divisible by p, one can apply Corollary A.3.1 and obtain that $P = 0$ and I is cyclic. Hence if we denote by π a uniformizer of V, the map $\sigma \mapsto (\sigma(\pi)/\pi)$ identifies I with the group μ_{e_i} of e_i-th roots of unity in function field L of Y_η.

Back to the decomposition $X_s = \coprod X_s^i$, for each i we get a ramification index e_i and if we consider the least common multiple e of these numbers, since $(e, p) = 1$, and k is an algebraically closed field of characteristic p one has $\mu_e \subset k$. By Hensel's lemma one can lift these roots of unity to V and hence K. This implies that $K[\sqrt[e]{\pi}]$ is a Galois extension of K with Galois group μ_e. Now replace V by $V' := V[\sqrt[e]{\pi}]$, and Y by normalization Y' of X in $Y \otimes_V K[\sqrt[e]{\pi}]$.

Claim 1.7.6. *Y' is étale over the generic points of connected components of the special fiber of $X_{V'}$.*

Proof. First consider the following diagram

$$\begin{array}{ccc} \mathcal{O}_{X,z} & \overset{g}{\hookrightarrow} & \mathcal{O}_{Y,z'} \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h'} \\ \mathcal{O}_{X,z}[\sqrt[e]{\pi}] & \overset{g'}{\hookrightarrow} & \mathcal{O}_{Y',z'} \end{array}$$

Now note that h is a totally ramified Galois extension with Galois group μ_e, g is a tamely ramified Galois extension with Galois group G and inertia group μ_{e_i}. Finally, apply Abhyankar's Lemma (see Lemma A.3.2) to deduce that g' is an unramified extension, hence the claim. □

We conclude by applying the Purity Theorem (see Theorem 1.2.10) to obtain that $Y' \to X_{V'}$ is a finite étale extension. □

Note that for any finite group G, isomorphism classes of Galois G-covers of a scheme are in bijection with surjective homomorphisms from the étale fundamental group to G. Therefore, the following statements are immediate corollaries of the above theorem.

Corollary 1.7.7. *With same hypothesis and notations as in above theorem, one has:*

1. *For any finite group G of order prime to p*

$$Hom_{cont}(\pi_1(X_{\bar{\eta}}), G) = Hom_{cont}(\pi_1(X_s), G).$$

2. *If we denote by $\pi_1^{(p)}$ the prime to p part of the étale fundamental group, i.e. the profinite group which classifies finite étale covers of prime to p degree, then*

$$\pi_1^{(p)}(X_{\bar{\eta}}) \cong \pi_1^{(p)}(X_s).$$

In particular, when k has characteristic zero, 'prime to p' condition is empty and one obtains an isomorphism between the whole étale fundamental groups of special and generic fibers.

1.8 Projective Curves

In this section, by a curve we mean a smooth, geometrically connected, projective curve, and we are going to study the étale fundamental group of such a curve X over an algebraically closed field k.

If $k = \mathbb{C}$ is the field of complex numbers, then the complex points $X(\mathbb{C})$ is a compact orientable Riemann surface, and finite étale covers $Y \to X$ correspond to finite coverings $Y(\mathbb{C}) \to X(\mathbb{C})$. By Riemann's existence theorem and GAGA, any finite covering $Y(\mathbb{C}) \to X(\mathbb{C})$ comes from an algebraic finite étale map $Y \to X$, and hence

$$\pi_1^{\text{alg}}(X, x) \cong \pi_1^{\text{top}}\widehat{(X(\mathbb{C}), x)},$$

where $\pi_1^{\text{top}}\widehat{(X(\mathbb{C}), x)}$ is the pro-finite completion of the topological fundamental group of the Riemann surface $X(\mathbb{C})$. On the other hand, it is well known that if the Riemann surface $X(\mathbb{C})$ has genus g, then

$$\pi_1^{\text{top}}(X(\mathbb{C}), x) = \{a_1, \ldots, a_g, b_1, \ldots, b_g \mid [a_1, b_1].[a_2, b_2]. \ldots .[a_g, b_g] = 1\}.$$

This implies that the étale fundamental group is topologically generated by $2g$ elements subject to one relation, and hence in particular is topologically finitely presented.

If k is any algebraically closed field of characteristic zero, we have exactly the same result. Note that X is defined over a finitely generated $\mathbb{Q}$-algebra $A \subset k$, and if we denote by k_0 the algebraic closure of the fraction field of A, k_0 can be embedded in $\mathbb{C}$, as well as in k. Now by Proposition 1.6.7, one has

$$\pi_1^{\text{alg}}(X/k) \cong \pi_1^{\text{alg}}(X/k_0) \cong \pi_1^{\text{alg}}(X/\mathbb{C}).$$

In positive characteristic we need to do more. We start by proving:

Theorem 1.8.1. *Let V be a complete discrete valuation ring with maximal ideal $\mathfrak{m}$ and algebraically closed residue field k, and let X_0 be a smooth and projective curve over k. Then there exists a smooth projective curve X over V with special fiber X_0.*

Proof. For any $n \geq 0$ consider $V_n := (V/\mathfrak{m}^{n+1})$, in particular $V_0 = k$. We first show that X_0 can be lifted to a smooth projective curve X_n over V_n for all $n \geq 0$. By induction suppose that we have a projective smooth curve X_n over V_n such that $X_n \otimes_{V_n} V_0 \cong X_0$. By Claim 1.2.7, we know that locally

$$\mathcal{O}_{X_n} = ((V_n[X_j])/(f_i))_g ,$$

where $(\partial f_i/\partial X_j)_{i,j}$ has an invertible full minor. These local pieces can be lifted modulo $\mathfrak{m}^{n+2}$, just by lifting the coefficients of f_i's, and noticing that any lift of an invertible element in V_n to V_{n+1} is still invertible. Hence we can find an open affine cover $X_n = \bigcup \mathcal{U}_{i,n}$ such that for all i, $\mathcal{U}_{i,n}$ can be lifted to a smooth affine scheme $\mathcal{U}_{i,n+1}$ over V_{n+1}. For any pair i, j the intersection $\mathcal{U}_{ij,n} := \mathcal{U}_{i,n} \bigcap \mathcal{U}_{j,n}$ is open and affine in both $\mathcal{U}_{i,n}$ and $\mathcal{U}_{j,n}$. $\mathcal{U}_{ij,n}$ can be lifted by the same reasoning to an open and affine scheme $\mathcal{U}_{ij,n+1}$ over V_{n+1}, which can be considered both as an open subscheme of $\mathcal{U}_{i,n+1}$ and as an open subscheme of $\mathcal{U}_{j,n+1}$. These two subscheme structures are isomorphic, but not uniquely. One can check that the difference between two isomorphisms is a derivation with values in $\mathcal{O}_{X_0} \otimes_k (\mathfrak{m}^{n+1}/\mathfrak{m}^{n+2})$ over $\mathcal{U}_{ij,0}$, i.e. an element of $\Gamma(\mathcal{U}_{ij,0}, \mathcal{T}_{X_0} \otimes (\mathfrak{m}^{n+1}/\mathfrak{m}^{n+2}))$ where $\mathcal{T}$ is the tangent sheaf. The gluing coming from these isomorphisms defines a global scheme X_{n+1} over V_{n+1} if and only if the circular difference of the above mentioned elements restricted to $\mathcal{U}_{ijk,0}$ vanishes in $\Gamma(\mathcal{U}_{ijk,0}, \mathcal{T}_{X_0})$. One can easily check that this circular difference satisfies the cocycle condition, and hence the obstruction of having a global lifting lies in $H^2(X_0, \mathcal{T}_{X_0} \otimes (\mathfrak{m}^{n+1}/\mathfrak{m}^{n+2}))$. But this second cohomology group vanishes since $dim(X_0) = 1$, and X_n can be lifted to X_{n+1} over V_{n+1}.

Now $\hat{X} := \varprojlim X_n$ is a formal scheme over V, which is projective and hence algebraic by Theorem 1.7.1. This gives us a smooth projective curve X over V with special fiber X_0 and hence we are done (see SGA 1, EXPOSÉ III, for more details). □

Now back to our original problem, consider a projective smooth curve X_k over an algebraically closed field k of characteristic $p > 0$. Denote by $W(k)$ the ring of Witt vectors, which is a complete discrete valuation ring with residue field k, and fraction field K of characteristic 0. Now by the above theorem X_k can be lifted to a projective smooth curve X over $W(k)$, and by Theorem 1.7.3 we have a surjective homomorphism

$$\pi_1(X_{\bar{K}}) \twoheadrightarrow \pi_1(X_k).$$

But by characteristic zero argument, we know the structure of $\pi_1(X_{\bar{K}})$, and in particular it implies that $\pi_1(X_k)$ is topologically finitely generated, but not necesarily finitely presented.

Remark 1.8.2. *By second part of Corollary 1.7.7, we obtain that* $\pi_1^{(p)}(X_{\bar K}) \cong \pi_1^{(p)}(X_k)$*, and hence the prime to p part of* $\pi_1^{(}X_k)$ *is topologically generated by* $2g$ *elements subject to a single relation.*

Remark 1.8.3. *In general, we have shown above that the étale fundamental group of any projective smooth curve over an algebraically closed field is topologically finitely generated. One can combine this result and a homotopical version of the weak Lefschetz Theorem for étale fundamental groups to obtain the same result for any smooth projective variety over an algebraically closed field. In particular, it shows that for any fixed finite group* G*, and any smooth projective variety* X *over an algebraically closed field, there are only finitely many isomorphism classes of Galois* G*-covers of* X*. Simply because* $Hom_{cont}(\pi_1(X), G)$ *has finite cardinality.*

1.9 Rational Points and the Étale Fundamental Group

Let $S = Spec(K)$, where K is a field, and $\pi : X \to S$ be a projective, smooth, geometrically connected variety. Fix a geometric point $\bar y : Spec(\bar K) \to X$, which automatically gives a geometric point of $\bar X := X \otimes_K \bar K$. As we have seen in Theorem 1.6.8, there is an exact sequence of profinite groups of the form:

$$0 \to \pi_1(\bar X, \bar y) \to \pi_1(X, \bar y) \xrightarrow{\pi_*} Gal(\bar K/K) \to 0.$$

Any K-rational point $x \in X(K)$ defines a section $s_x : Gal(\bar K/K) \to \pi_1(X)$ for the projection π_*, which is well defined up to conjugation by elements of $\pi_1(X_{\bar K}$. We make this assertion clear as follows.

Consider the point x as a map $x : Spec(K) \to X$, and denote by $\bar x$ its composition with $Spec(\bar K) \to Spec(K)$. Then by functoriality of étale fundamental groups (see Theorem 1.4.3), one obtains a map $x_* : Gal(\bar K/K) \to \pi_1(X, \bar x)$. On the other hand, by Theorem 1.4.4, $\pi_1(X, \bar x)$ is isomorphic to $\pi_1(X, \bar y)$, and this isomorphism is determined up to conjugation by an element of $\pi_1(X, \bar y)$. Denote by $\psi : \pi_1(X, \bar x) \xrightarrow{\sim} \pi_1(X, \bar y)$ one of these isomorphisms, and consider the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_1(\bar X, \bar y) & \longrightarrow & \pi_1(X, \bar y) & \xrightarrow{\pi_*} & Gal(\bar K/K) & \longrightarrow & 0 \\ & & & & \uparrow{\scriptstyle \psi,\ \cong} & & \downarrow{\scriptstyle x_*} & & \\ & & & & & & \pi_1(X, \bar x) & & \end{array}$$

Then s_x is defined to be the composition $\psi \circ x_*$. Until now, s_x is well defined up to conjugation by elements in $\pi_1(X, \bar y)$, but if one note that s_x must be a section for the projection π_*, then this ambiguity can be reduced to conjugation by elements in $\pi_1(X, \bar y)$ which map by π_* to the identity, i.e. by elements in $\pi_1(\bar X, \bar y)$.

Given such a section, one has a semi-direct decomposition

$$\pi_1(X,\bar{y}) \cong Gal(\bar{K}/K) \ltimes \pi_1(\bar{X},\bar{y}).$$

Fixing such a decomposition, any other section for π_* has the form $(Id,\rho) : Gal(\bar{K}/K) \to \pi_1(X,\bar{y})$, where $\rho : Gal(\bar{K}/K) \to \pi_1(\bar{X},\bar{y})$ is continuous. In orther to have a group homomorphism, for any $\sigma,\tau \in Gal(\bar{K}/K)$ we must have

$$(\sigma\tau, \rho(\sigma\tau)) = (\sigma, \rho(\sigma)).(\tau, \rho(\tau)) = (\sigma\tau, \rho(\sigma)\rho(\tau)^{\sigma}),$$

or equivalently $\rho(\sigma\tau) = \rho(\sigma)\rho(\tau)^{\sigma}$. This means that ρ is a (non-abelian) 1-cocycle with values in $\pi_1(\bar{X},\bar{y})$. On the other hand, conjugating by an element $g \in \pi_1(\bar{X},\bar{y})$ will change ρ by a 1-coboundry, simply because

$$(1, g^{-1}).(\sigma, \rho(\sigma)).(1, g) = (\sigma, g^{-1}\rho(\sigma)g^{\sigma}).$$

So we have proven the following result.

Proposition 1.9.1. *With all above notations and assumptions, set of sections $s : Gal(\bar{K}/K) \to \pi_1(X,\bar{y})$ up to conjugation by elements of $\pi_1(\bar{X},\bar{y})$ is in bijection with the non-abelian cohomology group $H^1(Gal(\bar{K}/K), \pi_1(\bar{X},\bar{y}))$.*

Continuing our investigation of the geometric étale fundamental group, we prove the following proposition which asserts that the étale fundamental group of a hyperbolic curve over an algebraically closed field has trivial center.

Proposition 1.9.2. *Let X be a projective, smooth, connected curve of genus $g > 1$, over an algebraically closed field K. Then $\pi_1(X)$ has trivial center.*

Proof. Take an arbitrary central element $g \in \pi_1(X)$. As finite quotients G of $\pi_1(X)$ correspond to G-covers $Y \to X$, for any such Y, g induces an automorphism of Y/X. Obviously it suffices to show all these automorphisms for all Y are trivial. Take a prime l different from the characteristic p of K, and consider the following diagram

$$\pi_1(Y) \twoheadrightarrow \pi_1^{ab}(Y) \twoheadrightarrow \pi_1^{ab}(Y)(l) \cong \mathbb{Z}_l^{2g_Y} \cong H_1(Y,\mathbb{Z}_l) := \left(H^1(Y,\mathbb{Z}_l)\right)^{\vee}.$$

On the other hand, $\pi_1(Y)$ is the kernel of the projection $\pi_1(X) \twoheadrightarrow G$, and the induced action of the automorphism g on $\pi_1(Y)$ is conjugation by g, and hence is trivial since g is central. In particular, g also acts trivially on $H^1(Y,\mathbb{Z}_l)$. Now if $g \neq Id$, then by Lefschetz trace formula one can compute the number of fixed points of g as

$$\#\mathrm{Fix}(g) = \sum_i (-1)^i tr(g \mid H^i(Y,\mathbb{Z}_l)) = 2 - 2g_Y = \#G.(2 - 2g_X) < 0,$$

which is a contradiction, and hence the claim. □

Remark 1.9.3. *One application of the above Proposition is as follows. Consider again a projective, smooth, geometrically connected curve X over a field K, and denote by $\bar{X}$ it's base extension to the algebraic closure $\bar{K}$ of K. By the exact sequence $0 \to \pi_1(\bar{X}) \to \pi_1(X) \to Gal(\bar{K}/K) \to 0$, $Gal(\bar{K}/K)$ acts on $\pi_1(\bar{X})$ via an outer automorphism, i.e. via a morphism $\vartheta : Gal(\bar{K}/K) \to Out(\pi_1(\bar{X}))$. Then we have the following commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_1(\bar{X}) & \longrightarrow & \pi_1(X) & \longrightarrow & Gal(\bar{K}/K) & \longrightarrow & 0 \\ & & \downarrow \psi & & \downarrow \varphi & & \downarrow \vartheta & & \\ 0 & \longrightarrow & Inn(\pi_1(\bar{X})) & \longrightarrow & Aut(\pi_1(\bar{X})) & \longrightarrow & Out(\pi_1(\bar{X})) & \longrightarrow & 0 \end{array}$$

where ψ and φ are induced by conjugation. But by the above proposition, $\pi_1(\bar{X})$ is centerless and hence ψ is an isomorphism. This implies that $\pi_1(X)$ is uniquely determined as an extension of $Gal(\bar{K}/K)$ by $\pi_1(\bar{X})$, by the homomorphism ϑ. Simply because one obtains from the above diagram that

$$\pi_1(X) \cong Aut(\pi_1(\bar{X})) \times_{Out(\pi_1(\bar{X}))} Gal(\bar{K}/K).$$

1.10 Open Curves

In this final section we are going to study the étale fundamental group of open curves. Let $\bar{X}$ be a smooth projective relative curve over the base scheme S with geometrically connected fibers, let $D \subset \bar{X}$ be a relative divisor which is finite and étale over S, and finally put $X := \bar{X} \setminus D$ (in the following, the base scheme S is going to be spectrum of either a field or a complete DVR). Our goal in this section is to study G-covers of X, where $\#G$ is invertible in $\mathcal{O}_S$. In particular, this gives us some information about the prime to p part of the étale fundamental group of an open curve over a field of characteristic p.

Let us begin by a rough sketch of the involving ideas. In order to lift G-covers of an open curve over a field of characteristic p to characteristic 0, we start by trying to projectivize the curve and it's cover so that we can apply algebraicity theorems to the formal lifts. But, obviously the difficulty is that in the process of projectivization, we loose étaleness and we might get ramifications at the points at infinity. On the other hand, if the order of the Galois group G is assumed to be prime to p, the resulting ramifications will be tame and we can unfold them by appropriate use of the Abhyankar's Lemma. Here is the formal and more detailed treatment of these ideas.

First assume $S = Spec(k)$, where k is a field of characteristic p, and $D = \{x_1, \ldots, x_r\}$ is a finite set of k-rational points. For any G-cover $Y \to X$, since Y is a smooth curve over k, it has a regular projective completion $\bar{Y}$. In fact, $\bar{Y}$ is the normalization of $\bar{X}$ in Y, hence $\bar{Y} \to \bar{X}$ is finite flat, and we have the following commutative diagram

$$\begin{array}{ccc} Y & \hookrightarrow & \bar{Y} \\ \downarrow & & \downarrow \\ X & \hookrightarrow & \bar{X} \end{array}$$

For any $1 \leq i \leq r$, consider $V_i := \mathcal{O}_{\bar{X},x_i}$, which is a discrete valuation ring, and let π_i be a uniformizer for V_i. For any point $y_i \in \bar{Y}$ lying above x_i, we can form the decomposition group D_i, the inertia group I_i, and the wild inertia group P_i of y_i over x_i. Since $\bar{Y}/\bar{X}$ has Galois group G and $\#G$ is not divisible by p, by general theory (see section A.3) one knows that $P_i = (1)$, and $I_i \hookrightarrow \kappa(y_i)^*$ is a cyclic group of order prime to p. Hence if we consider $W_i := \mathcal{O}_{\bar{Y},y_i}$ and take a uniformizer π'_i for it, then W_i/V_i is tamely ramified of ramification degree $e_i := \#I_i$ and thus one has $\pi_i = u_i(\pi'_i)^{e_i}$ where $u_i \in V_i$ is a unit. Let n be the least common multiple of e_i's, and consider the following diagram:

$$\begin{array}{ccc} V[\sqrt[n]{\pi_i}] & \overset{\psi}{\hookrightarrow} & W[\sqrt[n/e_i]{\sqrt[e_i]{u_i}.\pi'_i}] \\ \uparrow & & \uparrow \\ V & \hookrightarrow & W \end{array}$$

Then ψ is étale by Abhyankar's Lemma (see Lemma A.3.2). Using these ideas, we are going to show that the lift of X to characteristic zero admits the same G-covers as X.

Consider now $S = Spec(V)$, where V is a complete discrete valuation ring with algebraically closed residue field k of characteristic p. We denote the close point of S by s, and its open point by η. Let $X \to S$ be the complement in $\bar{X}$ of the relative divisor $D = \{x_1, \ldots, x_r\}$ with $x_i \in \bar{X}(V)$, and let t_i be a local parameter for a neighbourhood $\mathcal{U}_i$ of x_i. Finally assume $Y_s \to X_s$ is a given G-cover, which as mentioned above extends to a finite flat cover $\bar{Y}_s \to \bar{X}_s$. Let n be the cardinality of G, which by assumption is prime to p, and thus V contains n-th roots of unity μ_n. For any $1 \leq i \leq r$, let $\mathcal{V}_i \to \mathcal{U}_i$ be the finite cover defined by $\sqrt[n]{t_i}$. Extend the family of open sets $\mathcal{U}_i$ to a bigger family which forms an open covering of $\bar{X}$, and for these extra open sets $\mathcal{U}_i$, take $\mathcal{V}_i = \coprod_{\mu_n} \mathcal{U}_i$ to be the trivial cover on which μ_n acts by permuting the copies of $\mathcal{U}_i$. Consider the following coverings:

$$\mathcal{U} := \coprod_i \mathcal{U}_i \twoheadrightarrow \bar{X}\ , \quad \mathcal{V} := \coprod_i \mathcal{V}_i \xrightarrow{\text{finite flat}} \mathcal{U}$$

For any i, let $\bar{Z}_{i,s}$ be the normalization of $\bar{Y}_{i,s} := \bar{Y}_s \times_{\bar{X}_s} \mathcal{U}_{i,s}$ in $\mathcal{V}_{i,s}$, and consider the following diagram:

$$\begin{array}{ccccc} \bar{Y}_s & \longleftarrow & \bar{Y}_{i,s} & \longleftarrow & \bar{Z}_{i,s} \\ \downarrow & & \downarrow & & \downarrow {\scriptstyle \text{étale}} \\ \bar{X}_s & \longleftarrow & \mathcal{U}_{i,s} & \underset{\mu_n}{\longleftarrow} & \mathcal{V}_{i,s} \end{array}$$

Now, in order to lift the covering $\bar{Y}_s \to \bar{X}_s$ to a covering $\hat{\bar{Y}}_s \to \hat{\bar{X}}_s$ of formal schemes, we find liftings of $\bar{Y}_{i,s}$ on $\hat{\mathcal{U}}_{i,s}$, which are étale outside $\{t_i = 0\}$, and glue them to get a finite flat covering of $\hat{\bar{X}}_s$. For this purpose, we use the equivalence between étale schemes over a base and over it's nilpotent thickenings (see Lemma 1.5.1) to lift $\bar{Z}_{i,s}$ to $\hat{\bar{Z}}_{i,s}$ over $\hat{\mathcal{V}}_{i,s}$, which admits a μ_n action. Then the desired covering of $\hat{\mathcal{U}}_{i,s}$ will be the quotient of $\hat{\bar{Z}}_{i,s}$ by μ_n. More presicely, $\hat{\bar{Z}}_{i,s}$ is given by a complete ring B_i, and $\hat{\mathcal{V}}_{i,s}$ is also given by a complete ring A_i (which, locally at x_i, looks like $\hat{\mathcal{O}}_{\bar{X},x_i}[\sqrt[n]{t_i}] \supset \hat{\mathcal{O}}_{\bar{X},x_i}$). Now A_i is a regular 2-dimensional complete ring, and B_i is an étale algebra over it which admits a μ_n action. Take μ_n-invariants to obtain a finite flat $A_i^{\mu_n}$-algebra $B_i^{\mu_n}$. As an upshot, we get an algebraic finite cover $\bar{Y} \to \bar{X}$ such that for any i, the normalization of $\bar{Y} \times_{\mathcal{U}_i} \mathcal{V}_i$ is étale over $\mathcal{V}_i$.

Now we show that the above process of lifting is fully faithful. Let $\bar{Y}_i$ be the lift of $\bar{Y}_{i,s}$ for $i = 1, 2$. Then, by the usual bijection between homomorphisms and particular subsets of the product, one has:

$$Hom(\bar{Y}_{1,s}, \bar{Y}_{2,s}) = Hom(\bar{Y}_1, \bar{Y}_2) = Hom(\bar{Y}_{1,\bar{\eta}}, \bar{Y}_{2,\bar{\eta}}).$$

The last equlity is by replacing η by a finite extension, i.e. passing to the normalization V' of V in a finite field extension K' of K, and replacing $\mathcal{U}_i$ and $\mathcal{V}_i$ by their base extensions to V'. Hence one gets a fully faithful functor from the category of G-covers of X_s to the category of G-covers of $X_{\bar{\eta}}$.

Finally, we show that the above functor is an equivalence of categories. To do this, note that any G-cover $Y_{\bar{\eta}}$ of $X_{\bar{\eta}}$ comes from a G-cover $Y_{\eta'}$ of $X_{\eta'}$ for a finite separable extension η' of η. This étale cover extends to a finite flat cover $\bar{Y}_{\eta'} \to \bar{X}_{\eta'}$ such that the normalization of $\bar{Y}_{\eta'} \times_{\bar{X}_{\eta'}} \mathcal{U}_{i,\eta'}$ in $\mathcal{V}_{i,\eta'}$ is étale over $\mathcal{V}_{i,\eta'}$. After adjoining a proper root of a uniformizer π' of V', where V' corresponds to η', we may assume that the normalization $\bar{Y}$ of $\bar{X}$ in $\bar{Y}_{\eta'}$ is étale in all hight 1-points of $\mathcal{V}_i$, and hence by Purity (see Theorem 1.2.10), it actually is étale everywhere (see the proof of Theorem 1.7.5). Therefore, $\bar{Y}$ can be used as a model for lifting Y_s, which means that the original G-cover $Y_{\bar{\eta}}$ of $X_{\bar{\eta}}$ is induced by a G-cover Y_s of X_s.

The upshot of above arguments is the following result.

Proposition 1.10.1. *Let $S = Spec(V)$, where V is a complete discrete valuation ring with algebraically closed residue field of characteristic p, and let $\bar{X} \to S$ be a projective smooth relative curve with geometrically connected fibers with a relative divisor $D \subset \bar{X}$ which is finite and étale over S. Let s and η denote the close and open points of S, let $X \to S$ be the complement of D in $\bar{X}$, and let G be a finite group of order prime to p. Then*

$$Hom(\pi_1(X_s), G) = Hom(\pi_1(X_{\bar{\eta}}), G).$$

Now for any algebraically closed field of characteristic $p > 0$, by looking at the complete discrete valuation ring $V := W(k)$, one concludes that G-covers of an open curve over k are in bijection with G-covers over an algebraically closed field of characteristic 0. On the other hand, for any algebraically closed field k of characteristic 0, and any curve X over k, X is defined over a finitely generated subfield. Hence

$$Hom(\pi_1(X_k), G) = \varinjlim Hom(\pi_1(X_{\bar{k}_\alpha}), G),$$

where the direct limit is taken over all finitely generated subfields k_α of k. But since $\bar{k}_\alpha$ has finite trancendence degree over $\mathbb{Q}$, it can be identified with a subfield of the complex numbers and so we are reduced to $X/\mathbb{C}$.

From now on we may assume that we are over the field of complex numbers. So let X be the complement of finitely many points $\{x_1, \ldots, x_r\}$ in a projective smooth curve over $\mathbb{C}$, and $Y \to X$ be a G-cover. For any i, let Δ_i be a small disc around x_i which does not contain any of the other points at infinity. Then we know that finite topological G-covers of $\mathring{\Delta} := \Delta_i \setminus x_i$ are of the following form.

$$\coprod \mathring{\Delta} \xrightarrow{z \mapsto z^n} \mathring{\Delta},$$

and hence extend to ramified covers of Δ by adjoining proper roots of a local coordinate. Hence $Y \to X$ extends also to $\bar{Y} \to \bar{X}$ as a finite map of projective complex manifolds, which by GAGA is algebraic. As a result

$$\pi_1^{\mathrm{alg}} \cong \widehat{\pi_1^{\mathrm{top}}(X)}.$$

On the other hand, it is a well known fact that the topological fundamental group of the open Riemann surface X has the following presentation:

$$\pi_1^{\mathrm{top}}(X) = \{a_1, \ldots, a_g, b_1, \ldots, b_g, c_1, \ldots, c_r \mid (\prod_{i=1}^{g}[a_i, b_i]).(\prod_{j=1}^{r} c_j) = 1\}.$$

In particular, in the open case, that is when $r \geq 1$, the topological fundamental group of X is free on $2g + r - 1$ generators. If we denote by $\pi_1^{(p)}$ the maximal prime to p quotient of the étale fundamental group, which is the same as the full étale fundamental group if $p = 0$, we can put all the above together to obtain the following result.

Theorem 1.10.2. *Let k be an algebraically closed field of characteristic p, and let X be the complement of $r \geq 1$ points in a projective smooth curve of genus g over k. Then $\pi_1^{(p)}(X)$, as a topological group, is freely generated by $2g+r-1$ elements.*

Appendix A

Appendix

A.1 Kähler Differentials

In this short appendix, we briefly review the basic definitions and properties of the module of Kähler differentials culminating in the proof of the two fundamental exact sequences.

Definition A.1.1. *Let A be a ring and M be an A-module. A derivation d of A into M is an additive map $d : A \to M$ which satisfies the Leibnitz identity, that is for every $a, b \in A$ one has $d(ab) = ad(b) + bd(a)$. The A-module of all derivations of A into M is denoted by $Der(A, M)$. If R is a ring and A is an R-algebra, the set of derivations of A into M over R, denoted by $Der_R(A, M)$, is consists of those derivations which vanish on $R.1_A$.*

Definition A.1.2. *Let A be an R algebra. The module of Kähler differentials of A over R is an A-module $\Omega_{A/R}$ equipped with a derivation $d : A \to \Omega_{A/R}$ which satisfies the following universal property. For any derivation $D \in Der_R(A, M)$ there exists a unique A-linear map $f : \Omega_{A/R} \to M$ such that $D = f \circ d$.*

Definition A.1.3. *For any ring A and any A-module M, we define the ring $A * M$ to be the ring with underlying additive group $A \oplus M$ and the following product:*

$$(a_1, m_1).(a_2, m_2) := (a_1a_2, a_1m_2 + a_2m_1).$$

Note that M is an ideal in the ring with $M^2 = 0$ and one has a split exact sequence:

$$0 \to M \to A * M \to A \to 0.$$

Theorem A.1.4. *For any R-algebra A, the module of Kähler differentials $\Omega_{A/R}$ exists and is unique.*

Proof. Uniqueness follows easily from the universal property. For the existence, consider the algebra $B := A \otimes_R A$ and the homomorphisms $\epsilon : B \to A$, $\lambda_1, \lambda_2 : A \to B$ defined as:

$$\epsilon(a \otimes a') = aa' \ ; \ \ \lambda_1(a) = a \otimes 1 \ ; \ \ \lambda_2(a) = 1 \otimes a.$$

Let $I := Ker(\epsilon)$, put $\Omega_{A/R} := I/I^2$, and let $d : A \to \Omega_{A/R}$ be the differential induced by the map:

$$A \xrightarrow{\lambda_2 - \lambda_1} I \twoheadrightarrow I/I^2.$$

Now for any derivation $D : A \to M$ over R, we want to show the existence of a unique A-linear map $f : \Omega_{A/R} \to M$ such that $D = f \circ d$. For every $a, b \in A$, we have

$$a \otimes b = ab \otimes 1 + a(1 \otimes b - b \otimes 1) = \lambda_1 \epsilon(a \otimes b) + a(\lambda_2(b) - \lambda_1(b)).$$

Therefore, every element $\sum a_i \otimes b_i \in I$ has the form $\sum a_i(\lambda_2(b_i) - \lambda_1(b_i))$. This implies that $\overline{\sum a_i \otimes b_i} \in \Omega_{A/R}$ has the form $\sum a_i d(b_i)$ and thus the A-module $\Omega_{A/R}$ is generated by elements of the form $\{d(a)|a \in A\}$. This proves the uniqueness of f.

For the existence of f, consider the map $\varphi : B \to A * M$ defined as $\varphi(a \otimes b) = (ab, aD(b))$. Since $\varphi(I) \subset M$ and $M^2 = 0$, φ induces a homomorphism $\bar{\varphi} : B/I^2 = A * \Omega_{A/R} \to A * M$ and we have:

$$\bar{\varphi}(d(a)) = \varphi(1 \otimes a - a \otimes 1) = (0, D(a)).$$

So $f := \bar{\varphi}_{|\Omega_{A/R}}$ does the job. □

Example A.1.5. *Consider the polynomial R-algebra $A = R[X_1, \dots, X_n]$. We have seen in the proof of the above theorem that the A-module $\Omega_{A/R}$ is generated by $dX_1, \dots, dX_n$. We want to show that these elements are actually independent over A and thus $\Omega_{A/R} \cong \oplus_{i=1}^n A.dX_i$. First note that for every $1 \leq i \leq n$ taking partial derivative with respect to X_i defines a derivation $\partial/\partial X_i : A \to A$. Therefore, by the universal property, there exists an A-linear map $\lambda_i : \Omega_{A/R} \to A$ such that $\lambda_i(dX_j) = \partial X_j/\partial X_i = \delta_{ij}$. Now assume that $\sum_{i=1}^n c_i dX_i = 0$ in $\Omega_{A/R}$ with $c_i \in A$. For every $1 \leq i \leq n$, by applying λ_i to this relation we obtain that $c_i = 0$ and hence $\{dX_i\}_{i=1}^n$ forms a basis for $\Omega_{A/R}$ over A.*

The following results provide us with two fundamental exact sequences concerning Kähler differentials.

Theorem A.1.6. *Let $R \xrightarrow{\phi} A \xrightarrow{\psi} B$ be a sequence of ring homomorphisms. Then there is an exact sequence of B-modules of the form:*

$$\Omega_{A/R} \otimes_A B \xrightarrow{\alpha} \Omega_{B/R} \xrightarrow{\beta} \Omega_{B/A} \to 0.$$

Proof. The maps α and β are exident, namely $\alpha(d(a) \otimes b) = bd(\psi(a))$ and $\beta(bd(b')) = bd(b')$. Clearly, β is surjective and one has $\beta \circ \alpha = 0$. So it remains to show exactness in the middle. For this, it suffices to show that for every B-module M,

$$Hom(\Omega_{B/A}, M) \to Hom(\Omega_{B/R}, M) \to Hom(\Omega_{A/R} \otimes_A B, M)$$

is exact, or equivalently that

$$Der_A(B, M) \to Der_R(B, M) \to Der_R(A, M)$$

is exact. But the latter sequence is exact by definition, as every derivation of B over R which vanishes on A is a derivation over A. □

Theorem A.1.7. *Let R be a ring, A be an R-algebra, I be an ideal of A, and put $B := A/I$. Then the map $x \mapsto d(x) \otimes 1$ from I to $\Omega_{A/R} \otimes_A B$ induces a map $\delta : I/I^2 \to \Omega_{A/R} \otimes_A B$ and the following sequence of B-modules is exact:*

$$I/I^2 \xrightarrow{\delta} \Omega_{A/R} \otimes_A B \xrightarrow{\alpha} \Omega_{B/R} \to 0.$$

Proof. Surjectivity of α follows from that of $A \to B$ and obviously $\alpha \circ \delta = 0$. So it suffices to show that for every B-module M,

$$Hom(\Omega_{B/R}, M) \to Hom(\Omega_{A/R} \otimes_A B, M) \to Hom(I/I^2, M)$$

is exact, or equivalently that

$$Der_R(A/I, M) \to Der_R(A, M) \to Hom_A(I, M)$$

is exact. But in this form, the exactness is evident. □

A.2 Faithfully Flat Descent

Let $f : A \to B$ be a ring homomorphism. Then taking tensor product $- \otimes_A B$ defines a functor from the category of A-modules to the category of B-modules. Now, a natural question to ask is what objects and morphisms in the category of B-modules are in the image of this functor? In this section we show that the obvious necessary conditions are also sufficient if f is a faithfully flat map.

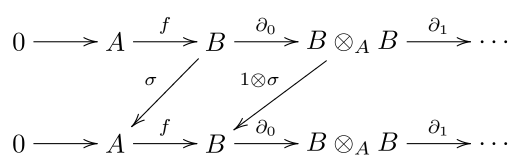
For a general ring homomorphism $f : A \to B$ define the Amitsur complex of B over A as follows:

$$\mathcal{A}_{B/A} : 0 \to A \xrightarrow{f} B \xrightarrow{\partial_0} B \otimes_A B \xrightarrow{\partial_1} B \otimes_A B \otimes_A B \xrightarrow{\partial_2} \dots,$$

where the differential maps are the alternating sums of the partial diagonals (for example $\partial_0(b) = b \otimes 1 - 1 \otimes b$).

Lemma A.2.1. *If $f : A \to B$ is faithfully flat, then the Amitsur complex $\mathcal{A}_{B/A}$ is exact.*

Proof. Let C be an arbitrary faithfully flat A-algebra. Since $\mathcal{A}_{(B \otimes_A C)/C} \cong \mathcal{A}_{B/A} \otimes_A C$, it suffices to show that $\mathcal{A}_{(B \otimes_A C)/C}$ is exact. By taking $C = B$, and noticing that the map $B \xrightarrow{1 \otimes f} B \otimes_A B$ admits a section $\sigma : B \otimes_A B \to B$, $\sigma(b_1 \otimes b_2) = b_1 b_2$, we may assume that the map $f : A \to B$ admits a section $\sigma : B \to A$ such that $\sigma \circ f = 1_A$. Now, σ defines the following homotopy:



between the zero and identity maps on the Amitsur complex. But this implies the desired exactness. □

Remark A.2.2. *Using the same notations and assumptions as above, a similar argument shows that for every A-module M, the corresponding Amitsur complex:*

$$0 \to M \to B \otimes_A M \xrightarrow{\partial_0} B \otimes_A B \otimes_A M \xrightarrow{\partial_1} \dots$$

is exact.

Corollary A.2.3. *Let $f : A \to B$ be a faithfully flat ring homomorphism and let M and N be A-modules. Then the following sequence is exact:*

$$0 \to Hom_A(M,N) \to Hom_B(B\otimes_A M, B\otimes_A N) \to Hom_{B\otimes_A B}(B\otimes_A B\otimes_A M, B\otimes_A B\otimes_A N).$$

Proof. Apply the left exact functor $Hom_A(M, -)$ to the exact sequence (Lemma A.2.1):

$$0 \to N \to B \otimes_A N \to B \otimes_A B \otimes_A N,$$

and use the identification $Hom_A(M,P) = Hom_C(C \otimes_A M, P)$ for every A-algebra C and every C-module P. □

Now suppose that M' is a B-module and $\varphi : p_1^*(M') \xrightarrow{\sim} p_2^*(M')$ is an isomorphism of $B\otimes_A B$-modules, where $p_1, p_2 : B \to B \otimes_A B$ send b to $b \otimes 1$ and $1 \otimes b$ respectively. For every $1 \le i < j \le 3$, define $\varphi_{ij} := p_{ij}^*(\varphi) : p_{ij}^*(p_1^*(M')) \xrightarrow{\sim} p_{ij}^*(p_2^*(M'))$. The following result is the faithfully flat descent for modules.

Lemma A.2.4. *With the above notations, if $\varphi_{13} = \varphi_{23} \circ \varphi_{12}$, then the A-module*

$$M := Ker(\varphi \circ p_1 - p_2) = \{u \in M' : \varphi(u \otimes 1) = 1 \otimes u\}$$

has the property that the canonical map $\lambda : B \otimes_A M \to M'$, $x \otimes u \mapsto xu$, is an isomorphism.

Proof. First, note that the $B\otimes_A B$-modules $p_1^*(M')$ and $p_2^*(M')$ can be identified in a canonical way with $M' \otimes_A B$ and $B \otimes_A M'$, where the latter modules are equipped with the diagonal action. So, taking these identifications into account, we consider φ as an isomorphism $\varphi : M' \otimes_A B \xrightarrow{\sim} B \otimes_A M'$. On the other hand, by definition of M, we have an exact sequence of the form:

$$0 \to M \to M' \xrightarrow{\tau} B \otimes_A M',$$

where $\tau : M' \to B \otimes_A M'$ is defined as $\tau(u) := 1 \otimes u - \varphi(u \otimes 1)$. Therefore, we have a diagram with exact rows (exactness of the second row follows from Lemma A.2.1):

$$\begin{array}{ccccccc}
0 & \longrightarrow & M \otimes_A B & \longrightarrow & M' \otimes_A B & \longrightarrow & B \otimes_A M' \otimes_A B \\
 & & \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle B\otimes_A \varphi} \\
0 & \longrightarrow & M' & \longrightarrow & B \otimes_A M' & \longrightarrow & B \otimes_A B \otimes_A M'
\end{array}$$

Finally, commutativity of the above diagram follows from the cocycle condition $\varphi_{13} = \varphi_{23} \circ \varphi_{12}$ and thus λ is an isomorphism. □

Remark A.2.5. *In the end, we would like to mention that there is a series of basic results in the theory of faithfully flat descent which assert that many classes of properties of the module M' are also satisfied by the descent module M. For example, this is the case for flatness, smoothness, unramifiedness, and étaleness. Let us finish up with sketches of the ideas. That flatness is preserved by faithfully flat descent is evident. So is the case of being unramified, as it is equivalent to the vanishing of Kähler differentials, which can be checked after a faithfully flat base change. For smoothness, note that knowing that the descent is flat, all we need to do is to check that the fibres are regular. But this can be checked after a faithfully flat base change. Finally, by combining the smooth and unramified cases, one gets the étale case.*

A.3 Higher Ramification Theory

Let A be a Noetherian normal integral domain with field of fractions K. Consider a finite Galois extension L over K with Galois group $G = Gal(L/K)$, and denote by B the normalization of A in L. Then the following basic facts are easy to check.

B is finite over A and for any prime ideal $\mathfrak{p}$ in A, there are finitely many primes $\mathfrak{q}_i \subset B$ which are above $\mathfrak{p}$ and moreover they are all conjugate under G. Let $D \subset G$ be the stabilizer of some $\mathfrak{q}$ above $\mathfrak{p}$. Call D the decomposition group of $\mathfrak{q}$. D operates on $B_\mathfrak{q}$ and also on $\kappa(\mathfrak{q})$. We denote the kernel of this action by I and call it the inertia group of $\mathfrak{q}$.

Now assume that $\mathfrak{p}$ has hight one, hence $B_\mathfrak{q}$ is a DVR. Let π be a generator of $\mathfrak{q}B_\mathfrak{q}$. For every $\sigma \in I$, we have $\sigma(\pi)/\pi \in B_\mathfrak{q} \twoheadrightarrow \kappa(\mathfrak{q})$ and this gives a homomorphism $I \to \kappa(\mathfrak{q})^*$. The image of this homomorphism is a finite cyclic group of order prime to $p = char(\kappa(\mathfrak{q}))$. The kernel is called the wild inertia which is a p-group by following argument.

Define subgroups $I_n \subset I$ as

$$I_n := \{\sigma \in I : \sigma \overset{\mathfrak{q}^{n+1}}{\equiv} Id\}.$$

For a given $\sigma \in I_n$, we have an $A_\mathfrak{p}$-linear derivation $B_\mathfrak{q} \to \mathfrak{q}^{n+1}/\mathfrak{q}^{n+2}$ which sends $b \in B_\mathfrak{q}$ to $\sigma(b) - b$. It is a derivation simply because $\sigma(b_1 b_2) - b_1 b_2 = (\sigma(b_1) - b_1).b_2 + \sigma(b_1).(\sigma(b_2) - b_2)$, and $\pi \mid (\sigma(b_1) - b_1)$. By definition it gives an injection

$$I_n/I_{n+1} \hookrightarrow Der(B_\mathfrak{q}, \mathfrak{q}^{n+1}/\mathfrak{q}^{n+2})$$

But $Der(B_\mathfrak{q}, \mathfrak{q}^{n+1}/\mathfrak{q}^{n+2})$ is annihilated by $p = char(\kappa(\mathfrak{p}))$, and hence the result is an exhustive filtration of the wild inertia with sub-quotients being p-groups. It is clear by definition that this filtration is separated as well, i.e. $\bigcap_n I_n = (Id)$, and hence the wild inertia, which is the same as I_0 in our notation, is a p-group.

Corollary A.3.1. *If $p \nmid \#G$, then $I_0 = (0)$ and hence the inertia group is cyclic.*

The following result is very useful in constructing some étale extensions (see 1.7)

Lemma A.3.2. (Abhyankar's Lemma) *Let V be a discrete valuation ring with fraction field K, and let L and K' be two tamely ramified Galois extensions of K with compositum L' over K. Suppose that n and m are the orders of inertia groups of L and K' respectively. If $n \mid m$, then L' is unramified over localizations of the normal closure V' of V in K'.*

Proof. Let W' be the normalization of V' in L', $\mathfrak{m}'$ a maximal ideal of V', $\mathfrak{n}'$ a maximal ideal of W' above $\mathfrak{m}'$, and $\mathfrak{n}$ the induced maximal ideal in the normalization W of V in L. Let G (resp. H, resp. M) be the Galois groups of L (resp. K', resp. L') over K, and G_i (resp. H_i, resp. M_i) the corresponding inertia groups of choosen places. Then M (resp. M_i) maps to $G \times H$ (resp. $G_i \times H_i$) in such a way that the projections $M \to G$ and $M \to H$ (resp. $M_i \to G_i$ and $M_i \to H_i$) are surjective. Since by hypothesis G_i and H_i are cyclics of orders n and m respectively, and $(mn, p) = 1$, M_i has order prime to p and hence is cyclic as well by above corollary. This is then easy to see that $M_i \xrightarrow{\sim} H_i$, but the kernel of this map is the inertia group of n' over m'. □

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