LIMIT GROUPS FOR RELATIVELY HYPERBOLIC GROUPS, I: THE BASIC TOOLS.

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Abstract. We begin the investigation of Γ-limit groups, where Γ is a torsion-free group which is hyperbolic relative to a collection of free abelian subgroups. Using the results of [16], we adapt the results from [21] and [22] to this context. Specifically, given a finitely generated group G, and a sequence of pairwise non-conjugate homomorphisms $\{h_n : G \to \Gamma\}$, we extract an $\mathbb{R}$-tree with a nontrivial isometric G-action.

This, along with the analogue of Sela's shortening argument allows us to prove the main result of this paper, that Γ is Hopfian.

In his remarkable series of papers [38, 39, 41], Z. Sela has classified those finitely generated groups with the same *elementary theory* as the free group of rank 2 (see also [40] for a summary). This class includes all nonabelian free groups, most surface groups, and certain other hyperbolic groups. In particular, Sela answers in the positive some long-standing questions of Tarski (Kharlampovich and Miasnikov have another approach to these problems; see [29]).

In [38], Sela begins with a study of *limit groups*. Sela's definition of a limit group is geometric, though it turns out that a group is a limit group if and only if it is a finitely generated fully-residually free group. Sela then produces *Makanin-Razborov diagrams*, which give a parametrization of $\mathrm{Hom}(G, \mathbb{F})$, where G is an arbitrary finitely generated group and $\mathbb{F}$ is a nonabelian free group (such a parametrisation is also given in [28]). Over the course of his six papers, two of the main tools Sela uses are the theory of isometric actions on $\mathbb{R}$-trees and the *shortening argument*.

Sela's work naturally raises the question of which other classes of groups can be understood using Sela's approach. Many of Sela's methods (and, more strikingly, some of the answers) come from geometric group theory. Thus it seems natural to consider, when looking for classes of groups to apply these methods to, groups of interest in geometric group theory. In [42], Sela considers an arbitrary torsion-free

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hyperbolic group Γ, and characterises those groups with the same elementary theory as Γ. Strikingly, any group which has the same elementary theory as a torsion-free hyperbolic group is itself a torsion-free hyperbolic group. This result exhibits a deep connection between the logic of groups and geometric group theory. In [2], Alibegović constructs Makanin-Razborov diagrams for limit groups. In [21] and [22] the author began this study for certain torsion-free CAT(0) groups.

In this paper, we generalise the results of [21] and [22] to the context of torsion-free groups which are hyperbolic relative to a collection of free abelian subgroups.[1]. We construct a space closely related to the Cayley graph, and use the results of Druţu and Sapir from [16] to analyse an asymptotic cone of this space. We then follow [21] to extract an $\mathbb{R}$-tree from this asymptotic cone. Armed with this $\mathbb{R}$-tree, we then follow [22] to consider the analogue of Sela's shortening argument in this context (see Section 7 below). Recall that a group is *Hopfian* if any surjective endomorphism is also injective. It seems that being Hopfian is a prerequisite for a group to be fruitfully studied using Sela's approach. The main result of this paper is the following

Theorem A. *Suppose that* Γ *is a torsion-free group which is hyperbolic relative to a collection free abelian subgroups. Then* Γ *is Hopfian.*

Our proof follows the proof of [22, Theorem 1.1], which in turn largely follows the proof of [37, Theorem 3.3]. We also prove the following much more straightforward application of the shortening argument (see Section 7 for a definition of $\mathrm{Mod}(\Gamma)$)

Theorem B. *Suppose that* Γ *is a torsion-free group which is hyperbolic relative to a collection of free abelian subgoups. Then* $\mathrm{Mod}(\Gamma)$ *has finite index in* $\mathrm{Aut}(\Gamma)$*.*

The true context of this paper is as the beginning of the study of Γ-limit groups where Γ is torsion-free and hyperbolic relative to free abelian subgroups. In a continuation paper [24] we use the results of this paper and of [42] to construct *Makanin-Razborov diagrams* for such a group Γ. It is our hope that much, possibly all, of Sela's program can be carried out for these groups.

The results of this paper subsume those of [21] and [22]. However, rather than reprove many of the results from these two papers in this very similar context we show how the results from [16] can be applied in this context so that the proofs from [21] and [22] apply *mutatis mutandis* in this more general setting.

[1] See Section 2 below for a definition and discussion of relatively hyperbolic groups.

The outline of this paper is as follows: In Section 1 we recall the concepts of *limit groups* and Γ-*limit groups*. In Section 2 we recall the definition of relatively hyperbolic groups, and the basic results required for this paper. In Section 3 we recall the concept of asymptotic cones and some results of Druţu and Sapir from [16]. In Section 4 we define a space X, closely related to the Cayley graph of the relatively hyperbolic group Γ. The space X, equipped with a natural Γ-action, is the appropriate space with which to study Γ-limit groups. In Section 5 a particular asymptotic cone X_ω is extracted from a sequence of non-conjugate homomorphisms $\{h_n : G \to \Gamma\}$, where G is an arbitrary finitely generated group and the limiting action of G on X_ω is studied. In Section 6 we extract an action of G on an $\mathbb{R}$-tree with no global fixed point. In Section 7 we present a version of Sela's shortening argument in the context of this paper. Finally in Section 8 we obtain the main results of this paper and discuss future work.

Acknowledgements. *The paper* [21] *was written before* [16] *appeared. Although I had already had a cursory glance at* [16], *I would like to thank both Mark Sapir and Jon McCammond for suggesting that the results in* [16] *may allow the results from* [21] *to be generalised as they have been in this paper. I would also like to thank Jason Manning and Indira Chatterji for fruitful discussions on the construction in Section 4.*

1. Limit groups and Γ-limit groups

Recall the following two definitions, due to Bestvina and Feighn [5].

Definition 1.1. [5, Definition 1.5] *Let G and Ξ be finitely generated groups, and let $\{h_n : G \to \Xi\}$ be a sequence of homomorphisms. The* stable kernel *of $\{h_n\}$, denoted $\underrightarrow{Ker}(h_n)$, is the set of all elements $g \in G$ so that $g \in ker(h_n)$ for all but finitely many n.*

The sequence $\{h_n\}$ is stable *if for all $g \in G$, either $g \in \underrightarrow{Ker}(h_n)$ or for all but finitely many n $g \notin ker(h_n)$.*

Definition 1.2. [5, Definition 1.5] *A* Ξ-limit group *is a group of the form $G/\underrightarrow{Ker}(h_n)$ where G is a finitely generated group and $\{h_n : G \to \Xi\}$ is a stable sequence of homomorphisms.*

Remark 1.3. *If each of the h_n is equal to h, a single homomorphism, then the sequence $\{h_n\}$ is certainly stable and the associated Ξ-limit group is just $h(G)$. In particular, all finitely generated subgroups of Ξ are Ξ-limit groups.*

A *limit group* is an $\mathbb{F}$-limit group, where $\mathbb{F}$ is a finitely generated free group. This terminology is due to Sela [38], although the definition that Sela gave was in terms on an action of G on an $\mathbb{R}$-tree induced by the sequence $\{h_n : G \to \mathbb{F}\}$. Sela's geometric definition also makes sense for δ-hyperbolic groups (see [42]). In this paper, we pursue a geometric definition of Γ-limit groups, where Γ is a torsion-free group which is hyperbolic relative to free abelian subgroups.

In case $\Xi = \mathbb{F}$, the geometric and algebraic definitions of Ξ-limit groups are the same (see [38, Lemma 1.3]). The two definitions are also the same when Ξ is a torsion-free δ-hyperbolic group (see [42, Lemma 1.3]). When Ξ is a torsion-free toral CWIF group (see [21] for the definition of these groups), a geometric definition of Ξ-limit group was given in [21, Definition 3.21] and it was proved [21, Theorem 5.1] that these two definitions are the same.

Suppose that Γ is a torsion-free group hyperbolic relative to free abelian subgroups. In this paper, we provide an appropriate geometric definition of Γ-limit group, in analogy with the definition from [21] (see Definition 5.7 below). It is proved in Theorem 6.5 that this definition is equivalent to Definition 1.2. As in Sela's definition, along with the geometric definition comes a faithful action of a (strict) Γ-limit group on an $\mathbb{R}$-tree.

The utility of the algebraic Definition 1.2 is that it has implications for the logic of Γ. In the case of Sela's limit groups, the nonabelian limit groups are exactly those that have the same universal theory as a nonabelian free group. In general, if $T_\forall(H)$ denotes the *universal theory* of a group H then we have the following (see [10] for a detailed discussion of this issue)

Lemma 1.4. *Let Ξ be a finitely presented group and suppose that L is a Ξ-limit group. Then $T_\forall(\Xi) \subseteq T_\forall(L)$.*

The utility of Sela's geometric definition is that it allows the application of the (Rips) theory of isometric actions on $\mathbb{R}$-trees, and Sela uses this to make a very deep study of limit groups (and of Γ-limit groups, where Γ is a torsion-free hyperbolic group). It turns out that the class of limit groups is exactly the class of fully residually free groups, which has been widely studied in the past.

2. Relatively hyperbolic groups

Recently there has been a large amount of interest in relatively hyperbolic groups. Relatively hyperbolic groups were originally defined by Gromov in his seminal paper [20], and an alternative definition was given by Farb [18]. Bowditch [6] gave two definitions, equivalent to Gromov's and Farb's, respectively (see [12] for a proof of the equivalence of the definitions). Recently, Druţu and Sapir [16] gave a characterisation of relatively hyperbolic groups in terms of their asymptotic cones. The results of this paper rely heavily on the results of [16].

Examples of relatively hyperbolic groups include: (i) geometrically finite Kleinian groups (which are hyperbolic relative to their cusp subgroups); (ii) fundamental groups of hyperbolic manifolds of finite volume (hyperbolic relative to their cusp subgroups); (iii) hyperbolic groups (relative to the trivial group, or a finite collection of quasi-convex subgroups); (iv) free products (relative to the factors); and (v) limit groups (relative to their maximal noncyclic abelian subgroups). See [18], [6], [13], [43] for details.

For further recent work on relatively hyperbolic groups, see [1], [44], [16], [17], [14] and [11] (among others).

The definition of relatively hyperbolic which we give is a hybrid of Farb's definition and a definition of Bowditch.

Definition 2.1 (Coned-off Cayley graph)**.** *Suppose that Γ is a finitely generated group, with finite generating set $\mathcal{A}$, and that $\{H_1, \ldots, H_m\}$ is a collection of finitely generated subgroups of Γ. Let X be the Cayley graph of Γ with respect to $\mathcal{A}$. We form the* coned-off Calyey graph*, $\tilde{X}$, by adding to X a vertex c_{γ,H_i} for each coset γH_i of a parabolic subgroup, and for each coset γH_i, an edge from c_{γ,H_i} to γ' for each $\gamma' \in \gamma H_i$.*

Definition 2.2. *We say that Γ is* hyperbolic relative to $\{H_1, \ldots, H_m\}$ *if*

(1) the coned-off Cayley graph $\tilde{X}$ is δ-hyperbolic for some δ; and
(2) for each edge $e \in \tilde{X}$, and each $n \geq 1$, there are only finitely many loops of length at most n which contain e.

Terminology 2.3. *Suppose that Γ is a group which is hyperbolic relative to the collection $\{H_1, \ldots, H_m\}$ of subgroups. The subgroups H_i are called* parabolic subgroups.[2]

In this paper we are concerned with torsion-free relatively hyperbolic groups Γ where all the parabolic subgroups are free abelian.

Definition 2.4. *A subgroup K of a group G is* malnormal *if for all $g \in G \smallsetminus K$ we have $gKg^{-1} \cap K = \{1\}$.*

A group G is CSA *if any maximal abelian subgroup of G is malnormal.*

Lemma 2.5. *Suppose that Γ is a torsion-free group which is hyperbolic relative to a collection of free abelian subgroups. Then Γ is CSA.*

[2]Alternative terminology for these subgroups is *peripheral* subgroups. Sometimes, all conjugates of the H_i are also called parabolic subgroups.

Proof. Let A be a maximal abelian subgroup of Γ.

First recall that the Bounded Coset Penetration property implies that any conjugate of a parabolic subgroup is malnormal (see [18, Example 1, p.819]). This implies that if M is a conjugate of a parabolic subgroup and A intersects M nontrivially then $A = M$, which is malnormal.

Suppose that A is a maximal abelian subgroup of Γ and that $g \in A \smallsetminus \{1\}$. If g is not contained in a conjugate of a parabolic subgroup then a result of Osin (see [30, Theorem 1.14, p.10] and the comment thereafter) implies that the centraliser of $\langle g \rangle$ is virtually cyclic. Since Γ is torsion-free, this centraliser is cyclic. Therefore, in this case $A = \langle h \rangle$ for some h. Suppose now that $\gamma \in \Gamma$ is such that $\gamma h^k \gamma^{-1} = h^j$ for some $k, j \in \mathbb{Z} \smallsetminus \{0\}$. Then [30, Corollary 4.21, p.83] implies that $|k| = |j|$. Thus, γ^2 commutes with h^j. This implies that $\gamma^2 \in \langle h \rangle$, which also implies that $\gamma \in \langle h \rangle$, so A is malnormal. ☐

3. Druţu and Sapir's results

In [16], Druţu and Sapir find a characterisation of relatively hyperbolic groups in terms of their asymptotic cones. In this section, we recall the definition of asymptotic cones and then briefly summarise those of their results necessary for this paper.

3.1. Asymptotic cones.

Asymptotic cones were introduced by van den Dries and Wilkie in [15] in order to recast and simplify Gromov's Polynomial Growth Theorem from [19]. See [16] for a discussion of other results about asymptotic cones. We briefly recall the definition of asymptotic cones.

Definition 3.1. *A* non-principal ultrafilter*, ω, is a $\{0,1\}$-valued finitely additive measure on $\mathbb{N}$ defined on all subsets of $\mathbb{N}$ so that any finite set has measure 0.*

The existence of non-principal ultrafilters is guaranteed by Zorn's Lemma. We fix once and for all a non-principal ultrafilter ω.[3] Given any bounded sequence $\{a_n\} \subset \mathbb{R}$ there is a unique number $a \in \mathbb{R}$ so that for all $\epsilon > 0$ we have $\omega(\{a_n \mid |a - a_n| < \epsilon\}) = 1$. We denote a by $\omega\text{-lim}\{a_n\}$. This notion of limit exhibits most of the properties of the usual limit (see [15]).

Let (X, d) be a metric space. Suppose that $\{\mu_n\}$ is a sequence of real numbers with no bounded subsequence, and that $\{x_n\}$ is a collection

[3] A different choice of ultrafilter can chance the resulting asymptotic cone in interesting ways, but not in a way that affects our results. Thus, we are unconcerned which ultrafilter is chosen.

of points in X. Let (X_n, d_n) be the metric space which has set X and metric $\frac{1}{\mu_n} d_X$. The *asymptotic cone of* X *with respect to* $\{x_n\}, \{\mu_n\}$ *and* ω, denoted X_ω, is defined as follows. First, define the set $\tilde{X_\omega}$ to consist of all sequences $\{y_n \mid y_n \in X_n\}$ for which $\{d_{X_n}(x_n, y_n)\}$ is a bounded sequence. Define a pseudo-metric $\tilde{d}$ on $\tilde{X_\omega}$ by

$$\tilde{d}(\{y_n\}, \{z_n\}) = \omega\text{-}\lim\{d_{X_n}(y_n, z_n)\}.$$

The asymptotic cone X_ω is the metric space induced by the pseudo-metric $\tilde{d}$ on $\tilde{X_\omega}$:

$$X_\omega := \tilde{X_\omega}/\sim,$$

where the equivalence relation '$\sim$' on $\tilde{X_\omega}$ is defined by: $x \sim y$ if and only if $\tilde{d}(x, y) = 0$. The pseudo-metric $\tilde{d}$ on $\tilde{X_\omega}$ naturally descends to a metric on d_ω on X_ω.

3.2. **Tree-graded spaces and relatively hyperbolic spaces.**

Definition 3.2. [16, Definition 1.10] *Let* Y *be a complete geodesic metric space, and let* $\mathcal{P}$ *be a collection of closed subsets of* Y *(called* pieces*). We say that the space* Y *is* tree-graded with respect to $\mathcal{P}$ *if the following two conditions are satisfied:*

- (T_1) *Each pair of distinct pieces intersect in at most a point; and*
- (T_2) *Every simple geodesic triangle (a simple loop composed of three geodesics) in* X *is contained in a single piece.*

It is worth remarking that in [21] it is proved that if Y is a CAT(0) space with isolated flats and relatively thin triangles then a particular asymptotic cone Y_ω of Y is tree-graded with respect to its collection of maximal flats (the proof of this is essentially contained in [27]). It was this fact that inspired the current paper. In [26], Hruska and Kleiner prove that if a cocompact CAT(0) space has isolated flats then it has relatively thin triangles.

One of the main results of [16] is the following

Theorem 3.3. [16, Theorem 1.11] *A finitely generated group* G *is relatively hyperbolic with respect to finitely generated subgroups* $H_1, \ldots, H_n$ *if and only if every asymptotic cone of* G *(with respect to any non-principal ultrafilter, any sequence of scaling constants, where the base-points are the identity of* G*) is tree-graded with respect to* ω*-limits of sequences of cosets of the subgroups* H_i*.*

Theorem 3.4. [16, Theorem 5.1] *Let* Y *be a metric space and let* $\mathcal{Q}$ *be a collection of subsets of* Y*. Let* $\mathfrak{q} : Y \to Y'$ *be a quasi-isometry. If* Y *is asymptotically tree-graded with respect to* $\mathcal{Q}$ *then* Y' *is asymptotically tree-graded with respect to* $\mathfrak{q}(\mathcal{Q})$*.*

Definition 3.5. [16, Definition 3.19] *Let $(Y, dist)$ be a metric space and let $\mathcal{Q} = \{Q_i \mid i \in I\}$ be a collection of subsets of Y. In every asymptotic cone Y_ω, with choice of basepoints $\{x_n\}$, we consider the collection of subsets*

$$\mathcal{Q}_\omega = \left\{ lim^\omega(Q_{i_n}) \mid (i_n)^\omega \in I^\omega \text{ such that } \left\{ \frac{dist(x_n, Q_{i_n})}{d_n} \right\} \text{ is bounded} \right\}.$$

The space Y is asymptotically tree-graded with respect to $\mathcal{Q}$ *if every asymptotic cone Y_ω is tree-graded with respect to $\mathcal{Q}_\omega$.*

Theorem 3.6. [16, Theorem 4.1] *Let $(Y, dist)$ be a geodesic metric space and let $\mathcal{Q} = \{Q_i \mid i \in I\}$ be a collection of subsets of Y. The space Y is asymptotically tree-graded with respect to $\mathcal{Q}$ if and only if the following properties are satisfied:*

- (α_1) *For every $\xi > 0$, the diameters of the intersections $\mathcal{N}_\xi(Q_i) \cap \mathcal{N}_\xi(Q_j)$ are uniformly bounded for all $i \neq j$;*
- (α_2) *For every $\theta \in [0, \frac{1}{2})$ there exists $M(\theta) > 0$ so that for every geodesic $\mathfrak{q}$ of length l and every $Q \in \mathcal{Q}$ with $\mathfrak{q}(0), \mathfrak{q}(l) \in \mathcal{N}_{\theta l}(Q)$ we have $\mathfrak{q}([0,l]) \cap \mathcal{N}_M(Q) \neq \emptyset$;*
- (α_3) *For every $k \geq 2$ there exists $\zeta > 0$, $\nu \geq 8$ and $\chi > 0$ such that every k-gon P in X with geodesic edges which is (ζ, ν, χ)-fat satisfies $P \subseteq \mathcal{N}_\chi(Q)$ for some $Q \in \mathcal{Q}$.*

4. The space X

In this paragraph we find a space X, closely associated to the Cayley graph of a relatively hyperbolic groups, which will be the appropriate space for our analysis of Γ-limit groups in the subsequent sections.

Suppose that Γ is a group which is hyperbolic relative to a collection $\{H_1, \ldots, H_m\}$ of subgroups.

Choose a generating set $\mathcal{A}$ for Γ which intersects each of the subgroups H_i in a generating set $\mathcal{B}_i$ for H_i, for $1 \leq i \leq m$. Let $\mathcal{B} = \cup_{i=1}^m \mathcal{B}_i$. For each $i \in \{1, \ldots, m\}$.

Let $d_\mathcal{A}$ be the word metric on Γ induced by the generating set $\mathcal{A}$ of Γ, and let $d_{\mathcal{B}_i}$ be the word metric on H_i induced by $\mathcal{B}_i$.

Let Y denote the Cayley graph of Γ with respect to $\mathcal{A}$, where each edge is isometric to the unit interval $[0,1]$. The group Γ acts on itself by left multiplication, which induces an isometric action on Y.

Let γH_i be a coset of some parabolic subgroup of Γ. The set $\mathcal{B}_i$ also gives a metric on γH_i, which we denote by $d_{\mathcal{B}_i}$. Now, [16, Lemma 4.3] states that there is a constant $K \geq 0$ so that for any $x, y \in \gamma H_i$, any geodesic joining x and y in Y stays entirely in the K-neighbourhood of γH_i.

We now build a space Y^k out of Y. Indira Chatterji told me of a similar construction which she attirbuted to David Epstein. Epstein proved an analogue of Theorem 4.3 for his space.

Consider a coset γH_i, along with the set of edges labelled by elements of $\mathcal{B}_i$. The resulting subgraph $Z(\gamma, H_i)$ of Y is exactly the Cayley graph of H_i. We now form a new graph $Z(\gamma H_i)^1$, which is another copy of $Z(\gamma, H_i)$, except that each edge is isometric to the closed interval $[0, \frac{1}{4}]$. Denote this new graph by $Z(\gamma, H_i)^1$, and join it to $Z(\gamma, H_i)$ by joining corresponding vertices by edges of length $\frac{1}{4}$. Perform this construction for each coset γH_i of a parabolic subgroup.

To build Y^j from Y^{j-1}, form $Z(\gamma, H_i)^j$ with edges of length 2^{-2j}, and join it to $Z(\gamma, H_i)^{j-1}$ by edges of length 2^{-2j}. Denote by $C(\gamma, H_i)^k$ the union of the graphs $Z(\gamma, H_i), Z(\gamma, H_i)^1, \ldots, Z(\gamma, H_i)^k$, along with the sets of edges that join successive graphs in this sequence.

There is also a natural space Y^∞, which is the metric completion of $\cup_{s=1}^\infty Y^s$, where we consider Y^s to be a subset of Y^{s+1}. Each coset γH_i inherits a 'cone-point' $w_{\gamma,i}$ from this completion process. In Y^∞, the point $w_{\gamma,i}$ lies at distance η from the coset γH_i, where $\eta = \sum_{s=1}^\infty 2^{-2s} < \frac{1}{2}$. It is clear that the space Y^∞ is quasi-isometric to the coned-off Cayley graph of Γ. Let Y^∞ be Υ-hyperbolic, and suppose without loss of generality that $\Upsilon > 1$.

Suppose that H_i is a parabolic subgroup of Γ, and γH_i is a coset of H_i in Γ. Associated to γH_i is the space $P_{\gamma,i}$, which is formed from Y by performing the above construction of Y^∞ to all cosets *except* the coset γH_i.

Lemma 4.1. *Suppose that γH_i is a coset of a parabolic subgroup in Γ, and that $x, y \in \gamma H_i$. Let $[x, y]$ be a geodesic between x and y in $P_{\gamma,i}$ and $[x, y]$ does not intersect γH_i except at its endpoints. Then $[x, y]$ lies entirely in the 35Υ-neighbourhood of γH_i in $P_{\gamma,i}$.*

Proof. Let $\widehat{[x, y]}$ be the image of $[x, y]$ in the space Y^∞, under the inclusion $P_{\gamma,i} \subset Y^\infty$. For any R, the R-neighbourhood of $w_{\gamma,i}$ in Y^∞ naturally corresponds to the $(R-\eta)$-neighbourhood of γH_i in $P_{\gamma,i}$, and if $R > \eta$ then outside of these balls the two spaces are locally isometric.

Suppose that $\widehat{[x, y]}$ is not contained in the 10Υ of $w_{\gamma,i}$ in Y^∞. That part of $\widehat{[x, y]}$ which lies at least 10Υ from $w_{\gamma,i}$ is a 10Υ-local-geodesic. By [7, Theorem III.H.1.13, p. 405], outside the 10Υ ball around $w_{\gamma,i}$ in Y^∞, the path $\widehat{[x, y]}$ is a $(\frac{14\Upsilon}{6\Upsilon}, 2\Upsilon)$-quasi-geodesic. However, the distance in Y^∞ which it travels outside of the 10Υ ball around $w_{\gamma,i}$ is at most 20Υ (since the path starts and finishes at distance $\eta < \frac{1}{2}$ from $w_{\gamma,i}$).

Therefore, the total distance that $\widehat{[x,y]}$ travels outside the 10Υ ball about $w_{\gamma,i}$ is at most

$$\left(\frac{7}{3}\right) 20\Upsilon + 2\Upsilon < 50\Upsilon.$$

Therefore $\widehat{[x,y]}$ is contained in the 35Υ ball around $w_{\gamma,i}$ in Y^∞. As above, this implies that $[x,y]$ is contained in the 35Υ-neighbourhood of γH_i in $P_{\gamma,i}$, as required. □

Lemma 4.2. *There exists a constant K_1, depending only on Y, and the set $\{\gamma H_i\}$, so that for all $x, y \in \gamma H_i$*

$$d_{P_{\gamma,i}}(x,y) \leq d_{\gamma H_i}(x,y) \leq K_1 d_{P_{\gamma,i}}(x,y).$$

Proof. Since $\gamma H_i \subset P_{\gamma,i}$, and since $d_{P_{\gamma,i}}$ is a path metric, the first inequality is immediate.

Let $x, y \in \gamma H_i$, and let $[x,y]$ be a geodesic between x and y in $P_{\gamma,i}$. By Lemma 4.1 above, the path $[x,y]$ lies entirely within the 35Υ-neighbourhood of γH_i.

Let $c_1, c_2, \ldots, c_k$ be points along $[x,y]$ which are such that $\eta \leq d_{P_{\gamma,i}}(c_i, c_{i+1}) \leq 1$ (this can be ensured if we choose each c_i to be either a vertex from Y or a cone-point $w_{\gamma',j}$). For each $1 \leq i \leq k$, let $b_i \in \gamma H_i$ be a point in γH_i as close as possible to c_i, and choose a geodesic $[c_i, b_i]$ (which has length at most 35Υ). Possibly $c_i = b_i$, and so the path $[c_i, b_i]$ is a constant path Also, choose a path $[b_i, b_{i+1}] \subset \gamma H_i$ of shortest length.

Consider the paths $p_i = [b_i, b_{i+1}]$ and $q_i = [b_i, c_i, c_{i+1}, b_{i+1}]$. The path q_i has length at most $70\Upsilon + 1$. Also, unless $[c_i, c_{i+1}] \subset \gamma H_i$, the path q_i intersects γH_i only at its endpoints.

The path q_i corresponds to a path $q_i' \subset Y$, where any part of q_i which passes through a cone-point is replaced by a path through the corresponding coset. Now, q_i' is a relative $(70\Upsilon + 1)$-quasi-geodesic. Also, p_i is a relative 2η-quasi-geodesic, and can be considered as a path in Y. Note that p_i penetrates γH_i while q_i' does not. Therefore, Bounded Coset Penetration implies that there is a constant $c = c(70\Upsilon + 1)$ so that p_i travels distance at most c in γH_i, which is to say that p_i has length at most c.

We have seen that each $[b_i, b_{i+1}]$ has length at most c. Therefore $d_{\gamma H_i}(x,y) \leq ck$. However, $d_{P_{\gamma,i}}(x,y) \geq \eta k$, and it suffices to take $K_1 = \frac{c}{\eta}$. □

Now let $P_{\gamma,i}^{(k)}$ be the space formed from Y by adding the spaces $C(\gamma', H_j)^k$ for all cosets of parabolic subgroups except γH_i. Then for all $x, y \in \gamma H_i$ we have

$$d_{P_{\gamma,i}}(x,y) \leq d_{P^{(k)}_{\gamma,i}}(x,y),$$

and so by Lemma 4.2 we have

$$d_{\gamma H_i}(x,y) \leq K_1 d_{P^{(k)}_{\gamma,i}}(x,y).$$

Theorem 4.3. *There exists $k \geq 0$ so that each of the graphs $Z(\gamma, H_i)^k$ is isometrically embedded in Y^k.*

Proof. Let $k > \frac{\log_2 K_1}{2}$.

Suppose that there exists $u, v \in Z(\gamma, H_i)^k$ so that a geodesic $[u, v]$ between u and v does not lie entirely within $Z(\gamma, H_i)^k$. Since $Z(\gamma, H_i)^k$ is isometrically embedded in $C(\gamma, H_i)^k$, the path $[u, v]$ cannot be contained in $C(\gamma, H_i)^k$.

Suppose that x is the point furthest along $[u, v]$ so that $[u, x] \subset C(\gamma, H_i)^k$. Let y be the point furthest along $[u, v]$ so that $[x, y]$ intersects $C(\gamma, H_i)^k$ only in $Z(\gamma, H_i)$. Because of the way $C(\gamma, H_i)^k$ was built, that part of $[u, v]$ immediately before x consists entirely of edges joining the different $Z(\gamma, H_i)^i$, from $Z(\gamma, H_i)^k$ to $Z(\gamma, H_i)$. Similarly, that part of $[u, v]$ immediately after y consists of a 'vertical' path from y to $Z(\gamma, H_i)$. Let x_1 be the final point in $[u, x]$ contained in $Z(\gamma, H_i)^k$ and let y_1 be the first point in $[y, v]$ contained in $Z(\gamma, H_i)^k$. Then we have $d_{\gamma H_i}(x, y) = 2^{2k} d_{Z(\gamma,H_i)^k}(x_1, y_1)$. Now let $D = d_{P^{(k)}_{\gamma,i}}(x, y)$, and note that $D \geq \eta$.

$$\begin{aligned} D + \frac{1}{2} &\leq d_{Z(\gamma,H_i)^k}(x_1, y_1) \\ &= 2^{-2k} d_{\gamma H_i}(x, y) \\ &\leq 2^{-2k} K_1 D. \end{aligned}$$

Therefore, $D + \frac{1}{2} \leq 2^{-2k} K_1 D$, which implies in particular that $2^{-2k} K_1 - 1 > 0$, contradicting our choice of k. This completes the proof. □

Choose k so that Theorem 4.3 holds.

Lemma 4.4. *There exists a function $f_1 : \mathbb{N} \to \mathbb{N}$ so that if $x, y \in Y^k$ are such x and y lie in the N-neighbourhood of γH_i and $[x, y]$ does not intersect $Z(\gamma, H_i)^k$ then $d(x, y) \leq f_1(N)$.*

Proof. By Theorem 4.3, and the way Y^k is constructed, it suffices to bound the length of a geodesic $[w, z]$ where $w, z \in \mathcal{N}_N(\gamma H_i)$ and $[w, z]$ does not intersect $C(\gamma, H_i)^k \smallsetminus Z(\gamma, H_i)$.

For such a pair we have $d_{Y^k}(w,z) = d_{P^{(k)}_{\gamma,i}}(w,z)$ Denote this distance by E. Let w_1, z_1 be points in γH_i which are closest to w and z, respectively. Also, let w_2, z_2 be the points in $Z(\gamma, H_i)^k$ which are closest to w_1 and z_1, respectively. Also, let $\eta_k = \sum_{i=1}^k 2^{-2i}$ be the distance from γH_i to $Z(\gamma, H_i)^k$. Then we have

$$\begin{aligned}
E &= d_{Y^k}(w,z) \\
&\leq d_{Y^k}(w_1,z_1) + 2N \\
&= d_{Z(\gamma,H_i)^k}(w_2,z_2) + 2N + 2\eta_k \\
&= 2^{-2k} d_{\gamma H_i}(w_1,z_1) + 2N + 2\eta_k \\
&= 2^{-2k} K_1 d_{P^{(k)}_{\gamma,i}}(w_1,z_1) + 2N + 2\eta_k \\
&\leq 2^{-2k} K_1 \left(d_{P^{(k)}_{\gamma,i}}(w,z) + 2N \right) + 2N + 2\eta_k \\
&= 2^{-2k} K_1 \left(E + 2N \right) + 2N + 2\eta_k,
\end{aligned}$$

which implies (since the choice of k from Theorem 4.3 ensures that $1 - 2^{-2k}K_1 > 0$) that

$$E \leq \frac{2^{-2k}K_1 N + 2N + 2\eta_k}{1 - 2^{-2k}K_1}.$$

This completes the proof. □

We now assume that Γ is torsion-free and that each of the parabolic subgroups of Γ are free abelian.

Glue into each subspace $Z(\gamma, H_i)^k$ of Y^k a copy of $\mathbb{R}^{n_i}$ where n_i is the rank of H_i, and $\mathbb{R}^{n_i}$ is equipped with the standard Euclidean (ℓ_2-) metric.

The resulting space is denoted X, and $\mathcal{Q}$ is the collection of copies of the $\mathbb{R}^{n_i}$ glued onto the cosets γH_i (where $i \in \{1, \ldots, m\}$ and $\gamma \in \Gamma$).

The copies of $\mathbb{R}^n$ that have been glued to Y^k to form the space X now play the role of cosets. They are isometrically embedded, and Lemma 4.4 above holds for these subspaces also, since lengths of paths are unchanged outside of $Z(\gamma, H_i)^k$, and distances can only get shorter inside $Z(\gamma, H_i)^k$.

The action of Γ on X is defined in the obvious way. The stabiliser in Γ of any $Q \in \mathcal{Q}$ is a conjugate of a parabolic subgroup, which acts by translations on Q.

4.1. **Properties of X.**

Lemma 4.5. *Suppose $Q \in \mathcal{Q}$ is a copy of $\mathbb{R}^{n_i}$ in X as above. Then Q is isometrically embedded.*

Given this lemma, we call the elements $Q \in \mathcal{Q}$ 'flats'. (Note, since it is possible that some H_i is cyclic, it is possibly that some flat is isometric to $\mathbb{R}$, unlike in [25] and [21]).

Lemma 4.6. *Left multiplication of Γ on itself induces an isometric action of Γ on X. This action is proper and cocompact.*

Lemma 4.7. *X is asymptotically tree-graded with respect to the set $\mathcal{Q}$.*

Proof. The inclusion map $\Gamma \hookrightarrow X$ is a quasi-isometry.

We know that Γ is asymptotically tree-graded with respect to the set of cosets γH_i. By the proof of [16, Theorem 5.1, p.44], X is asymptotically tree-graded with respect to $\mathcal{Q}$. □

Note also that any asymptotic cone of Γ is bi-Lipschitz homeomorphic to the analogous asymptotic cone of X (taking the same basepoints, and the same scaling factors). The utility of using X rather than just Γ is the following

Lemma 4.8. *Suppose that X_ω is an asymptotic cone of X. Then each piece of X_ω is isometric to $(\mathbb{R}^k, d)$ for some k, where d is the standard Euclidean metric on $\mathbb{R}^k$.*

In analogy with Hruska's definition of *Isolated Flats* for CAT(0) spaces (see [25, 2.1.2]), we note the following

Lemma 4.9 (Isolated Flats)**.** *Let $\mathcal{Q}$ be the collection of flats in X. Then there is a function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that for every pair of distinct flats $Q_1, Q_2 \in \mathcal{Q}$ and for every $k \geq 0$, the intersection of the k-neighbourhoods of Q_1 and Q_2 has diameter less than $\phi(k)$.*

Proof. This is merely a restatement of Theorem 3.6.(α_1). □

Convention 4.10. *Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be as in Lemma 4.9. We suppose that $\phi(k) \geq k$ for all $k \geq 0$ and that ϕ is a nondecreasing function.*

We now prove a quasi-convexity result for the metric on X

Lemma 4.11. *There exists a function $N_1 ; \mathbb{N} \to \mathbb{N}$ so that for any K, if $x_1, x_2, y \in X$ so that $d_X(x_1, x_2) \leq K$ and $[x_1, y]$ and $[x_2, y]$ are geodesics, then $[x_1, y]$ is contained in the $N_1(K)$-neighbourhood of $[x_2, y]$ (and vice versa).*

Proof. Choose a geodesic $[x_1, x_2]$. Then the path $[y, x_1, x_2] = [y, x_1] \cup [x_1, x_2]$ is a $(1, K)$-quasi-geodesic.

By [16, Theorem 1.12] there are constants τ and M so that:

• $[y, x_1, x_2]$ is contained in the τ-tubular neighbourhood of the M-saturation of $[y, x_2]$ (see [16, Definition 8.9] for a definition of M-saturations); and

• the points at which $[y, x_1, x_2]$ enters and leaves the τ-neighbourhood of flats in the M-saturation of $[y, x_2]$ are at bounded distance from $[a, x_1]$.

By Lemma 4.4, and the fact the flats are isometric to $\mathbb{R}^n$, the path $[y, x_1, x_2]$ lies in the D_2-neighbourhood of $[y, x_2]$ for some constant D_2. A symmetric argument on $[y, x_2, x_1]$ and $[y, x_1]$ implies that $[y, x_2, x_1]$ lies in the D_2-neighbourhood of $[y, x_1]$. □

For our purposes, one of the most important properties of the space X is contained in the following theorem, which shows that geodesic triangles in X satisfy the *Relatively Thin Triangles Property* (see [25, Definition 3.1.1]).

Theorem 4.12. *Suppose that X is as constructed above. There exists $\delta > 0$ so that for any $a, b, c \in X$, and any $\Delta(a, b, c)$ is a geodesic triangle, either (i) $\Delta(a, b, c)$ is δ-thin in the usual sense; or else (ii) there is a unique flat $E \subset X$ so that each side of $\Delta(a, b, c)$ is contained in the δ-neighbourhood of the union of E and the other two sides.*

Proof. Choose a geodesic triangle $\Delta(a, b, c)$ in X, with a choice of geodesics $[a, b]$, $[b, c]$ and $[a, c]$.

By [16, Lemma 8.16] and [16, Lemma 8.17], there is a constant α such that one of two possibilities occurs: either

(i) there is a point $x \in X$ whose α-neighbourhood intersects all three of the geodesics $[a, b]$, $[b, c]$ and $[a, c]$ nontrivially; or

(ii) there is a flat $E \in \mathcal{Q}$ so that the α-neighbourhood of E intersects each of the three geodesics nontrivially.

In case (i), let x_1 be a point on $[a, b]$ which is within α of x, and let x_2 be a point on $[a, c]$ which is within α of x. Then $d_X(x_1, x_2) \le 2\alpha$.

In case (ii) let x_1 be the point on $[a, b]$ which is closest to a subject to being in the α-neighbourhood of E, and similarly for x_2 on $[a, c]$. Then [16, Corollary 8.14] implies that there is a constant D_1 so that $d_X(x_1, x_2) \le D_1$. We assume that $D_1 \ge 2\alpha$.

Therefore, in either case, there exist points $x_1 \in [a, b]$ and $x_2 \in [a, c]$ so that $d_X(x_1, x_2) \le D_1$. Denote by $[a, x_1]$ the sub-path of $[a, b]$ from a to x_1, and similarly for $[a, x_2] \subset [a, c]$. By Lemma 4.11, there is a constant D_2 so that $[a, x_1]$ lies in the D_2-neighbourhood of $[a, x_2]$, and vice versa.

We use a symmetric argument on the points b and c – finding points $y_1 \in [c, a]$, $y_2 \in [c, b]$ and $z_1 \in [b, a]$, $z_2 \in [b, c]$ as with x_1 and x_2.

Now, in case (i) above, we can take $x_1 = z_1$, $x_2 = y_1$ and $y_2 = z_2$, and we're done. In case (ii), we note that the path $[x_1, z_1] \subseteq [a, b]$ lies in the $N_1(\alpha)$neighbourhood of E, by Lemma 4.11, and similarly for $[x_2, y_1] \subseteq [a, c]$ and $[z_2, y_2] \subseteq [b, c]$.

Therefore, it suffices to take $\delta = \max\{D_2, N_1(\alpha)\}$. □

4.2. **Projecting to flats.** In this paragraph we record some results about projecting to flats which are required for the proofs in the subsequent sections.

Definition 4.13. [16, Definition 4.9] *Let $x \in X$ and $A \subset X$. The* almost projection *of x* onto *A is the set of points $y \in A$ so that $d_X(x, y) \leq d_X(x, A) + 1$.*

The following result follows immediately from [16, Corollary 8.14]

Lemma 4.14. *There exists a constant C_1 so that if $Q \in \mathcal{Q}$ and $x \in X$ then the almost projection of x onto Q has diameter at most C_1.*

The following result also follows immediately from [16, Corollary 8.14]

Lemma 4.15. *There exists a function $N_3 : \mathbb{N} \to \mathbb{N}$ so that if $x_1, x_2 \in X$, $Q \in \mathcal{Q}$ and $\pi(x_1), \pi(x_2)$ are in the almost projections of x_1 and x_2 to Q, respectively then $d_X(\pi(x_1), \pi(x_2)) \leq N_3(d_X(x_1, x_2))$.*

Again, we suppose that $N_3(x) \geq x$ for all $x \geq 0$ and that N_3 is a nondecreasing function.

Recall that δ is the constant from Theorem 4.12 and that $\phi : \mathbb{N} \to \mathbb{N}$ is the function from Lemma 4.9.

Lemma 4.16 (cf. Lemma 2.11, [21])**.** *Suppose that $\Delta = \Delta(a, b, c)$ is a geodesic triangle in X. If Δ is not $\left(\delta + \frac{\phi(\delta)}{2}\right)$-thin then Δ is δ-thin relative to a* unique *flat $Q \in \mathcal{Q}$.*

Proof. Given Lemma 4.9 and Theorem 4.12, the proof of [21, Lemma 2.11] applies directly. □

Lemma 4.17 (cf. Lemma 2.22, [21])**.** *Suppose that $Q \in \mathcal{Q}$, that $x, y \in Q$ and that $z \in X$. Let $[x, z]$ and $[y, z]$ be geodesics. Then there exist $u \in [x, z]$ and $v \in [y, z]$ that both lie in the 2δ-neighbourhood of Q such that*

$$d_X(u, v) \leq \phi(\delta).$$

Proof. Given Theorems 4.3 and 4.12 and Lemma 4.9, the proof of [21, Lemma 2.22] applies directly. □

Proposition 4.18 (cf. Proposition 2.23, [21]). *Suppose that $Q \in \mathcal{Q}$, that $x, y \in X$ and that some geodesic $[x, y]$ does not intersect the 4δ-neighbourhood of Q. Let $\pi(x), \pi(y)$ be in the almost projections of x and y, respectively. Then*

$$d_X(\pi(x), \pi(y)) \leq N_3(\phi(3\delta) + N_1(\phi(\delta)).$$

Proof. By Lemma 4.17 there exist $w_1 \in [\pi(x), y]$ and $w_2 \in [\pi(y), y]$, both in the 2δ-neighbourhood of Q such that $d_X(w_1, w_2) \leq \phi(\delta)$. By a similar argument as in the proof of Lemma 4.17 (see [21]), there are $u_1 \in [\pi(x), x]$ and $u_2 \in [\pi(x), y]$ which lie outside the 2δ-neighbourhood of E so that $d_X(u_1, u_2) \leq \phi(3\delta)$. Now $[w_1, y]$ is contained in the $N_1(\phi(\delta))$-neighbourhood of $[w_2, y]$, by Lemma 4.11. Therefore, there exists $u_3 \in [\pi(y), y]$ so that $d_X(u_2, u_3) \leq N_1(\phi(3\delta))$.

We can choose $\pi(u_1)$ and $\pi(u_3)$ in the almost projections of u_1, u_3 so that $\pi(u_1) = \pi(x)$ and $\pi(u_3) = \pi(y)$. Now, $d_X(u_1, u_3) \leq \phi(3\delta) + N_1(\phi(\delta))$, so by Lemma 4.15 we have

$$\begin{aligned} d_X(\pi(x), \pi(y)) &= d_X(\pi(u_1), \pi(u_3)) \\ &\leq N_3(d_X(u_1, u_3)) \\ &\leq N_3(\phi(3\delta) + N_1(\phi(\delta))), \end{aligned}$$

as required. □

5. Asymptotic cones and compactification

In this section we start with Γ, a finitely generated group which acts properly and cocompactly by isometries on a metric space (X, d_X), a finitely generated group G and a sequence $\{h_n : \Gamma \to G\}$ os homomorphisms. Using $\{h_n\}$ we construct a particular asymptotic cone X_ω, which is equipped with an isometric action of G with no global fixed point.

In the case of δ-hyperbolic groups and spaces, the construction we describe in this section is essentially due to Paulin [31, 32] (see also Bestvina [3] and Bridson–Swarup [8]), though was not cast there in terms of asymptotic cones. For CAT(0) spaces, this construction is performed by Kapovich and Leeb [27]. The general construction is similar. See [21] for more details about this construction and [15] or [16] for many properties about asymptotic cones.

Let G be a finitely generated group and Γ a torsion-free group which is hyperbolic relative to a collection of free abelian subgroups. Let $\mathcal{A}$ be a finite generating set for G, let X be the space constructed from a Cayley graph of Γ in Section 4, and let $x \in X$ correspond to the identity of Γ. If $h : G \to \Gamma$ is a homomorphism, define

$$\|h\| := \min_{\gamma\in\Gamma} \max_{g\in\mathcal{A}} d_X(x, (\gamma h(g)\gamma^{-1}).x),$$

and let γ_h be an element of Γ which realises this minimum.

Terminology 5.1. *We say that a pair of homomorphisms $h, h' : G \to \Gamma$ are* non-conjugate *if there is no inner automorphism $\tau : \Gamma \to \Gamma$ so that $h' = \tau \circ h$.*

Suppose that $\{h_i : G \to \Gamma\}$ is a sequence of pairwise non-conjugate homomorphisms. Then the sequence $\{\|h_n\|\}$ does not contain a bounded subsequence. Define the asymptotic cone X_ω with respect to some non-principal ultrafilter ω, the sequence of basepoints $x_n = x$ and the sequence of scaling factors $\mu_n = \|h_n\|$.

The action of G on X_ω is defined by $g.\{y_n\} = \{\gamma_{h_n} h_n(g)\gamma_{h_n}^{-1}.y_n\}$.

Lemma 5.2. *The action of G on X_ω has no global fixed point.*

Proof. See [21, Lemma 3.9], where the proof does not use the CAT(0) property. □

5.1. **The action of G on X_ω.** Suppose now that Γ is a torsion-free group which is hyperbolic relative to a collection $\{H_1, \ldots, H_m\}$ of free abelian subgroups. Let G be a finitely generated group with finite generating set $\mathcal{A}$ and let $\{h_i : G \to \Gamma\}$ be a sequence of pairwise non-conjugate homomorphisms.

Let (X, d_X) be the metric space constructed in Section 4 above.

Suppose that X_ω is the asymptotic cone of X constructed from $\{h_n\}$ as above, with basepoint x_ω corresponding to the constant sequence $\{x_n = x\}$. By Lemma 5.2 above, X_ω is endowed with an isometric action of G without global fixed points.

Define a separable G-invariant subspace $\mathcal{C}_\infty$ of X_ω to be the union of (i) the geodesic segments $[x_\omega, g.x_\omega]$ for all $g \in G$; and (ii) the flats $Q_\omega \subseteq X_\omega$ which contain simple geodesic triangle which are contained in $\Delta(g_1.x_\omega, g_2.x_\omega, g_3.x_\omega)$ for some g_1, g_2, g_3.

Lemma 5.3. *The space $\mathcal{C}_\infty$ is (i) separable; (ii) G-invariant; (iii) convex in X_ω; and (iv) tree-graded and the pieces are isometric to $(\mathbb{R}^n, \ell_2)$, for some n (which may vary according to the piece).*

Suppose that $\{(Y_n, \lambda_n)\}_{n=1}^\infty$ and (Y, λ) are pairs consisting of metric spaces, together with actions $\lambda_n : G \to \mathrm{Isom}(Y_n)$, $\lambda : G \to \mathrm{Isom}(Y)$. Recall (cf. [5, §3.4, p. 16]) that $(Y_n, \lambda_n) \to (Y, \lambda)$ in the *G-equivariant Gromov topology* if and only if: for any finite subset K of Y, any $\epsilon > 0$ and any finite subset P of G, for sufficiently large n, there are subsets

K_n of Y_n and bijections $\rho_n : K_n \to K$ such that for all $s_n, t_n \in K_n$ and all $g_1, g_2 \in P$ we have

$$|d_Y(\lambda(g_1).\rho_n(s_n), \lambda(g_2).\rho_n(t_n)) - d_{Y_n}(\lambda_n(g_1).s_n, \lambda_n(g_2).t_n)| < \epsilon.$$

To a homomorphism $h : G \to \Gamma$, we naturally associate a pair (X_h, λ_h) as follows: let X_h be the convex hull in X of $G.x$ (where x is the basepoint of X), endowed with the metric $\frac{1}{\mu_h} d_X$; and let $\lambda_h = \iota \circ h$, where $\iota : \Gamma \to \mathrm{Isom}(X)$ is the fixed homomorphism.

Let $\lambda_\infty : G \to \mathrm{Isom}(\mathcal{C}_\infty)$ denote the action of G on $\mathcal{C}_\infty$.

Proposition 5.4. [21, Lemma 3.15] *If there is a separable G-invariant subspace $\mathcal{C}$ of X_ω which contains the basepoint x_ω of X_ω then there is a subsequence $\{f_i\}$ of $\{h_i\}$ so that $(X_{f_i}, \lambda_{f_i}) \to (\mathcal{C}_\infty, \lambda_\infty)$ in the G-equivariant Gromov topology.*

For the remainder of the paper, we will assume that we have passed to the convergent subsequence $\{f_i\}$ of $\{h_i\}$. Thus, we will denote X_{f_i} by X_i, and λ_{f_i} by λ_i.

Lemma 5.5. [21, Corollary 3.17] *Let $\mathcal{F}_\infty$ be the set of flats in $\mathcal{C}_\infty$. For each $E \in \mathcal{F}_\infty$ there is a sequence $\{E_i \subset X_i\}$ so that $E_i \to E$ in the G-equivariant Gromov topology.*

Observation 5.6. *The action of G on $\mathcal{C}_\infty$ has no global fixed point.*

Definition 5.7. *Suppose that G and Γ are finitely generated groups and that $\{h_i : G \to \Gamma\}$ is a sequence of pairwise non-conjugate homomorphisms, leading to an isometric action of G on $\mathcal{C}_\infty$, where $\mathcal{C}_\infty$ is constructed from X_ω, the asymptotic cone of Γ, as above. Let K_∞ be the kernel of the action of G on $\mathcal{C}_\infty$:*

$$K_\infty = \{g \in G \mid \forall y \in \mathcal{C}_\infty, g.y = y\}.$$

The strict Γ-limit group *is $L_\infty = G/K_\infty$.*

A Γ-limit group *is a group which is either a strict Γ-limit group as above or else a finitely generated subgroup of Γ.*

The following result is clear from the definition of the Gromov topology.

Lemma 5.8. *Suppose that the sequence of homomorphisms $\{f_i : G \to \Gamma\}$ gives rise to a sequence of actions converging to an action of G on $\mathcal{C}_\infty$, and that K_∞ is the kernel of the action of G on $\mathcal{C}_\infty$. Then $\underrightarrow{Ker}(f_i) \subset K_\infty$.*

The following results give information about the flats in $\mathcal{C}_\infty$, and their stabilisers in G.

Proposition 5.9 (cf. [21], Lemma 3.18)**.** *Suppose $g \in G$ leaves a flat $E \subseteq \mathcal{C}_\infty$ (setwise) invariant, and that $\{E_j\}$ converges to E. Then for all but finitely many i we have $f_i(g).E_i = E_i$.*

Proof. The proof of [21, Lemma 3.18] applies directly. □

Proposition 5.10 (cf. [21], Lemma 3.19)**.** *Suppose $g \in G$ lies in Stab(E) for some flat $E \subseteq \mathcal{C}_\infty$. Then g acts (possibly trivially) by translation on E.*

Proof. The proof of [21, Lemma 3.19] applies directly, once we notice that an element of $\gamma \in \Gamma$ which leaves a flat in X invariant lies in a conjugate of a parabolic subgroup and acts by Euclidean translations on the flat. □

6. The $\mathbb{R}$-tree T

6.1. **Constructing the $\mathbb{R}$-tree.** We now follow [21] to construct from $\mathcal{C}_\infty$ an $\mathbb{R}$-tree T equipped with an isometric G-action with no global fixed point. Given the construction of $\mathcal{C}_\infty$ in the previous section, the construction of T is exactly the same as in [21]. We repeat the definition of T here.

Let $\mathcal{F}_\infty$ be the collection of all pieces in $\mathcal{C}_\infty$. By Definition 3.2, for any $g \in G$ exactly one of the following holds: (i) $g.E = E$; (ii) $|g.E \cap E| = 1$; or (iii) $g.E \cap E = \emptyset$. By Lemmas 5.9 and 5.10, Stab(E) is a finitely generated free abelian group, acting by translations on E (possibly not faithfully).

Let $\mathcal{D}_E$ be the set of directions of the translations of E by elements of Stab(E).

For each element $g \in G \setminus \text{Stab}(E)$, let $l_g(E)$ be the (unique) point where any geodesic from a point in E to a point in $g.E$ leaves E, and let $\mathcal{L}_E$ be the set of all $l_g(E) \subset E$. Note that if $g.E \cap E$ is nonempty (and $g \notin \text{Stab}(E)$) then $g.E \cap E = \{l_g(E)\}$.

Since G is finitely generated, and hence countable, both sets $\mathcal{D}_E$ and $\mathcal{L}_E$ are countable. Given a (straight) line $p \subset E$, let χ_E^p be the projection from E to p. Since $\mathcal{L}_E$ is countable, there is a line $p_E \subset E$ such that:

(1) the direction of p_E is not orthogonal to a direction in $\mathcal{D}_E$; and
(2) if x and y are distinct points in $\mathcal{L}_E$, then $\chi_E^{p_E}(x) \neq \chi_E^{p_E}(y)$.

Project E onto p_E using $\chi_E^{p_E}$. The action of Stab(E) on p_E is defined in the obvious way (using projection) – this is an action since the action of Stab(E) on E is by translations. Connect $\mathcal{C}_\infty \setminus E$ to p_E in the obvious way – this uses the following

Observation 6.1. *Suppose S is a component of $\mathcal{C}_\infty \smallsetminus E$. Then there is a (unique) point $x_S \in E$ so that S is a component of $\mathcal{C}_\infty \smallsetminus \{x_S\}$.*

Glue such a component S to p_E at the point $\chi_E^{p_E}(x_S)$.

Perform this projecting and gluing construction in an equivariant way for all flats $E \subseteq \mathcal{C}_\infty$ – so that for all $E \subseteq \mathcal{C}_\infty$ and all $g \in G$ the direction of the lines $p_{g.E}$ and $g.p_E$ is the same (this is possible since the action of $\mathrm{Stab}(E)$ on E is by translations, so doesn't change directions).

Having done this for all flats $E \subseteq \mathcal{C}_\infty$, we arrive at a space T which is endowed with the (obvious) path metric.

An isometric action of G on T is defined in the obvious way from the action of G on X_ω.

The space T has a distinguished set of geodesic lines, namely those of the form p_E, for $E \in \mathcal{F}_\infty$. Denote the set of such geodesic lines by $\mathbb{P}$.

The following lemma is [21, Lemma 4.2], and the proof there holds in the current situation.

Lemma 6.2. *The space T is an $\mathbb{R}$-tree and has an isometric G-action with no global fixed point.*

Remark 6.3. *Since $K_\infty \leq G$ acts trivially on $\mathcal{C}_\infty$, it also acts trivially on T, and the action of G on T induces an isometric action of L_∞ on T.*

6.2. **The actions of G and L_∞ on T.** Let G be a finitely generated group, and Γ a torsion-free group which is hyperbolic relative to a collection of free abelian subgroups. Suppose that $\{h_i : G \to \Gamma\}$ is a sequence of pairwise non-conjugate homomorphisms. Let X_ω, $\mathcal{C}_\infty$ and T be as in Sections 5 and 5.1 and Subsection 6.1, repsectively. Let $\{f_i : G \to \Gamma\}$ be the subsequence of $\{h_i\}$ as in the conclusion of Proposition 5.4. Let K_∞ be the kernel of the action of G on $\mathcal{C}_\infty$ and let $L_\infty = G/K_\infty$ be the associated strict Γ-limit group.

Theorem 6.4. *[cf.* [38], *Lemma 1.3;* [21], *Theorem 4.4] In the above situation, the following properties hold.*

1. *Suppose that $[A, B]$ is a nondegenerate segment in T. Then $Stab_{L_\infty}[A, B]$ is an abelian subgroup of L_∞;*
2. *If T is isometric to a real line then for all but finitely many n the group $f_n(G)$ is free abelian. Furthermore, in this case L_∞ is free abelian;*
3. *If $g \in G$ fixes a tripod in T pointwise then $g \in \underrightarrow{Ker}(f_i)$;*

(4) *Let* $[y_1, y_2] \subset [y_3, y_4]$ *be a pair of non-degenerate segments of* T *and assume that* $Stab_{L_\infty}[y_3, y_4]$ *is non-trivial. Then*

$$Stab_{L_\infty}[y_1, y_2] = Stab_{L_\infty}[y_3, y_4].$$

In particular, the action of L_∞ *on the* $\mathbb{R}$*-tree* T *is stable;*

(5) *Let* $g \in G \setminus K_\infty$*. Then for all but finitely many* n *we have* $g \notin ker(f_n)$*;*

(6) L_∞ *is torsion-free; and*

(7) *If* T *is not isometric to a real line then* $\{f_i\}$ *is a stable sequence of homomorphisms.*

Proof. The proof of [21, Theorem 4.4] relies on a number of different results. In each case, we have an exact analogue in the setting of Theorem 6.4 here.

The results we need are: Proposition 5.9, Lemma 5.8, Lemma 5.5, Lemma 4.16, Lemma 4.17, Proposition 5.4, Proposition 4.18 and the fact that stabilisers in Γ of flats in X are malnormal, which is [18, Example 1, p. 819].

Given these results, the proof of [21, Theorem 4.4] applies directly, except that since some of the constants have changed, some of the counting has to be changed. This is straightforward. □

The following are two immediate applications of the above construction of the $\mathbb{R}$-tree T, and of Theorem 6.4. See [21] for proof which apply without change in the current setting.

Theorem 6.5 (cf. Theorem 5.1, [21])**.** *Suppose that* Γ *is a torsion-free group which is hyperbolic relative to a collection of free abelian subgroups. A group* L *is a* Γ*-limit group in the sense of Definition 1.2 if and only if it is a* Γ*-limit group in the sense of Definition 5.7.*

Theorem 6.6 (cf. Theorem 5.9, [21])**.** *Suppose that* Γ *is a torsion-free group which is hyperbolic relative to a collection of free abelian subgroups, and suppose that* $Out(\Gamma)$ *is infinite. Then* Γ *admits a non-trivial splitting over a finitely generated free abelian group.*

The results of [10] now imply the following

Lemma 6.7 (cf. Corollary 5.7, [21])**.** *Suppose that* Γ *is a torsion-free group hyperbolic relative to a collection of free abelian subgroups, and suppose that* L *is a* Γ*-limit group. Then*

(1) *Any finitely generated subgroup of* L *is a* Γ*-limit group;*
(2) L *is torsion-free;*
(3) L *is commutative-transitive and CSA; and*
(4) *Every solvable subgroup of* L *is abelian.*

7. The Shortening Argument

Sela's shortening argument is a powerful tool which yields certain finiteness results. It was first developed in [33] and [36], and has been put to use in [37], [38] and [42], among other places.

In the context of torsion-free toral CWIF groups, the author considered the shortening argument in [22], and found a version of it powerful enough for the work there. For limit groups, Alibegović developed a version of the shortening argument in [2]. It seems to possible to take the approach either of [22] or of [2] here, and they will lead to equivalent statements.

Recall the definition of $\mathrm{Mod}(G)$ for a finitely generated group G.

Definition 7.1. *Let G be a finitely generated group. A* Dehn twist *is an automorphism of one of the following two types:*

1. *Suppose that $G = A *_C B$ and that c is contained in the centre of C. Then define $\phi \in Aut(G)$ by $\phi(a) = a$ for $a \in A$ and $\phi(b) = cbc^{-1}$ for $b \in B$;*
2. *Suppose that $G = A*_C$, that c is in the centre of C, and that t is the stable letter of this HNN extension. Then define $\phi \in Aut(G)$ by $\phi(a) = a$ for $a \in A$ and $\phi(t) = tc$.*

Definition 7.2 (Generalised Dehn twists)**.** *Suppose G has a graph of groups decomposition with abelian edge groups, and A is an abelian vertex group in this decomposition. Let $A_1 \leq A$ be the subgroup generated by all edge groups connecting A to other vertex groups in the decomposition. Any automorphism of A that fixes A_1 elementwise can be naturally extended to an automorphism of the ambient group G. Such an automorphism is called a* generalised Dehn twist *of G.*

Definition 7.3. *Let G be a finitely generated group. We define* $\mathrm{Mod}(G)$ *to be the subgroup of* $\mathrm{Aut}(G)$ *generated by:*

1. *Inner automorphisms;*
2. *Dehn twists arising from splittings of G with abelian edge groups; and*
3. *Generalised Dehn twists arising from graph of groups decompositions of G with abelian edge groups.*

Similar definitions are made in [38, §5] and [5, §1].

We will try to shorten homomorphisms by precomposing by elements of $\mathrm{Mod}(G)$. However, as seen in [22, §3], this is not sufficient to get the most general result. Thus, we also define a further kind of move (very similar to Alibegović's *bending* move, [2, §2].

Definition 7.4. *Suppose that Γ is a torsion-free group which is hyperbolic relative to free abelian subgroups, and that G is a finitely generated group and that $h : G \to \Gamma$ is a homomorphism. We define two kinds of 'bending' moves as follows:*

- *(B1) Let Λ be a graph of groups decomposition of G, and let A be an abelian vertex group of G. Suppose that $h(A)$ is properly contained in a parabolic subgroup $P \leq G$. A move of type (B1) replaces h by a homomorphism h' which is such that (i) $h'(A) \leq P$; and (ii) h' agrees with h on all edge groups adjacent to A, and all vertex groups other than A.*
- *(B2) Let Λ be a graph of groups decomposition of G, and let A be an abelian edge group associated to an edge e. Suppose that $h(A)$ is properly contained in a parabolic subgroup $P \leq \Gamma$. A move of type (B2) replaces h by a map which either (i) conjugates a component of $\pi_1(\Lambda \smallsetminus e)$ by an element of P, in case e is separating; or (ii) multiplies the stable letter associated to e by an element of P, in case e is non-separating.*

Remark 7.5. *The bending moves of Type (B1) and (B2) can be replaced by embedding G in a group $\hat{G}$ and precomposing with certain elements of Mod($\hat{G}$), as in* [22, §7]. *These two approaches are equivalent. However, we choose to follow Alibegović's approach.*

Suppose that Γ is a torsion-free group which is hyperbolic relative to free abelian subgroups, and that X is the space constructed in Section 4. Suppose also that G is a finitely generated group, with finite generating set $\mathcal{A}$. Let $h : G \to \Gamma$ be a homomorphism. Recall that in Section 5 we defined the *length* of h by

$$\|h\| := \max_{g \in \mathcal{A}} \left\{ d_X(x, h(g).x) \right\}.$$

We will state two versions of the shortening argument here, and for this we need two equivalence relations on homomorphisms from G to Γ.

Definition 7.6. *We define an equivalence relation on the set of homomorphisms $h : G \to \Gamma$ by setting $h_1 \sim_1 h_2$ if there is $\alpha \in$ Mod(G) and $\gamma \in \Gamma$ so that $h_1 = \tau_\gamma \circ h_2 \circ \alpha$, where τ_γ is the inner automorphism of Γ induced by γ.*

A homomorphism $h : G \to \Gamma$ is $\sim_1$-short *if for any h' such that $h' \sim_1 h$ we have $\|h\| \leq \|h'\|$.*

Definition 7.7 (cf. Definition 4.2, [5])**.** *We define a relation ' $\sim_2'$ on the set of homomorphisms $h : G \to \Gamma$ to be the equivalence relation generated by setting $h_1 \sim_2 h_2$ if h_2 is obtained from h_1 by:*

(1) precompositing with an element of Mod(G);
(2) postcompositing with a conjugation in Γ; or
(3) a bending move of type (B1) or (B2).

*A homomorphism $h : G \to \Gamma$ is $\sim_2$-*short *if for any h' such that $h \sim_2 h'$ we have $\|h\| \leq \|h'\|$.*

Though it is used in various contexts in somewhat different forms, the shortening argument is encapsulated in the following two theorems.

Theorem 7.8. *Suppose that Γ is a non-abelian, freely indecomposable and torsion-free group which is hyperbolic relative to a collection of free abelian subgroups. Suppose that the sequence of automorphisms $\{h_n : \Gamma \to \Gamma\}$ converges to an action $\eta : \Gamma \to Isom(\mathcal{C}_\infty)$ as in Section 5 above. Then, for all but finitely many n, the automorphism h_n is not $\sim_1$-short.*

Theorem 7.9. *Suppose that Γ is a non-abelian, freely indecomposable and torsion-free group which is hyperbolic relative to a collection of free abelian subgroups. Let G be a finitely generated group and $\{h_n : G \to \Gamma\}$ be a sequence of homomorphisms which converge to a faithful action of G on $\mathcal{C}_\infty$ as in Section 5 above. Then, for all but finitely many n, the homomorphism h_n is not $\sim_2$-short.*

Remark 7.10. *Theorem 7.9 is not true if we replace $\sim_2$-short by $\sim_1$-short – see* [22, Theorem 3.6].

The proof of Theorem 7.8 is almost exactly the same as in [22]. In fact, given the work we've done in Section 4, all changes are entirely straightforward, and the reader is referred to the proof contained in [22].

In order to prove Theorem 7.9, the idea that images in flats get 'denser and denser' no longer holds. This is where the bending moves in Definition 7.4 above. Although it is quite elementary, given the proof of Theorem 7.8 and the concept of bending, we postpone the proof of Theorem 7.9 until [24].

8. Applications and future work

Theorem B *Suppose that Γ is a torsion-free group which is hyperbolic relative to free abelian subgroups. Then* $\mathrm{Mod}(\Gamma)$ *has finite index in* $\mathrm{Aut}(\Gamma)$.

Proof. This is an immediate corollary of Theorem 7.8. The proof is identical to that of [22, Theorem 1.2] (see Section 4 of [22]). □

Theorem A *Suppose that* Γ *is a torsion-free group which is hyperbolic relative to free abelian subgroups. Then* Γ *is Hopfian.*

Proof. The proof of this result is almost identical to that of [22, Theorem 1.1] (see Section 7 of [22]). Given the previous results in this paper, all adaptations are entirely straightforward. □

In the continuation paper, [24], we construct Makanin-Razborov diagrams for Γ when Γ is a torsion-free group which is hyperbolic relative to a collection of free abelian subgroups (the techniques of Alibegović may allow this problem to be solved; see [2, Remark 3.7]). In [23] we consider which such groups Γ can be co-Hopfian, and also conjugacy classes of subgroups of such a Γ (Dahmani [14] has an alternative approach to both of these problems for more general relatively hyperbolic groups). Finally, we expect that the tools developed in this paper will allow much, if not all, of Sela's work on the elementary theory of torsion-free hyperbolic groups to be generalised to relatively hyperbolic groups with free abelian parabolic subgroups.

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