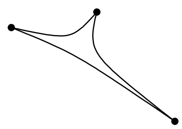
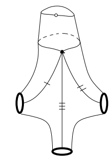


**Research Statement:**

## Daniel Groves



# 1. **Introduction**

In the last few years, my research has been in **geometric group theory**, which attempts to understand groups via the spaces upon which they act. The last two decades have seen tremendous progress in our understanding of infinite discrete groups, and many long-standing open problems have been solved using geometric methods.

I am particularly interested in coarse negative and non-positive curvature, in decision problems and in the logic of infinite discrete groups. I am also interested in the relationships of discrete groups to topology (low and high dimensional); to formal languages; and to analysis and K-theory, to mention a few. I will discuss the bearing of my work on these ideas and fields throughout this statement.

First, I outline my most important results in geometric group theory (I discuss my earlier work in Section 8). These results (and others) will be discussed in more detail later.

In the series of papers [2, 3, 4], Martin Bridson and I proved (see Section 2):

**Theorem A** (Bridson–G.; [4]). *Let $F$ be a finitely generated free group and $\phi \in Aut(F)$. Then the mapping torus $F \rtimes_\phi \mathbb{Z}$ admits a quadratic isoperimetric inequality.*

The *Isomorphism Problem* [46] is one of the three fundamental problems in combinatorial group theory. In [5, 6], François Dahmani and I proved (see Section 6):

**Theorem B** (Dahmani–G.; [5]). *The Isomorphism Problem is solvable for toral relatively hyperbolic groups.*

A *Makanin-Razborov diagram* encodes the set $\mathrm{Hom}(G, H)$, for f.g. groups $G$ and $H$; using quotients of $G$, and certain automorphisms (see Section 5). In [11], I proved (see Section 5):

**Theorem C** (G.; [11]). *Makanin-Razborov diagrams exist for toral relatively hyperbolic groups.*

In [14], Jason Manning and I proved an algebraic analogue of the Gromov-Thurston $2\pi$-theorem (see Section 7):

**Theorem D** (G.–Manning; [14]). *Let $G$ be a torsion-free group, which is hyperbolic relative to a collection $\mathcal{P} = \{P_1, \ldots, P_n\}$ of finitely generated subgroups. Let $S$ be a generating set for $G$ so that $P_i = \langle P_i \cap S \rangle$ for each $1 \le i \le n$.*

*There exists a constant $\Upsilon$ depending only on $(G, \mathcal{P})$ so that for any collection $\{K_i\}_{i=1}^n$ of subgroups satisfying*

- *$K_i \trianglelefteq P_i$; and*
- *$|K_i|_{P_i} \ge \Upsilon$,*

*the following hold, where $K$ is the normal closure in $G$ of $K_1 \cup \cdots \cup K_n$:*

*(1) The map $P_i/K_i \xrightarrow{\iota_i} G/K$ given by $pK_i \mapsto pK$ is injective for each $i$.*
*(2) $G/K$ is hyperbolic relative to the collection $\mathcal{Q} = \{\iota_i(P_i/K_i) \mid 1 \le i \le n\}$.*

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1.1. **Background and motivation.** Gromov's definition of *hyperbolic groups* beautifully captures the notion of *negative curvature* for discrete groups, and there is now a well-developed theory of these groups.

This naturally raises the question: What broader classes of groups can geometric methods be applied to? One class which has received a lot of attention recently is *relatively hyperbolic groups*. These act on $\delta$-hyperbolic spaces, so there is a flavour of negative curvature, but the action needn't be cocompact, so there are many more groups in this class than just hyperbolic groups, and the theory becomes more complicated. Sections 3–7 of this statement are about my work on relatively hyperbolic groups.

Once 'negative curvature' is well understood for discrete groups, it is natural to ask the following:

**Question 1.** *What does it mean for a discrete group to be* non-positively curved*?*

There are many candidate classes of groups: CAT$(0)$ groups, automatic groups, semi-hyperbolic groups, groups with a quadratic isoperimetric function, .... However, none of these capture the notion of non-positive curvature (NPC) as beautifully as hyperbolic groups do negative curvature. Probably there is no one notion of NPC for discrete groups, but more understanding of Question 1 is still a central problem for geometric group theory. This is the context of Theorem A (see Section 2).

Recently, in his solution of the Tarski problem (see Section 5), Sela has defined a new class of groups: *limit groups*. It turns out that these are exactly the class of finitely generated fully residually free groups, which have been studied by many people for many years. [1]

Sela's work raises many questions and opens new areas for study. One way that I have approached this area is to try to understand larger classes of groups from this point of view. This is the context of Theorem C and is discussed in Section 5.

I am also interested in decision problems for group theory. In 1911, Dehn [45, 46] formulated three basic questions: the *word problem*, the *conjugacy problem* and the *isomorphism problem*. In general these are all unsolvable, but it is interesting to restrict these questions to specific classes of groups. In recent years geometric group theory has given new methods for attacking these questions for large classes of groups. Theorem B is one of the most general positive results about the Isomorphism Problem. This is discussed more in Section 6. Other algorithmic questions are discussed in Sections 2 and 5.

The final direction of my current research described in this statement is my joint work with Jason Manning (see Theorem D) about Dehn filling in relatively hyperbolic groups, which is discussed in Sections 3 and 7.

## 2. Free-by-cyclic groups

In this section I discuss my joint work with Martin Bridson [2, 3, 4, 17], and also some future directions for this work. The main result of the papers [2, 3, 4] is Theorem A.

*Free-by-cyclic groups* (which are groups of the form $F \rtimes_\phi \mathbb{Z}$ as in Theorem A) exhibit many of the algebraic properties of groups which are regarded as non-positively curved (such as CAT$(0)$ groups). However, not all free-by-cyclic groups are CAT$(0)$ [51], and they are not all automatic either [42].

[1] Sela's solution of the Tarski problem [65, 66, 67, 68, 69, 70, 71] has now all appeared. There is another solution by Kharlampovich and Miasnikov [54]. However, as far as I am aware, their work is still in the process of being refereed. Whilst we have no reason to disbelieve any of their claims, we consider the status of their work to be unsettled.

Whilst it is unclear whether there *is* a good notion for what it means for a discrete group to be non-positively curved, certainly a prerequisite should be a quadratic isoperimetric inequality. Therefore, Theorem A places free-by-cyclic groups at least *near* to the class of non-positively curved groups. This class of groups provides a natural testing ground for better understanding Question 1, since free-by-cyclic act like NPC groups in many ways, but also exhibit an extremely diverse range of behaviour.

Our strategy to prove Theorem A is to understand their van Kampen diagrams, which consist entirely of *corridors*. The key is to understand the way that corridors behave under a natural time flow, which amounts to gaining a thorough understanding of how words in $F$ behave under iteration by $\phi$. In [2], we restricted attention to *positive automorphisms*, which allowed us to focus on the large-scale dynamics of van Kampen diagrams. In [3], we further developed the train track theory of Bestvina, Feighn and Handel [41, 39], in order to prove the *Beaded Decomposition Theorem*, which gives strong evidence that the strategy from [2] is applicable in the general case. For background on free-group automorphisms, see [26, 33]. Finally, in [4], this approach comes to fruition, and Theorem A is proved.

In the course of this project, we have gained great insight into the geometry of free-by-cyclic groups. A sample outcome of this insight is given in the paper [17], where we understand the growth of words and conjugacy classes under iteration by a free-group automorphism. However, I strongly believe that there will be many further applications of this work. In particular, I hope that we can make an attack on the following:

**Conjecture 1.** *The Isomorphism Problem is solvable for free-by-cyclic groups.*

A solution to this conjecture would likely involve a solution to the conjugacy problem for $\text{Out}(F_n)$. Martin Lustig has claimed a solution to this problem, and has posted preprints. However, no solution has appeared in print, and it is important to clarify the situation with a published proof.

Other important questions which I believe our methods give insight into are:

**Question 2.**

*(1) Which free-by-cyclic groups are CAT$(0)$?*
*(2) Which free-by-cyclic groups are automatic?*
*(3) What can be said about quasi-isometries of free-by-cyclic groups?*

A student of mine, Peter Samuelson, has made some progress on the first of these questions [63], but there is still much work to be done.

Finally, the work of [3] gives some significant improvements to the train track technology of Bestvina, Feighn and Handel. I hope that these technical improvements can be used for solving other questions about free-group automorphisms.

## 3. Relatively hyperbolic groups: Introduction

According to Gromov [52], a group is *relatively hyperbolic* if it acts properly on a hyperbolic space with quotient quasi-isometric to a finite collection of rays, joined at $0$ (the stabilisers of the endpoints of lifts of these rays are *parabolic* subgroups).

Farb [50] gave an alternative definition of relatively hyperbolic groups, and proved many theorems about them. Since then, there has been a lot of work on these groups, including a recent explosion. There are now many definitions and characterisations of these groups (for example, see [40, 47, 60, 74])

In [14], Manning and I gave a new characterisation, which is essentially a combinatorial realisation of Gromov's original definition. The idea behind our space is to take a finitely generated group $G$ and a collection $\mathcal{P}$ of finitely generated subgroups of $G$. Take the Cayley graph of $G$ and add a 'combinatorial horoball' to each coset of each element of $\mathcal{P}$. The resulting space $X$ is called the 'cusped space'. We prove:

**Theorem 1** (G.–Manning; [14])**.** *The cusped space is hyperbolic if and only if $G$ is hyperbolic relative to $\mathcal{P}$.*

The cusped space has already been used in [14, 20, 6], and I expect it to become one of the standard models for relatively hyperbolic groups.

## 4. Toral Relatively hyperbolic groups, $\mathbb{R}$-trees and splittings

A class of relatively hyperbolic groups which I have been particularly interested in are the *toral* relatively hyperbolic groups, which are torsion-free relatively hyperbolic groups with abelian parabolic subgroups.

Key examples of toral relatively hyperbolic groups are limit groups and the fundamental groups of finite-volume hyperbolic manifolds with torus cusp cross sections.

4.1. **Actions on $\mathbb{R}$-trees.** In [10, 13], I constructed an $\mathbb{R}$-tree $T$ starting from a toral relatively hyperbolic group $\Gamma$, an arbitrary finitely generated group $G$ and a sequence of non-conjugate homomorphisms $\{h_n : G \to \Gamma\}$. The tree $T$ comes equipped with an isometric $G$-action.

This is analogous to the Bestvina-Paulin construction in case $\Gamma$ is a hyperbolic group [38, 62].

The utility of such a construction is that if $L$ is the quotient of $G$ by the kernel of the $G$-action on $T$, then the Rips theory of groups acting on $\mathbb{R}$-trees implies that $L$ admits a nontrivial splitting over an abelian subgroup.

After I defined my construction, there were a number of other constructions of $\mathbb{R}$-trees from such a sequence (where $\Gamma$ is a relatively hyperbolic group). See, for example, [37, 49]. However, some of the most important applications of my construction involve the *shortening argument* (due to Sela for hyperbolic groups), and so far mine is the only construction for relatively hyperbolic groups for which the shortening argument has been implemented. An example where the shortening argument is required is

**Theorem 2** (G.; [13])**.** *Any toral relatively hyperbolic group is Hopfian.*

The construction of such an $\mathbb{R}$-tree allows us to understand abelian splittings. Once we understand abelian splittings, there is a natural abelian JSJ decomposition for toral relatively groups (see [30] for more on JSJ decompositions). That such a splitting exists was proved by myself and Dahmani in [5] (though it is also implicit in my papers [11, 13]).

Once the JSJ decomposition is defined, there is a natural subgroup, $\text{Mod}(\Gamma)$, of $\text{Aut}(\Gamma)$ which is generated by inner automorphisms, and automorphisms which can be easily seen from the JSJ decomposition. The most straightforward application of the shortening argument is the following

**Theorem 3** (G.; [13])**.** *Let $\Gamma$ be a toral relatively hyperbolic group. Then Mod$(\Gamma)$ has finite index in Aut$(\Gamma)$.*

It would be very interesting to know whether the shortening argument can be made to work for Druţu and Sapir's construction from [49], as this would have many similar implications for more general relatively hyperbolic groups.

## 5. The Elementary Theory of groups

Given a group $G$, the *elementary theory* of $G$, denoted $\mathrm{Th}(G)$, is the set of all first-order sentences which are true in $G$, where the language involves $\forall, \exists, \wedge, \neg, \vee, =$ has a constant for each element of $G$, and the variables are all interpreted as elements of $G$. In the 1940's, Tarski asked whether $\mathrm{Th}(F_2) = \mathrm{Th}(F_3)$ (where $F_n$ is the free group of rank $n$).

Sela [71] has answered Tarski's question (see [32] for a summary of this work). In fact, he has classified all finitely generated groups with the same elementary theory as $F_2$. Sela has also proved the following remarkable theorem:

**Theorem 4** (Sela; [72])**.** *Suppose $G$ is a torsion-free hyperbolic group and that a finitely generated group $H$ is such that Th$(G)$ = Th$(H)$. Then $H$ is hyperbolic.*

However, it is far from true that $H$ must be isomorphic to $G$ in the above situation .

Some of the key tools for Sela's work are actions on $\mathbb{R}$-trees and the shortening argument. The motivation for my work described in the previous paragraph was to generalise Sela's results to toral relatively hyperbolic groups. The start of this project are the papers [10, 12, 13, 11]. This project remains a central part of my research, and is continued in [21, 22].

The first step to understanding the elementary theory of a group $\Gamma$ is to understand sets of solutions to equations over $\Gamma$. This study leads to an analogue of algebraic geometry over groups, a field of study that is in its infancy. In this study, one is naturally lead to the class of $\Gamma$-*limit groups*, which arise as limits of homomorphisms from an arbitrary finitely generated group to $\Gamma$. A consequence of my $\mathbb{R}$-tree construction is that if $\Gamma$ is a toral relatively hyperbolic group then any $\Gamma$-limit group comes equipped with a nontrivial faithful action on an $\mathbb{R}$-tree. In [11], I proved the following:

**Theorem 5** (G.; [11])**.** *Let $\Gamma$ be a toral relatively hyperbolic group. Any descending sequence of $\Gamma$-limit groups terminates.*

A *Makanin-Razborov* diagram is a parametrisation of $\mathrm{Hom}(G, \Gamma)$, where $\Gamma$ is a group of interest (free group, hyperbolic gorup, toral relatively hyperbolic group, etc.), and $G$ is an arbitrary finitely generated group. It is a finite rooted tree of groups, with vertices labelled by groups (all except $G$ will be $\Gamma$-limit groups), the root labelled by $G$, and edges corresponding to proper quotient maps. Using an equivalence relation (which is obtained by twisting by inner automorphisms in the target, elements of $\mathrm{Mod}(L)$ in the range, and a couple of extra moves in case $\Gamma$ is toral relatively hyperbolic), we obtain the required parametrisation. Sela proved that Makanin-Razborov diagrams exist for free groups [65], and torsion-free hyperbolic groups [72], whilst Alibegović [36] proved that they exist for limit groups.

In [11], I proved Theorem C, that Makanin-Razborov diagrams exist for toral relatively hyperbolic groups. This, along with other structural results in [11], form the beginning of our approach to generalising Sela's results (though there is a very long way to go in this project). One result I believe is attainable (though far from straightforward) is the following analogue of Theorem 4:

**Conjecture 2.** *Suppose that $\Gamma$ is a toral relatively hyperbolic group and that $H$ is a finitely generated group such that Th$(\Gamma)$ = Th$(H)$. Then $H$ is toral relatively hyperbolic.*

### 5.1. **Stability.**

Sela [73] has proved the elementary theory of a torsion-free hyperbolic group is *stable*.

This raises the question of the stability of other groups of interest to geometric group theorists. (Though one would expect it to be rare for an infinite group to have a stable theory.) In [18], Martin Bridson and I investigate this question. In particular, we prove

**Theorem 6** (Bridson–G.; [18])**.** *Thompson's group $F$ and $GL_n(\mathbb{Z})$ ($n \geq 4$) have unstable theories.*

It would be interesting to find an example of a $\mathrm{CAT}(0)$ group with an unstable theory (Sela asked this question, and I hope to answer it). However, note that Thompson's group $F$ has a quadratic Dehn function [53], so Theorem 6 above provides a sharp boundary between hyperbolic groups and non-hyperbolic groups in terms of stability.

## 6. Constructing JSJ decompositions and the Isomorphism Problem

One of the most important results on the isomorphism problem is Sela's solution to the isomorphism problem for torsion-free hyperbolic groups. Unfortunately, although he published a key case [64], Sela never published a complete proof of his result.

Starting with my work on $\mathbb{R}$-trees mentioned above, and Dahmani's work on equations over groups [44], François Dahmani and I found significant simplifications to Sela's proof, and also substantial generalisations of Sela's result. The outcome of this was Theorem B. This immediately implies Sela's unpublished result (and is the first time this result is publically available). It also implies a solution to the isomorphism problem for limit groups, previously obtained by Bumagin, Kharlampovich and Miasnikov [43]. Our proof is substantially simpler than the one in [43].

Another outcome of this work is the following theorem.

**Theorem 7** (Dahmani–G.; [5])**.** *The homeomorphism problem is solvable for finite-volume hyperbolic manifolds of dimension at least* $3$*.*

One of the key results in [5], crucial for the proof of Theorem B is the following, which should have many future applications.

**Theorem 8** (Dahmani–G.; [5])**.** *There is an algorithm to construct the JSJ decomposition of a freely indecomposable toral relatively hyperbolic group.*

As well as asking the above question about $\mathrm{Th}(F_2)$, Tarski also asked whether the theory of $F_2$ is *decidable* (ie whether there exists an algorithm to decide whether or not a given sentence is lies in $\mathrm{Th}(F_2)$). Theorems 8 and B suggest a a possible attack on this second question of Tarski. [2]

In [72], Sela generalised his work on the elementary theory of free groups to torsion-free hyperbolic groups. Thus it is of great interest whether Sela's results can be made algorithmic:

**Question 3.** *Is the elementary theory of a torsion-free hyperbolic group decidable?*

I intend to investigate this in joint work with Henry Wilton. This question is also interesting for other classes of groups where we have some chance of understanding the elementary theory (such as hyperbolic groups in general, or toral relatively hyperbolic groups). I am very confident that our methods will eventually answer Question 3 in the affirmative. The first part of this project is [25].

[2] A solution to Tarski's second question has already been claimed by Kharlampovich and Miasnikov [54]. As we mentioned earlier, their work is still going through the refereeing process, and we regard the status as uncertain.

# 7. Dehn filling in relatively hyperbolic groups

A classical theorem of Thurston states that if one starts with a finite-volume hyperbolic $3$-manifold, and performs Dehn filling on the cusps, then for all but finitely many choices in each cusp the resulting manifold is a closed hyperbolic $3$-manifold. Gromov and Thurston proved that if all of the filling slopes are more than $2\pi$ then this result holds. In [35] and [55], Agol and Lackenby (independently) strengthened $2\pi$ to $6$, but weakened the conclusion to saying that $\pi_1(M)$ (where $M$ is the resulting manifold) is word-hyperbolic.

It is natural to ask whether there is a version of this theorem where the hypothesis, as well as the conclusion, is algebraic. This is Theorem D. This theorem has also been proved by Osin [61]. He does not need the hypotheses of torsion-free or finitely generated. In [15] we explain why the infinitely generated version follows from the finitely generated version.

One of the important tools in our proof of Theorem D is the cusped space described in Section 3 above. However, by far the most important innovation in [14] is *preferred paths*. These give a bicombing of the cusped space by uniform quasi-geodesics which is equivariant with respect to isometries (in particular with respect to the action of the group $G$), and give very strong combinatorial control over triangles made from these paths.

The construction of preferred paths is very robust, and can be applied in many situations. We apply it in the proof of Theorem D. However, it is also a key tool in [20] and in [23].

In [56], Lafforgue proves that if a group has *Rapid Decay* and also acts properly on a uniformly locally finite, weakly geodesic, strongly bolic metric space then it satisfies the Baum-Connes conjecture. Druţu and Sapir [48] prove that a relatively hyperbolic group has Rapid Decay if and only if all of the parabolic subgroups do. In [20], I prove:

**Theorem 9.** *Let $\Gamma$ be a relatively hyperbolic group with parabolic subgroups $P_1, \ldots, P_n$. Suppose that each of the $P_i$ act properly on a strongly bolic space satisfying the hypothesis of Lafforgue's theorem* [56]*. Then $\Gamma$ also acts properly on such a strongly bolic space.*

As far as I am aware, this is the first result about relatively hyperbolic groups whose conclusion implies the Baum-Connes conjecture (though there are various other results which imply the weaker Novikov conjecture). See [34] and references therein for background on the Novikov conjecture and the Baum-Connes conjecture.

In [23], we construct a bounded area bicombing on the coned-off Cayley graph of a relatively hyperbolic group. This implies

**Theorem 10** (G.–Manning, [23])**.** *Suppose that $G$ is a relatively hyperbolic group with $\mathcal{P}$ its parabolic subgroups. Then with any bounded $\mathbb{C}G$-module as coefficients, the second relative cohomology of $G$ is bounded.*

A more general theorem has been proved by Mineyev and Yaman [59], but we believe our approach is more direct. See [29] and the references therein for more information about bounded cohomology.

There are many future directions in which the work from [14] can be pursued. First, many of the applications to hyperbolic groups of Mineyev's construction from [57] (of which there are myriad) have a relatively hyperbolic version which can be proved using appropriate versions of preferred paths (and homological bicombings). This is the context of [20] and [23], and there should be many other applications (for instance, to measure and orbit equivalence rigidity, as in [58]). Theorem D raises many questions, which I am investigating with Manning and Osin. For example:

**Question 4.**

(1) *What does it mean to 'drill' a hyperbolic group and obtain a relatively hyperbolic group?*
(2) *What happens to various boundaries under Dehn filling? (e.g. to dimesnion)*
(3) *What kinds of groups can be built by successively applying Theorem D and taking a direct limit?*

## 8. Early work

My thesis, and the work done just afterwards, was motivated by Zelmanov's solution of the Restricted Burnside Problem (see [28] for a description of this work).

In [7], I found an example of a *non-identical Lie relator*, answering a 20 year old question.

In [16], Michael Vaughan-Lee and I simplifed a key step in Zelmanov's proof, and thereby improved the known bounds on the size of finite groups of fixed exponent (in terms of the exponent and the number of generators). Our bound is still the best known.

In [8, 9] I investigated relatively free groups defined by outer commutator identities. I proved that many of these are torsion-free and residually nilpotent. However not all of them are torsion-free.

In [1], Don Barnes and I laid the foundations for the study of the Wielandt subalgebra of a Lie algebra (in analogy with the Wielandt subgroup of a group).

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