Notes/exercises on the functor of points.

Due Monday, February 8 in class.

For a more careful treatment of this material, consult the last chapter of Eisenbud and Harris’ book, The Geometry of Schemes, or Section 9.1.6 in Vakil’s notes, although I would encourage you to try to work all this out for yourself.

Let $C$ be a category. For each object $X \in Ob(C)$, define a contravariant functor $F_X : C \to \text{Sets}$ by setting $F_X(S) = \text{Hom}(S,X)$ and for each morphism in $C$, $f : S \to T$ define $F_X(f) : \text{Hom}(S,X) \to \text{Hom}(T,X)$ by composition with $f$.

Given a morphism $g : X \to Y$, we get a natural transformation of functors, $F_X \to F_Y$ defined by mapping $\text{Hom}(S,X) \to \text{Hom}(S,Y)$ using composition with $g$.

**Lemma 1** (Yoneda’s Lemma). Every natural transformation $\phi : F_X \to F_Y$ is induced by a unique morphism $f : X \to Y$.

Proof: Given a natural transformation $\phi$, set $f_\phi = \phi_X(id)$.

**Exercise 1.** Check that $\phi$ is the natural transformation induced by $f_\phi$.

This implies that the functor $F$ that takes an object $X$ of $C$ to the functor $F_X$ and takes a morphism in $C$ to the corresponding natural transformation of functors defines a fully faithful functor from $C$ to the category whose objects are functors from $C$ to $\text{Sets}$ and whose morphisms are natural transformations of functors.

In particular, $X$ is uniquely determined by $F_X$. However, not all functors are of the form $F_X$.

**Definition 1.** A contravariant functor $F : C \to \text{Sets}$ is set to be representable if there exists an object $X$ in $C$ such that $F \cong F_X$.

Now we specialize to the case where $C$ is the category of schemes. The goal here is going to be to give a characterization of representable functors. This can then be used as a way to construct schemes – first write down a functor, then show that it is representable.

Given a contravariant functor $F : \text{Sch} \to \text{Sets}$, we say that $F$ is a sheaf in the Zariski topology, if for each scheme $X$, the presheaf on $X$ induced by $F$ is a sheaf.

(The presheaf in question is the one that assigns to each open set $U \subset X$ the set $F(U)$, and has restriction maps induced by the maps $F(i)$ where $i : V \to U$ is an inclusion of open sets.)

**Exercise 2.** Prove that a representable functor on the category of schemes is a sheaf in the Zariski topology.

We now want to prove that a functor is representable if it is a sheaf and if it is “locally representable.” For this, we need a notion of an open subset of a functor. The definition we use here is not completely standard, but is probably the simplest approach.

**Definition 2.** An open subset $U$ of a contravariant functor $F : \text{Sch} \to \text{Sets}$ is an association to each element $\xi \in F(T)$ an open set $U_\xi \subset T$ which is compatible with pullbacks. That is, if $f : S \to T$ is a morphism of schemes, then $f^{-1}(U_\xi) = U_{F(f)(\xi)}$. 

1
Lemma 2. If $F = F_X$ then open subsets of $X$ are in natural bijection with open subsets of $F$.

Given an open subset $U$ of $X$, we get an open subset of $F_X$ by associating to an element of $F_X(S)$ the pullback of $U$ under the associated morphism $S \to X$. Conversely, given an open subset of $F_X$, we let $U$ be the open subset of $X$ associated to the identity element of $F_X(X)$.

Exercise 3. Check that this gives the desired bijection by showing that one can reconstruct the entire subset of $F_X$ from this data. This should be very similar to the proof of Yoneda’s Lemma.

Definition 3. A collection of open subsets $U_i$ is said to be an open covering of $F$ if for every scheme $S$ and every element $\xi \in F(S)$, the resulting collection of open subsets of $S$, $\{U_\xi\}$ is an open covering of $S$.

Given an open subset $U \subset F$, we define the corresponding “open subfunctor” to be $F_U(S) = \{\xi \in F_X(S) | U_\xi = S\}$. Note that if $F = F_X$, this agrees with the functor of points of the corresponding open subscheme. (This just says that a map to $U$ is the same as a map to $X$ whose image is contained in $U$.)

Definition 4. We say that an open subset $U \subset F$ is representable if the associated open subfunctor is representable.

Now we can state a basic theorem which can be a useful tool to construct interesting schemes.

Theorem 1. A contravariant functor from schemes to sets is representable if and only if it is a sheaf in the Zariski topology and has a covering by representable open subsets.

Exercise 4. Prove this. Probably by letting $X_i$ be the collection of schemes associated to the representable open subfunctors that cover $F$. Now show that one can construct the gluing data in Hartshorne Exercise II.2.12. (To start, note that the pullback of the open set associated to $X_i$ gives an open subset of $X_i$, so for all pairs $i, j$ one gets an open set $U_{ij}$ in $X_i$.) Use the sheaf property of $F$ to show that $F$ is represented by the glued scheme $X$.

Here we will consider an example of a scheme which can be constructed this way.

Exercise 5. Construct the Grassmannian $G(r, n)$ by showing that the following functor is representable:

$$F(X) = \{\mathcal{O}_X^\oplus n \to \mathcal{E} \to 0\}$$

where $\mathcal{E}$ is a rank $r$ locally free sheaf on $X$. (Here these are taken up to isomorphism, but isomorphisms which act trivially on $\mathcal{O}_X^\oplus n$.) Consider the open subsets of $F$ indexed by subsets $\{i_1, \ldots, i_r\} \subset \{1, \ldots, n\}$ which associate to an object of $F(X)$ the open subset of $X$ on which the map $\mathcal{O}_X^\oplus r \to \mathcal{E}$ given by restricting the original map to the $r$ factors corresponding the the $i_j$ is surjective. Check that this is an open subset, that the corresponding open subfunctor is representable, and indeed represented by an affine space, and that they cover $F$. Also check that $F$ is a sheaf, and thus conclude the existence of a scheme $G(r, n)$. (It might be helpful to think about the case $r = 1$ where this recovers the standard open cover of projective space by affine spaces.)
If you haven’t had enough of these yet, you can also construct the projective bundle associated to a locally free sheaf by starting from the desired functor, and then covering it by the open subsets which correspond to the preimages of open sets over which the bundle is trivial. Or you could combine these, and construct once and for all the grassmannian bundle parametrizing rank $r$ quotients of a coherent sheaf over a scheme $X$. Or see section 9.1.6 in Vakil’s notes to see how to construct fibered products of morphisms of schemes in these terms (although be warned that he is using a fancier and possibly more confusing (although equivalent) definition of the notion of open subfunctor.)