ALGEBRAIC GEOMETRY, PROBLEM SET 7

Due Wednesday, December 2.

(1) Show that if a hypersurface X in $\mathbb{P}^n$ contains a linear subspace of dimension strictly bigger than $n/2$, then X must be singular if it is not itself a linear space.

(2) Recall that in class we showed that given a dominant morphism $f : X \to Y$ of irreducible varieties, there was a dense open set $U \subset Y$ so that for all $y \in U$ the fiber $f^{-1}(u)$ contained exactly d points where d was the separable degree of the field extension $K(Y) \subset$K(X). Show how this implies that there is a well defined answer to the question "How many lines are there on a general cubic surface over a fixed algebraically closed base field?" . Also, prove that although the answer appears to depend on the choice of field, it actually depends only on the characteristic of the field. (It actually is independent of this as well, but you don't need to prove this.) Hint: It will probably be useful to show that if $K \subset L$ is an inclusion of algebraically closed fields, and $X \subset \mathbb{A}^n$ is an affine variety over K, then if we think of X as a variety over L (just using the same defining equations) then the points of X with coordinates in K are Zariski dense in the resulting variety over L.

(3) Let M_n denote the set of $n \times n$ matrices (which we identify with $\mathbb{A}^{n^2}$ in the obvious way.) Let $A_k \subset M_n$ denote the set of matrices with rank less than or equal to k. Show that the singular locus of A_k consists of precisely those matrices M for which the rank of M is strictly less than k. (Except for the trivial case $k = n$.)

(4) (Kleiman's Bertini Theorem) We work over a field of characteristic zero and suppose G is an irreducible algebraic group and Y is a homogeneous space for G. Let X be a smooth irreducible variety, $f : X \to Y$ a regular morphism, and Z a smooth irreducible subvariety of Y.
   (a) Show that the regular morphism $v : G \times X \to Y$ defined by $v(g, x) = g \cdot f(x)$ is smooth. (Use generic smoothness.)
   (b) Deduce that for general $g \in G$ the preimage $f^{-1}(g \cdot Z)$ is either empty or smooth of pure dimension equal to $\dim X - \operatorname{codim} Z$. (Hint: apply generic smoothness to the map $v^{-1}(Z) \to G$.)
   (c) Show that this implies the usual Bertini Theorem for smoothness of the preimage of a general hyperplane under a morphism $f : X \to \mathbb{P}^n$.