ALGEBRAIC GEOMETRY, PROBLEM SET 6

Due Wednesday, November 18.

(1) Let $C \subset \mathbb{A}^2$ be the curve defined by $y^2 = x^3$. We have seen before that C is birational to $\mathbb{A}^1$ via the birational map $f : \mathbb{A}^1 \to C$ given by $f(t) = (t^2, t^3)$. Under the resulting identification of $K(C)$ with $k(t)$, identify the local ring of C at $(0, 0)$ as a subring of $k(t)$.

(2) Suppose $f : X \to Y$ is a morphism of affine varieties. Given a point p in X with $f(p) = q$, show that the differential $D_p f : T_p X \to T_q Y$ defined via the pullback on maximal ideals agrees with the differential defined via partial derivatives. (i.e. with $X \subset \mathbb{A}^m$ and $Y \subset \mathbb{A}^n$ the second version is write out the map as $(f_1, \ldots, f_n)$ and define the derivative using the matrix of partial derivatives to define a derivative map from $k^m \to k^n$ and restrict it to $T_p X$.)

(3) Show that a quadric is smooth if and only if it has maximal rank. (Depending on your conventions, this is actually false - bonus points for correcting the statement.)

(4) Suppose $X = X_1 \cup X_2$ is a union of affine varieties and $p \in X_1 \cap X_2$. Show that $T_p X_1 + T_p X_2 \subset T_p X$. Find an example where one does not have equality.

(5) For which values of a does the curve $x^3 + y^3 + z^3 + a(x + y + z)^3 = 0$ have a singular point? What are the singular points? When is the curve reducible?