

ALGEBRAIC GEOMETRY, PROBLEM SET 5

Due Wednesday, November 11.

- (1) Show that the image of a nonconstant regular morphism $\mathbb{P}^n \rightarrow \mathbb{P}^m$ has dimension n . In particular there are no such morphisms if $m < n$.
- (2) Show that if $\binom{d+r}{r} > (r+1)(n-r)$, a general hypersurface of degree d in \mathbb{P}^n does not contain a linear space of dimension r . (As in class, for the case $r = 1, n = 3$, use the incidence correspondence parametrizing pairs of a degree d homogeneous polynomial in $n + 1$ variables, and a linear space on which that polynomial vanish and consider the projection maps in two directions.)
- (3) Let M be the mn dimensional vector space of m by n matrices, and let M_r be the subset of matrices of rank less than or equal to r . Show that M_r is an irreducible subvariety of dimension $r(m+n-r)$. (Hint: Consider the incidence correspondence

$$\Sigma = \{(A, [L]) \in M \times G(n-r, n) \mid L \subset \ker(A)\}$$

and look at the projection maps to the two factors.)

- (4) Suppose $X \subset \mathbb{P}^n$ is a projective variety of dimension d . Let the secant variety of X , $S(X)$ denote the closure of the union of all lines joining pairs of distinct points of X . (Note, $S(X) \subset \mathbb{P}^n$, I don't mean that it is the corresponding subset of the Grassmannian.) Show that $\dim(S(X)) \leq 2d+1$.
- (5) Use the previous result to show that for any projective variety X of dimension d , there exists an injective map from X to \mathbb{P}^n with $n \leq 2d + 1$.