ALGEBRAIC GEOMETRY, PROBLEM SET 4

Due Wednesday, November 4.

Do 4 of the following 5 problems.

(1) Show that $\mathbb{P}^1 \times \mathbb{A}^1$ is not (isomorphic to) an affine variety or a projective variety.

(2) Suppose X and Y are closed subsets of $\mathbb{P}^n$ and $\Phi : X \to Y$ is a regular morphism with the property that $\Phi(x) \neq x$ for all x in X. (For example, if X and Y are disjoint.) Show that the union over all x in X of the line spanned by x and $\Phi(x)$ is closed. What is this closed set in case X and Y are skew lines in $\mathbb{P}^3$ and Φ is an isomorphism between them?

(3) Consider the $\mathbb{P}^5$ parameterizing nonzero, homogeneous, degree 2 polynomials in 3 variables. This contains two natural closed subsets - the locus X of reducible polynomials and the locus Y of squares of linear polynomials. Naturally, we have $Y \subset X$. Show that X is exactly the closure of the locus of lines which meet Y in at least 2 points. (i.e. X is the secant variety to Y.) (Hint: One approach to this problem would make essential use of the idea to consider a quadratic polynomial as a symmetric matrix - the three types of plane conics - as you classified them earlier - correspond to the different possible ranks of that matrix.) In fact, the word "closure" in the problem is unnecessary, so you can ignore it if you like.

(4) Let $X \subset \mathbb{A}^r$ be a hypersurface. For a line L through the origin, let π_L be the linear projection map projecting X "along L" to an $r-1$ dimensional affine space complementary to L. Show that the set of L (thought of as a subset of the $\mathbb{P}^{r-1}$ of lines through the origin) such that π_L is not finite is Zariski closed. (We have shown in class that it is not all of $\mathbb{P}^{r-1}$.) Hint: Show that this set can be identified with the intersection of $\overline{X}$ with the hyperplane at infinity in $\mathbb{P}^r$. Work this out explicitly for the curve defined by $xy = 1$ in $\mathbb{A}^2$. (It might be useful to do this case first just to make sense of the problem.)

(5) Let $f : X \to Y$ be a dominant morphism of quasi-projective varieties, such that the induced extension of function fields is finite, that is $K(X)$ is a finite dimensional $K(Y)$ vector space. Show that there exists an open subset $V \subset Y$ such that if we set $U = f^{-1}(V)$ then the induced map $f_U : U \to V$ is finite.