ALGEBRAIC GEOMETRY, PROBLEM SET 2

Due Wednesday, October 14.

(1) Show that a morphism $\phi : X \to Y$ of algebraic subsets of $\mathbb{A}^n$ gives an isomorphism of X with a closed subset of Y if and only if $\phi^* : A(Y) \to A(X)$ is surjective.

(2) Show that if $X \subset H \subset \mathbb{P}^n$ is a closed subset of a hyperplane in $\mathbb{P}^n$ and $p \in \mathbb{P}^n$ is a point not in H, that the union of all the lines through X meeting p is also a closed subset of $\mathbb{P}^n$. This is called the cone over X with vertex p.

(3) Show that any set of d points in $\mathbb{P}^n$ can be defined by a collection of polynomials of degree at most d. If the d points are not all on a line, show that they can be defined by polynomials of degree at most $d-1$.

(4) Assume that the characteristic of k is not equal to 2. Prove that every degree 2 hypersurface in $\mathbb{P}^n$ is projectively equivalent to $\{X_0^2+X_1^2+\cdots+X_r^2 = 0\}$ for some $r \leq n$. To say it differently, after a linear change of coordinates, every homogeneous polynomial of degree 2 can be put into this form. (It may be easier, or more familiar, to relate this to the problem of classifying symmetric bilinear forms, by associating to $F = \sum_{i \leq j} f_{ij} X_i X_j$ the symmetric matrix A where the diagonal entries $a_{ii} = f_{ii}$ and off-diagonal entries are given by $a_{ij} = \frac{1}{2} f_{ij}$ so that writing $X = [X_0, \ldots, X_n]$ we have $F(X) = XAX^t$.

(5) *The Veronese embedding* Set $N = \binom{n+d}{d} - 1$ and consider the mapping $\nu_d : \mathbb{P}^n \to \mathbb{P}^N$ given by $\nu_d([X_0, \ldots, X_n]) = [X_0^d, X_0^{d-1}X_1, \ldots X_n^d]$ where the ellipses mean list every possible degree d monomial in the X_i.
- Prove that $\nu_d(\mathbb{P}^n)$ is a closed subset by exhibiting a collection of defining equations. (To avoid getting lost in hopeless notational difficulties, I suggest denoting the homogeneous variables on the big projective space by X_I where I runs over the set of degree d monomials in the X_i, or equivalently, the set of $n+1$-tuples of nonnegative integers which sum to d.)
- Show that in fact ν_d gives an isomorphism of $\mathbb{P}^n$ with its image.
- Deduce that for any homogeneous polynomial F, the quasi-projective variety $\mathbb{P}^n \setminus Z(F)$ is (isomorphic to) an affine variety.