ALGEBRAIC GEOMETRY, PROBLEM SET 1

Due Wednesday, October 7.

(1) If $X \subset \mathbb{A}^n$ is an arbitrary subset, show that $Z(I(X)) = \overline{X}$, that is, the closure of X in the Zariski topology.

(2) Show that if X_1 and X_2 are algebraic subsets of $\mathbb{A}^n$ that

$$I(X_1 \cup X_2) = I(X_1) \cap I(X_2) \quad \text{and} \quad I(X_1 \cap X_2) = \sqrt{I(X_1) + I(X_2)}.$$

Show that the radical in the second part is necessary - i.e. find examples of X_1 and X_2 where $I(X_1 \cap X_2) \neq I(X_1) + I(X_2)$.

(3) Show that if X is a Noetherian topological space that the irreducible decomposition $X = X_1 \cup X_2 \cup \cdots \cup X_n$ whose existence we proved in class is unique, provided we have $X_i \not\subset X_j$ for $i \neq j$.

(4) Let Z be the algebraic set defined by the polynomials $x^2 - yz$ and $xz - x$. Show that Z has 3 irreducible components. Find the corresponding prime ideals.

(5) Prove that if $X \subset \mathbb{A}^2$ is an irreducible algebraic set, then either
- $X = \mathbb{A}^2$
- X is a single point
- or $X = Z(f)$ where f is an irreducible polynomial.

(6) Let $X \subset \mathbb{A}^m$ and $Y \subset \mathbb{A}^n$ be algebraic sets.
    (a) Show that $X \times Y \subset \mathbb{A}^{m+n}$ is an algebraic set.
    (b) If X and Y are irreducible, then so is $X \times Y$.
    (c) Show that $X \times Y$ satisfies the universal property for products - namely the projections $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are morphisms and given any other algebraic set Z and morphisms $f : Z \to X$ and $g : Z \to Y$, there is a unique morphism $f \times g : Z \to X \times Y$ so that $f = \pi_1 \circ (f \times g)$ and $g = \pi_2 \circ (f \times g)$.

Warning!! The Zariski topology on $X \times Y$ is *not* the product topology induced by the Zariski topology on the factors separately. (You should be able to see this in the case $X = Y = \mathbb{A}^1$.)