ALGEBRAIC GEOMETRY FINAL

Due Friday, December 11.

(1) Prove the following fact which has been stated and used in class. Suppose C_1 and C_2 are two curves in $\mathbb{A}^2$ which both contain the point $p = (0, 0)$, and which are both smooth there. Then if we let X be the blow up of $\mathbb{A}^2$ at p, with exceptional divisor E, and we let $\tilde{C}_i$ be the curve in X obtained by considering the irreducible components of $\pi^{-1}(C_i)$ other than E, show that $\tilde{C}_1 \cap \tilde{C}_2 \cap E$ is nonempty if and only if $T_p(C_1) = T_p(C_2)$.

(2) If C is a smooth curve of degree d in $\mathbb{P}^2$, define a map from C to the dual projective plane by the formula $\Phi(p) = T_p(C)$. Check that this is a regular morphism. Assuming that Φ is birational onto its image (this is always true in characteristic zero) compute the degree of $\Phi(C)$.

(3) Show that for a general point p on a general cubic threefold X, there are exactly six lines contained in X which contain p. (Hint: any such line must be contained in T_pX.)

(4) If C is a curve, and p is a point of C, show that p is locally defined by a single equation in C if and only if p is a smooth point of C. (Here "locally defined by" is meant in the strong sense of "locally has ideal generated by".)

(5) Suppose $X \subset \mathbb{P}^n$ is irreducible of dimension r and degree 2. Prove that X is contained in an $r + 1$ dimensional linear space, and hence is isomorphic to a quadric hypersurface.