

# STRONG WAVEFRONT LEMMA AND COUNTING LATTICE POINTS IN SECTORS

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ABSTRACT. We compute the asymptotics of the number of integral quadratic forms with prescribed orthogonal decompositions. This is a partial case of our main result on counting lattice point in sectors in affine symmetric spaces. The crucial ingredient of the proof is the strong wavefront lemma, which shows that the Cartan decomposition is uniformly Lipschitz.

## 1. INTRODUCTION

One of the motivations of this paper is a certain counting problem in the space of quadratic forms. Let  $\mathcal{Q}_W$  the set of real quadratic forms on a Euclidean space  $W$  with determinant equal to  $\pm 1$ . It follows from the main result of Duke, Rudnick, Sarnak [DRS] or Eskin, McMullen [EM] that

$$\#\{q \in \mathcal{Q}_W(\mathbb{Z}) : \|q\| < T\} \sim_{T \rightarrow \infty} c \cdot T^{d(d-1)/2}$$

where  $d = \dim W$  and  $c > 0$ . Here we consider a refinement of this problem that concerns counting quadratic forms with prescribed structure. Fix an orthogonal decomposition

$$(1.1) \quad W = \bigoplus_{i=0}^n W_i,$$

and for  $\Omega \subset \mathrm{SO}(W)$  and  $\Omega' \subset \mathcal{Q}_{W_0} \times \cdots \times \mathcal{Q}_{W_n}$ , set

$$N_T(\Omega, \Omega') = \# \left\{ q \in \mathcal{Q}_W(\mathbb{Z}) : \begin{array}{l} \|q\| < T, \\ q(k \cdot x) = \alpha_0 q_0(x) \perp \cdots \perp \alpha_n q_n(x) \\ \text{for some } k \in \Omega, (q_0, \dots, q_n) \in \Omega', \\ \text{and } \alpha_0 > \cdots > \alpha_n > 0 \end{array} \right\}.$$

Assuming that  $\Omega$  and  $\Omega'$  are bounded measurable sets such that the subset  $\Omega\Omega'$  has positive measure and boundary of measure zero<sup>1</sup>, we prove that

**Theorem 1.2.**

$$N_T(\Omega, \Omega') \sim_{T \rightarrow \infty} c \cdot T^{d(d-\dim W_n)/2}$$

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<sup>1</sup>The measure of  $\Omega\Omega'$  is understood in terms of the identification (2.1).

for some  $c = c(\Omega, \Omega') > 0$ .

Theorem 1.2 is an example of our main result on counting lattice points in sectors on affine symmetric spaces. Let  $G$  be a connected noncompact semisimple Lie group with finite center. A subgroup  $H$  of  $G$  is called symmetric if its Lie algebra is the set of fixed points of an involution. In this case, the homogeneous space  $G/H$  is called an affine symmetric space. Affine symmetric spaces have many similar features with Riemannian symmetric spaces. In particular, a Cartan decomposition holds

$$G = KAH$$

where  $K$  is a maximal compact subgroup of  $G$  compatible with  $H$ , and  $A$  is a Cartan subgroup corresponding to the pair  $(K, H)$ . The Cartan subgroup supports a system of simple roots  $\Delta_\sigma$ . An element  $g = kah \in KAH$  will be called  $c$ -regular if the distance of  $a$  from the walls  $\{\alpha = 0\}$ ,  $\alpha \in \Delta_\sigma$ , is at least  $c$ . Otherwise, we call such element  $c$ -singular.

We fix a Riemannian metric on  $G$  and denote by  $\mathcal{O}_\varepsilon$  the  $\varepsilon$ -ball at identity.

**Theorem 1.3** (Strong wavefront lemma). *Given  $c > 0$ , there exist  $\ell > 1$  and  $\varepsilon_0 > 0$  such that for every  $c$ -regular  $g = kah \in KAH$  and  $0 < \varepsilon < \varepsilon_0$ ,*

$$\mathcal{O}_\varepsilon \cdot g \subset (K \cap \mathcal{O}_{\ell\varepsilon})k \cdot (A \cap \mathcal{O}_{\ell\varepsilon})a \cdot (H \cap \mathcal{O}_{\ell\varepsilon})h.$$

The wavefront lemma, which is a weaker form of Theorem 1.3, was introduced by Eskin and McMullen [EM, Theorem 3.1] (see also [EMM, Lemma 5.11]). It says that for any neighborhood  $\mathcal{O}$  of  $e$  in  $G$  there exists a neighborhood  $\tilde{\mathcal{O}}$  of  $e$  in  $G$  such that that for every  $g \in KA$ ,

$$\tilde{\mathcal{O}}g \subset g\mathcal{O}H.$$

The continuity of the Cartan decomposition for Riemannian symmetric spaces (i.e., when  $H = K$ ) was independently shown in Nevo [N, Proposition 7.3] and by Gorodnik, Oh [GO, Theorem 2.1]. While the proof of [N] uses embeddings of  $G$  in linear groups, the proof of [GO] is based on geometric properties of the Riemannian symmetric spaces. The strong wavefront lemma was used in [N] to prove maximal inequalities for cube averages on semisimple groups and in [GO] to compute the asymptotics of the number of lattice points lying in sectors.

Theorem 1.3 fails on the set of singular elements (for example, consider the polar coordinates on the hyperbolic plane). To state a version of the strong wavefront lemma that holds for singular elements, we introduce additional notation. Let  $\Delta_\sigma$  be the set of simple roots of  $A$  and  $I \subset \Delta_\sigma$ . We call an element  $kah \in KAH$   $(J, c)$ -regular if the distance of  $a$  from the walls  $\{\alpha = 0\}$ ,  $\alpha \in J$ , is at least  $c$ . For  $I \subset \Delta_\sigma$ , we set  $A_I = \ker(I) \subset I$ . There exists a connected semisimple subgroup  $M_I$  centralizing  $A_I$  such that

$$G = KM_I A_I H \quad \text{and} \quad M_I \cap A_I = 1.$$

**Theorem 1.4** (Strong wavefront lemma: general form). *Let  $\Delta_\sigma = I \sqcup J$ . Given  $c > 0$ , there exist  $\ell > 1$  and  $\varepsilon_0 > 0$  such that for every  $(J, c)$ -regular  $g = kah \in KAH$  and  $0 < \varepsilon < \varepsilon_0$ ,*

$$\mathcal{O}_\varepsilon \cdot g \subset (K \cap \mathcal{O}_{\ell\varepsilon})k \cdot (M_I \cap \mathcal{O}_{\ell\varepsilon}) \cdot (A_I \cap \mathcal{O}_{\ell\varepsilon})a \cdot (H \cap \mathcal{O}_{\ell\varepsilon})h.$$

We use Theorem 1.3 to solve the lattice counting problem for sectors with respect to the decomposition:

$$G = KM_I A_I H.$$

Let  $\iota : G \rightarrow \mathrm{GL}(W)$  be a representation of  $G$  and  $v_0 \in W$  such that the orbit  $V = Gv_0$  is an affine symmetric space. We set  $H = \mathrm{Stab}_G(v_0)$ . Let  $\Gamma$  be a lattice in  $G$  such that  $H \cap \Gamma$  is a lattice in  $H$ . In particular,  $\Gamma v_0$  is discrete. For a norm  $\|\cdot\|$  on  $W$ , we set

$$B_T = \{w \in W : \|w\| < T\}.$$

It was shown by Duke, Rudnick, Sarnak [DRS] and Eskin, McMullen [EM] that the orbit  $\Gamma v_0$  is equidistribution with respect to the sets  $V \cap B_T$ :

$$(1.5) \quad \#(\Gamma v_0 \cap B_T) \sim_{T \rightarrow \infty} \mathrm{Vol}(V \cap B_T)$$

where  $\mathrm{Vol}$  is the normalized  $G$ -invariant measure on  $V$ . In fact, it was shown there that (1.5) holds for any *well-rounded* family of sets  $S_T \subset V$ . Recall that a family  $S_T$  is called well-rounded if for any  $\varepsilon > 0$  there exists a neighborhood  $\mathcal{O}$  of  $e$  in  $G$  such that

$$\frac{\mathrm{Vol}(\mathcal{O} \cdot \partial S_T)}{\mathrm{Vol}(S_T)} < \varepsilon$$

for all sufficiently large  $T > 0$ . For  $\Omega \subset KM_I/(M_I \cap H)$ , we consider a family of sets

$$S_T(\Omega) = \Omega A_I^+ v_0 \cap B_T$$

where

$$A_I^+ = \{a \in A_I : \alpha(\log a) > 0 \text{ for } \alpha \notin I\}.$$

Using the strong wavefront lemma, we deduce that

**Theorem 1.6.** *For every bounded measurable set  $\Omega \subset KM_I/(M_I \cap H)$  with positive measure and boundary of measure zero<sup>2</sup>, the family  $S_T(\Omega)$  is well-rounded. In particular,*

$$\#(\Gamma v_0 \cap S_T(\Omega)) \sim_{T \rightarrow \infty} \mathrm{Vol}(S_T(\Omega)).$$

**Remark 1.7.** (1) Although a similar counting question was considered in [GOS], the sets  $S_T(\Omega)$  don't fit into the framework of [GOS]. For the space of quadratic forms  $\mathcal{Q}_W$ , the counting results in [GOS] are always of order  $T^{(\dim W)(\dim W - 1)/2}$  (see [GOS, Section 2.3]). On the other hand,

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<sup>2</sup>The measure on  $KM_I/(M_I \cap H)$  is understood in terms of the identification (2.1).

Theorem 1.2 exhibits different asymptotic behaviors depending on the choice of the decomposition (1.1).

- (2) It can be deduced from the volume computation in [GOS] (see Proposition 4.1 below) that for measurable bounded  $\Omega \subset KM_I/(M_I \cap H)$ ,

$$\text{Vol}(S_T(\Omega)) \sim_{T \rightarrow \infty} c_I(\Omega) \cdot T^{a_I} (\log T)^{b_I - 1}$$

where  $a_I \in \mathbb{Q}^+$ ,  $b_I \in \mathbb{N}$ , and  $c_I(\Omega)$  is positive when the set  $\Omega$  has positive measure. We give explicit formulas for  $a_I$ ,  $b_I$ , and  $c_I$  in Remark 4.6 below. In particular,  $c_I$  can be considered as a measure supported on one of the components of the Satake boundary.

- (3) We mention the following issue that requires extra care in the proof of Theorem 1.6: usually,

$$\liminf_{T \rightarrow \infty} \frac{\text{Vol}(B_T \cap \{(J, c)\text{-singular elements in } V\})}{\text{Vol}(B_T \cap V)} > 0$$

for all  $c > 0$ . Nonetheless, we show that

$$\limsup_{T \rightarrow \infty} \frac{\text{Vol}(B_T \cap \{(J, c)\text{-singular elements in } V\})}{\text{Vol}(B_T \cap V)} \ll c.$$

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## 2. CARTAN DECOMPOSITION

In this section, we set up notations and review basic properties of affine symmetric spaces (see [Sc, Ch. 7], [HS, Part II] or [GOS] for more details).

Let  $G$  be a connected noncompact semisimple Lie group with finite center and  $\mathfrak{g}$  its Lie algebra. A closed subgroup  $H$  of  $G$ , with the Lie algebra  $\mathfrak{h} \subset \mathfrak{g}$ , is called *symmetric* if  $\mathfrak{h}$  is the set of fixed points of an involution  $\sigma$  of  $\mathfrak{g}$ . Then the factor space  $G/H$  is called an *affine symmetric space*.

There exists a Cartan involution  $\theta$  of  $\mathfrak{g}$  which commutes with  $\sigma$ . We denote by  $K$  the maximal compact subgroup of  $G$  that corresponds to  $\theta$  and by  $\mathfrak{k}$  its Lie algebra. We have decompositions

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q} \quad \text{and} \quad \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

into  $+1$  and  $-1$  eigenspaces of  $\sigma$  and  $\theta$  respectively.

A *Cartan subalgebra*  $\mathfrak{a}$  corresponding to the pair  $(K, H)$  is a maximal abelian subalgebra of  $\mathfrak{p} \cap \mathfrak{q}$ , and  $A = \exp(\mathfrak{a})$  is called a Cartan subgroup. We have the decomposition

$$\mathfrak{g} = \sum_{\beta \in \Sigma_\sigma \cup \{0\}} \mathfrak{g}_\beta$$

into the root spaces  $\mathfrak{g}_\beta$  of  $\mathfrak{g}$ , where  $\Sigma_\sigma$  is a root system on  $\mathfrak{a}$ . We choose a system  $\Delta_\sigma$  of simple roots and denote by  $\Sigma_\sigma^+$  the corresponding set of positive roots.

We have the *Cartan decomposition*

$$G = KAH.$$

Note that the  $A$ -component is uniquely defined, and the  $K$ -component of a regular element is uniquely defined modulo  $M = C_{K \cap H}(A)$ . For a finite  $\mathcal{W} \subset N_K(A)$ ,

$$G = KA^+\mathcal{W}H.$$

For  $I \subset \Delta_\sigma$ , we set

$$A_I = \{a \in A : \alpha(\log a) = 0 \text{ for } \alpha \in I\} \quad \text{and} \quad Z_I = C_G(A_I).$$

There exists a connected semisimple subgroup  $M_I \subset Z_I$  such that

$$G = KM_I A_I H \quad \text{and} \quad M_I \cap A_I = 1$$

(see [GOS, Section 4] for the definition of  $M_I$ ). Note that  $A_I$ -component is uniquely defined, and for  $(\Delta_\sigma - I)$ -regular elements, and the  $KM_I$ -component is uniquely defined modulo  $M_I \cap H$ . Also, we have decomposition

$$G = KM_I A_I^+ \mathcal{W}H.$$

The group  $M_I$  also has the Cartan decomposition (see [GOS, Proposition 4.22])

$$M_I = (M_I \cap K) \tilde{A}^{I,+} (M_I \cap H)$$

where  $\tilde{A}^{I,+}$  is a cone in the complement of  $A_I$ . This decomposition defines a diffeomorphism

$$(2.1) \quad KM_I / (M_I \cap H) \cong K / C_{K \cap H}(A) \times \text{int}(\tilde{A}^{I,+})$$

between a dense open subset of  $KM_I / (M_I \cap H)$  and the right hand side.

### 3. STRONG WAVEFRONT LEMMA

This section is devoted to the proof of Theorems 1.3 and 1.4. Since any two Riemannian metrics are bi-Lipschitz in a neighborhood of identity, it suffices to prove the theorem for one such metric. It will be convenient to work with the right-invariant Riemannian metric  $d$  induced by the positive definite form

$$B(X, Y) = -\text{Tr}(\text{ad } X \circ \text{ad}(\theta(Y))), \quad X, Y \in \mathfrak{g}.$$

We will use the following properties of  $B$ :

$$\begin{aligned} B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) &= 0 \quad \text{for all } \alpha \neq \beta \in \Sigma_\sigma \cup \{0\}, \\ B^\theta &= B^\sigma = B. \end{aligned}$$

For  $\varepsilon > 0$  and  $S \subset G$ , we set

$$S_\varepsilon = \{s \in S : d(s, e) < \varepsilon\}.$$

For  $I \subset \Delta_\sigma$  and  $c > 0$ , we define

$$A_I^+(c) = \{a \in A^+ : \beta(\log a) \geq c \text{ if } \beta \in \Delta_\sigma - I \text{ and } \beta(\log a) < c \text{ if } \beta \in I\}.$$

For instance, if  $I = \{\beta\}$ , then  $A_I^+(c)$  forms a system of neighborhoods of the wall  $\{a \in A^+ : \beta(\log a) = 0\}$  in  $A^+$ .

**Theorem 3.1.** *For  $I \subset \Delta_\sigma$  and  $c > 0$ , there exist  $\varepsilon_0 > 0$  and  $\sigma > 1$  such that for every  $0 < \varepsilon < \varepsilon_0$  and  $a \in A_I^+(c)$ ,*

$$G_\varepsilon \cdot a \subset K_{\sigma\varepsilon} \cdot Z_{I, \sigma\varepsilon} \cdot a \cdot H_{\sigma\varepsilon}.$$

We consider the Lie subalgebras

$$\mathfrak{n}_I^+ = \bigoplus_{\beta \in \Sigma_\sigma^+ : \beta|_{\mathfrak{a}_I} \neq 0} \mathfrak{g}_\beta \quad \text{and} \quad \mathfrak{n}_I^- = \bigoplus_{\beta \in \Sigma_\sigma^+ : \beta|_{\mathfrak{a}_I} \neq 0} \mathfrak{g}_{-\beta},$$

and the corresponding analytic subgroups  $N_I^+$  and  $N_I^-$ . Note that the Lie algebra of  $Z_I$  is given by

$$\mathfrak{z}_I = \bigoplus_{\beta \in \Sigma_\sigma \cup \{0\} : \beta|_{\mathfrak{a}_I} = 0} \mathfrak{g}_\beta,$$

and we have the decomposition

$$(3.2) \quad \mathfrak{g} = \mathfrak{n}_I^- \oplus \mathfrak{z}_I \oplus \mathfrak{n}_I^+.$$

**Lemma 3.3.** *There exist  $\theta > 1$  and  $\varepsilon_0 > 0$  such that for every  $0 < \varepsilon < \varepsilon_0$ ,*

$$G_\varepsilon \subset N_{I, \theta\varepsilon}^- Z_{I, \theta\varepsilon} H_{\theta\varepsilon} \quad \text{and} \quad G_\varepsilon \subset K_{\theta\varepsilon} Z_{I, \theta\varepsilon} N_{I, \theta\varepsilon}^+.$$

*Proof.* Since  $\sigma|_{\mathfrak{a}} = -id$ , we have  $\sigma(\mathfrak{n}_I^-) \subset \mathfrak{n}_I^+$ , and for every  $x \in \mathfrak{n}_I^+$ ,

$$x = (x + \sigma(x)) - \sigma(x) \in \mathfrak{h} + \mathfrak{n}_I^-.$$

Hence, it follows from (3.2) that

$$\mathfrak{g} = \mathfrak{n}_I^- + \mathfrak{z}_I + \mathfrak{h}.$$

Since  $\mathfrak{n}_I^- \cap \mathfrak{h} = 0$ , there exists a subspace  $\mathfrak{z}_0$  of  $\mathfrak{z}_I$  such that

$$\mathfrak{g} = \mathfrak{n}_I^- \oplus \mathfrak{z}_0 \oplus \mathfrak{h}.$$

Then the product map  $N_I^- \times \exp(\mathfrak{z}_0) \times H \rightarrow G$  is a diffeomorphism at a neighborhood of the identity. In particular, it is bi-Lipschitz, and the first claim follows. The proof of the second claim is similar.  $\square$

**Lemma 3.4.** *For  $I \subset \Delta_\sigma$  and  $c > 0$ , there exist  $\varepsilon_0 > 0$  and  $\alpha \in (0, 1)$  such that for every  $0 < \varepsilon < \varepsilon_0$  and  $a \in A_I^+(c)$ ,*

$$a^{-1} N_{I, \varepsilon}^+ a \subset N_{I, \alpha\varepsilon}^+ \quad \text{and} \quad a N_{I, \varepsilon}^- a^{-1} \subset N_{I, \alpha\varepsilon}^-.$$

*Proof.* For

$$X = \sum_{\beta \in \Sigma_\sigma^+ : \beta|_{\mathfrak{a}_I} \neq 0} X_\beta \in \mathfrak{n}_I^+, \quad X_\beta \in \mathfrak{g}_\beta,$$

we have

$$\text{Ad}(a^{-1})X = \sum_{\beta} \text{Ad}(a^{-1})X_\beta = \sum_{\beta} e^{-\beta(\log a)} X_\beta.$$

Note that if  $\beta = \sum_{\alpha \in \Delta_\sigma} n_\alpha \alpha \in \Sigma_\sigma^+$  with  $n_\alpha \geq 0$  satisfies  $\beta|_{\mathfrak{a}_I} \neq 0$ , then  $n_\alpha \geq 1$  for some  $\alpha \in \Delta_\sigma - I$ . Hence, for  $a \in A_I^+(c)$ , we have  $\beta(\log a) \geq c$  and

$$\|\mathrm{Ad}(a^{-1})X_\beta\| \leq e^{-c}\|X_\beta\|.$$

Since the root spaces  $\mathfrak{g}_\beta$  are orthogonal to each other,

$$(3.5) \quad \|\mathrm{Ad}(a^{-1})X\| \leq e^{-c}\|X\|.$$

Since the differential of the exponential map  $\exp : \mathfrak{n}_I^+ \rightarrow N_I^+$  is identity at 0, we can find a small ball  $U$  at 0 in  $\mathfrak{n}_I^+$  such that for every  $Y \in U$ ,

$$(3.6) \quad e^{-c/3}\|Y\| \leq d(\exp(Y), e) \leq e^{c/3}\|Y\|.$$

Note that for  $a \in A^+$ , we have  $\mathrm{Ad}(a^{-1})U \subset U$ . Combining (3.5) and (3.6), we deduce that for  $a \in A_I^+(c)$  and  $n = \exp(X) \in \exp(U)$ ,

$$\begin{aligned} d(a^{-1}na, e) &= d(\exp(\mathrm{Ad}(a^{-1})X), e) \leq e^{c/3}\|\mathrm{Ad}(a^{-1})X\| \\ &\leq e^{-2c/3}\|X\| \leq e^{-c/3}d(n, e). \end{aligned}$$

This proves the claim for  $N_I^+$ . The claim for  $N_I^-$  is proved similarly.  $\square$

**Lemma 3.7.** *For  $I \subset \Delta_\sigma$  and  $\tau > 1$ , there exists  $\varepsilon_0 > 0$  such that for every  $z \in Z_{I, \varepsilon_0}$  and  $0 < \varepsilon < \varepsilon_0$ ,*

$$zN_\varepsilon^+ z^{-1} \subset N_{\tau\varepsilon}^+ \quad \text{and} \quad zN_\varepsilon^- z^{-1} \subset N_{\tau\varepsilon}^-.$$

*Proof.* It is easy to check that  $L_I$  normalizes  $N_I^\pm$ .

We can choose  $\varepsilon_0 > 0$  so that

$$\begin{aligned} \|\mathrm{Ad}(z)X\| &\leq \tau^{1/3}\|X\|, & z \in Z_{I, \varepsilon_0}, X \in \mathfrak{n}_I^+, \\ \tau^{-1/3}\|X\| &\leq d(\exp(X), e) \leq \tau^{1/3}\|X\|, & X \in \mathrm{Ad}(Z_{I, \varepsilon_0})\exp^{-1}(N_{I, \varepsilon_0}^+). \end{aligned}$$

Then for every  $n = \exp(X) \in N_{I, \varepsilon_0}^+$ ,

$$\begin{aligned} d(znz^{-1}, e) &= d(\exp(\mathrm{Ad}(z)X), e) \leq \tau^{1/3}\|\mathrm{Ad}(z)X\| \\ &\leq \tau^{2/3}\|X\| \leq \tau d(n, e). \end{aligned}$$

This proves the first part of the lemma. The proof of the second part is similar.  $\square$

**Lemma 3.8.** *For  $I \subset \Delta_\sigma$  and  $\gamma > 1$ , there exists  $\varepsilon_0 > 0$  such that for every  $0 < \varepsilon < \varepsilon_0$ ,*

$$N_{I, \varepsilon}^+ \subset N_{I, \gamma\varepsilon}^- Z_{I, \varepsilon} H_{2\gamma\varepsilon} \quad \text{and} \quad N_{I, \varepsilon}^- \subset K_{2\gamma\varepsilon} Z_{I, \varepsilon} N_{I, \gamma\varepsilon}^+.$$

*Proof.* As in the proof of Lemma 3.3, we choose a subspace  $\mathfrak{z}_0$  of  $\mathfrak{z}_I$  such that the product map  $N_I^- \times \exp(\mathfrak{z}_0) \times H \rightarrow G$  is a diffeomorphism in a neighborhood of the identity. Denote by  $f$  the local inverse the product map:

$$f = (f_1, f_2, f_3) : U \rightarrow N_I^- \times \exp(\mathfrak{z}_0) \times H$$

where  $U$  is a neighborhood of identity in  $G$ . For  $X \in \mathfrak{n}_I^+$ , the derivative  $(df)_e$  is given by

$$(df)_e(X) = (-\sigma(X), 0, X + \sigma(X)) \in \mathfrak{n}_I^- \oplus \mathfrak{z}_0 \oplus \mathfrak{h}.$$

Since the Riemannian metric at identity is invariant under  $\sigma$ , we have for  $X \in \mathfrak{n}_I^+$ ,

$$\|(df_1)_e(X)\| = \|X\|, \quad (df_2)_e = 0, \quad \|(df_3)_e(X)\| \leq 2\|X\|.$$

This implies that for sufficiently small  $\varepsilon > 0$ ,

$$f(N_{I,\varepsilon}^+) \subset N_{I,\gamma\varepsilon}^- \times Z_{I,\varepsilon} \times H_{2\gamma\varepsilon}.$$

This proves the first claim. The proof of the second claim is similar.  $\square$

**Lemma 3.9.** *For  $I \subset \Delta_\sigma$  and  $c > 0$ , there exist  $0 < \beta < 1$  and  $\varepsilon_0 > 0$  such that for every  $0 < \varepsilon, \delta < \varepsilon_0$  and  $a \in A_I^+(c)$ ,*

$$K_\varepsilon Z_{I,\varepsilon} a Z_{I,\varepsilon} N_{I,\delta}^+ H_\varepsilon \subset K_{\varepsilon+4\delta} Z_{I,\varepsilon+4\delta} a Z_{I,\varepsilon+4\delta} N_{I,\beta\delta}^+ H_{\varepsilon+4\delta}.$$

*Proof.* For simplicity, we write  $N_I^\pm = N^\pm$  and  $Z_I = Z$ .

Choose  $\alpha = \alpha(c) \in (0, 1)$  as in Lemma 3.4,  $\gamma \in (1, 2)$  so that  $\alpha\gamma^2 < 1$ , and  $\tau > 1$  so that  $\tau^5\alpha\gamma^2 < 1$ . Let  $\varepsilon_0 > 0$  be such that Lemmas 3.4, 3.7, 3.8 hold. Fixing  $0 < \varepsilon < \varepsilon_0$ , let  $k_0 \in K_\varepsilon$ ,  $x_0, y_0 \in Z_\varepsilon$ ,  $n_0^+ \in N_\delta^+$ , and  $h_0 \in H_\varepsilon$ . Then

$$\begin{aligned} & k_0 x_0 a y_0 n_0^+ h_0 \\ &= k_0 x_0 a y_0 (n_1^- y_1 h_1) h_0 && \text{by Lemma 3.8} \\ & && \text{with } n_1^- \in N_{\gamma\delta}^-, y_1 \in Z_\delta, h_1 \in H_{2\gamma\delta} \\ &= k_0 n_2^- x_0 a y_0 y_1 h_1 h_0 && \text{by Lemmas 3.7 and 3.4} \\ & && \text{with } n_2^- \in N_{\tau^2\alpha\gamma\delta}^- \\ &= k_0 (k_2 x_2 n_2^+) x_0 a y_0 y_1 h_1 h_0 && \text{by Lemma 3.8,} \\ & && \text{with } k_2 \in K_{2\tau^2\alpha\gamma^2\delta}, x_2 \in Z_{\tau^2\alpha\gamma\delta}, n_2^+ \in N_{\tau^2\alpha\gamma^2\delta}^+ \\ &= k_0 k_2 (x_2 x_0 a y_0 y_1) n_3^+ h_1 h_0 && \text{by Lemmas 3.7 and 3.4} \\ & && \text{with } n_3^+ \in N_{\tau^5\alpha^2\gamma^2\delta}^+. \end{aligned}$$

Since  $\tau^5\alpha\gamma^2 < 1$ , we have

$$k_0 k_2 \in K_{\varepsilon+4\delta}, \quad x_2 x_0, y_0 y_1 \in Z_{\varepsilon+4\delta}, \quad n_3^+ \in N_{\beta\delta}^+, \quad h_1 h_0 \in H_{\varepsilon+4\delta}.$$

where  $\beta = \tau^5\alpha^2\gamma^2 < 1$ .  $\square$

*Proof of Theorem 3.1.* Set  $N_I^\pm = N^\pm$  and  $Z_I = Z$  for simplicity.

We choose  $\varepsilon_0 > 0$  so that Lemmas 3.3 (for some  $\theta > 1$ ), 3.4, 3.9 hold. Because of Lemma 3.3, it suffices to show that

$$K_\varepsilon Z_\varepsilon N_\varepsilon^+ \cdot a \subset K_{\sigma\varepsilon} (Z_{\sigma\varepsilon} a) H_{\sigma\varepsilon}$$

for some  $\sigma > 1$ . Also by Lemma 3.4,

$$K_\varepsilon Z_\varepsilon N_\varepsilon^+ \cdot a \subset K_\varepsilon (Z_\varepsilon a Z_\varepsilon) N_\varepsilon^+ H_\varepsilon.$$

Now we can apply Lemma 3.9 inductively. We consider  $\varepsilon > 0$  such that

$$(3.10) \quad \varepsilon + \frac{4\varepsilon}{1-\beta} < \varepsilon_0.$$

Setting  $\varepsilon_0 = \delta_0 = \varepsilon$ , we apply Lemma 3.9 to find

$$\varepsilon_{i+1} < \varepsilon_i + 2\delta_i, \quad \delta_{i+1} < \beta\delta_i$$

such that for every  $a \in A_I^+(c)$ ,

$$K_{\varepsilon_i} Z_{\varepsilon_i} a Z_{\varepsilon_i} N_{\delta_i}^+ H_{\varepsilon_i} \subset K_{\varepsilon_{i+1}} Z_{\varepsilon_{i+1}} a Z_{\varepsilon_{i+1}} N_{\delta_{i+1}}^+ H_{\varepsilon_{i+1}}.$$

Note that

$$\delta_i < \varepsilon\beta^i \quad \text{and} \quad \varepsilon_i < \varepsilon + 4\varepsilon \frac{1-\beta^i}{1-\beta}.$$

Hence by (3.10),  $\varepsilon_i, \delta_i < \varepsilon_0$ , and we can continue this process indefinitely.

It follows that for every  $g \in K_\varepsilon(Z_\varepsilon a Z_\varepsilon)N_\varepsilon^+ H_\varepsilon$ , there exist sequences  $k_i \in K_{\varepsilon_i}, x_i, y_i \in Z_{\varepsilon_i}, n_i \in N_{\delta_i}^+, h_i \in H_{\varepsilon_i}$  such that  $g = k_i x_i a y_i n_i h_i$  for all  $i \geq 1$ . Since  $\delta_i \rightarrow 0, n_i \rightarrow e$ . Also, passing to a subsequence, we may assume that  $k_i \rightarrow k, x_i \rightarrow x, y_i \rightarrow y, h_i \in h$ . Then

$$g = k x a y h \subset K_{\rho\varepsilon} Z_{\rho\varepsilon} a Z_{\rho\varepsilon} H_{\rho\varepsilon}$$

with  $\rho = 1 + 4(1-\beta)^{-1}$ . We have decomposition  $a = a_1 a_2$  where  $a_1 \in A_I^+$  and  $a_2$  is in the fixed compact set determined by  $c$ . This implies that for some  $\tau > 1$ ,

$$a Z_{\rho\varepsilon} a^{-1} \subset Z_{\tau\rho\varepsilon},$$

and the theorem follows.  $\square$

*Proof of Theorem 1.4.* There exists  $\zeta > 1$  such that  $k^{-1}\mathcal{O}_\varepsilon k \subset \mathcal{O}_{\zeta\varepsilon}$  for every  $k \in K$ . Then for  $g = kah \in KAH$ , we have

$$\mathcal{O}_\varepsilon \cdot g \subset k(\mathcal{O}_{\zeta\varepsilon} a)h.$$

Hence, it suffices to prove the the theorem for  $g = a \in A$ . Without loss of generality, we may assume that  $a \in A^+$ .

Since  $M_{I_1} \subset M_{I_2}$  for  $I_1 \subset I_2$ , we may assume that  $J$  is maximal such that  $a$  is  $(J, c)$ -regular. Then  $a \in A_I^+(c)$ . We have the decomposition

$$(3.11) \quad \mathfrak{z}_I = (\mathfrak{z}_I \cap \mathfrak{k}) \oplus (\mathfrak{m}_I \cap \mathfrak{p} \cap \mathfrak{q}) \oplus \mathfrak{a}_I \oplus (\mathfrak{z}_I \cap \mathfrak{h})$$

(see [GOS, equation (4.24)]). Hence, the product map

$$(Z_I \cap K) \times \exp(\mathfrak{m}_I \cap \mathfrak{p} \cap \mathfrak{q}) \times A_I \times (Z_I \cap H) \rightarrow Z_I$$

is a diffeomorphism in a neighborhood of identity, and there exists  $\eta > 1$  such that for sufficiently small  $\varepsilon > 0$ ,

$$Z_{I,\varepsilon} \subset (Z_I \cap K)_{\eta\varepsilon} \exp(\mathfrak{m}_I \cap \mathfrak{p} \cap \mathfrak{q})_{\eta\varepsilon} A_{I,\eta\varepsilon} (Z_I \cap H)_{\eta\varepsilon}.$$

Therefore, it follows from Theorem 3.1 that

$$\mathcal{O}_\varepsilon \cdot a \subset K_{\sigma\varepsilon} Z_{\sigma\varepsilon} a H_{\sigma\varepsilon} \subset K_{(\sigma+\sigma\varepsilon)\varepsilon} M_{I,\sigma\varepsilon} (A_{I,\sigma\varepsilon} a) H_{(\sigma+\sigma\varepsilon)\varepsilon}.$$

This proves the theorem.  $\square$

*Proof of Theorem 1.3.* Suppose that in Theorem 1.4 we have  $J = \Delta_\sigma$ . Then  $Z = C_G(A)$  is  $\sigma$ - and  $\theta$ -invariant, and

$$\mathfrak{z} = (\mathfrak{z} \cap \mathfrak{k}) \oplus (\mathfrak{z} \cap \mathfrak{p} \cap \mathfrak{q}) \oplus (\mathfrak{z} \cap \mathfrak{h}).$$

Since  $\mathfrak{a}$  is a maximal abelian subspace of  $\mathfrak{p} \cap \mathfrak{q}$ ,  $\mathfrak{z} \cap \mathfrak{p} \cap \mathfrak{q} = \mathfrak{a}$ . Hence, decomposition (3.11) becomes

$$\mathfrak{z} = (\mathfrak{z} \cap \mathfrak{k}) \oplus \mathfrak{a} \oplus (\mathfrak{z} \cap \mathfrak{h}),$$

and we complete the proof as in Theorem 1.4.  $\square$

#### 4. WELL-ROUNDEDNESS OF SECTORS

In this section, we prove Theorem 1.6. We start by computing the volume asymptotics of sectors:

**Proposition 4.1.** *There exist  $a_I \in \mathbb{Q}^+$  and  $b_I \in \mathbb{N}$  such that for any bounded measurable set  $\Omega \subset KM_I/(M_I \cap H)$ ,*

$$\text{Vol}(S_T(\Omega)) \sim_{T \rightarrow \infty} c_I(\Omega) \cdot T^{a_I} (\log T)^{b_I-1}$$

where  $c_I(\Omega)$  is positive when the set  $\Omega$  has positive measure.

*Proof.* It is easy to deduce from [HS, Theorem 2.5] (see also [GOS]) that a  $G$ -invariant measure on  $G/H$  is given by

$$(4.2) \quad \int_{G/H} f d\mu = \sum_{w \in \mathcal{W}} \int_{KM_I/(M_I \cap wHw^{-1})} \int_{A_I^+} f(\omega awH) \xi_I(a) da d\omega, \quad f \in C_c(G/H),$$

where  $da$  is a Haar measure on  $A_I$ ,  $d\omega$  is a measure on  $KM_I/(M_I \cap H)$  induced from Haar measures, and

$$(4.3) \quad \xi_I(a) = \prod_{\alpha \in \Sigma_\sigma^+ - \langle I \rangle} \sin(\alpha(\log a))^{l_\alpha^+} \cos(\alpha(\log a))^{l_\alpha^-}.$$

Here  $l_\alpha^\pm$  denote the dimensions of the  $(\pm 1)$ -eigenspaces of  $\sigma\theta$  in  $\mathfrak{g}_\alpha$ . Applying [GOS, Theorem 6.1], we deduce that there exist  $a_I \in \mathbb{Q}^+$  and  $b_I \in \mathbb{N}$  such that for every  $f \in C_c(W)$ ,

$$(4.4) \quad \lim_{T \rightarrow \infty} \frac{1}{T^{a_I} (\log T)^{b_I-1}} \int_{A_I^+} f(av_0/T) \xi_I(a) da = \int_W f d\nu$$

where  $\nu$  is a nonzero locally finite measure on  $W$ . Note that  $\nu$  is homogeneous of degree  $a_I$ .

Let  $c > 1$  and take a continuous function  $\psi : [0, \infty] \rightarrow [0, 1]$  such that  $\text{supp}(\psi) \subset [0, c]$  and  $\psi = 1$  on  $[0, 1]$ . Setting  $f(y) = \psi(\|y\|)$ , we have

$$\int_{A_I^+} \chi_{B_T}(av_0)\xi_I(a)da \leq \int_{A_I^+} f(av_0/T)\xi_I(a)da,$$

and by (4.4),

$$\limsup_{T \rightarrow \infty} \frac{1}{T^{a_I}(\log T)^{b_I-1}} \int_{A_I^+} \chi_{B_T}(av_0)\xi_I(a)da \leq \int_W f d\nu \leq c^{a_I} \nu(B_1).$$

The lower estimate for  $\liminf$  is proved similarly.

Hence, taking  $c \rightarrow 1^+$ , we obtain

$$\lim_{T \rightarrow \infty} \frac{1}{T^{a_I}(\log T)^{b_I-1}} \int_{A_I^+} \chi_{B_T}(av_0)\xi_I(a)da = \nu(B_1).$$

Let  $s : KM_I/(M_I \cap H) \rightarrow KM_I$  be a measurable section of the factor map such that  $s(\Omega)$  is measurable and bounded (for example, one can define  $s$  using the identification (2.1)). Since

$$S_T(\Omega) = \Omega A_I^+ v_0,$$

we deduce from (4.2), using the dominated convergence theorem, that

$$(4.5) \quad \lim_{T \rightarrow \infty} \frac{\text{Vol}(S_T(\Omega))}{T^{a_I}(\log T)^{b_I-1}} = \int_{\Omega} \nu(s(\omega)^{-1}B_1)d\omega.$$

Note that there exists  $\delta > 0$  such that  $s(\omega)^{-1}B_1 \supset B_\delta$  for all  $\omega \in \Omega$ , and because  $\nu$  is homogeneous,  $\nu(B_\delta) > 0$ . This implies the last claim.  $\square$

**Remark 4.6.** When the representation  $\iota$  is irreducible, parameters  $a_I, b_I, c_I$  can be described explicitly in terms of the highest weight

$$\lambda_\iota = \sum_{\alpha \in \Delta_\sigma} n_\alpha \alpha$$

and the sum of positive roots (with multiplicities)

$$2\rho = \sum_{\alpha \in \Delta_\sigma} u_\alpha \alpha.$$

We apply [GOS, Theorem 6.1] to (4.4) decomposing  $\xi_I$  as a linear combination of functions  $\exp(\chi(a))$  where  $\chi$ 's are characters of  $A_I$ . Note that  $2\rho$  is the maximal character in this decomposition. We have

$$a_I = \max\left\{\frac{u_\alpha}{n_\alpha} : \alpha \in \Delta_\sigma - I\right\},$$

$$b_I = \#\left\{\alpha \in \Delta_\sigma - I : \frac{u_\alpha}{n_\alpha} = a_I\right\}.$$

Let

$$J_0 = \left\{\alpha \in \Delta_\sigma - I : \frac{u_\alpha}{n_\alpha} < a_I\right\} \quad \text{and} \quad I_0 = I \cup J_0.$$

The computation as in [GOS, Theorem 6.4] gives that for every  $f \in C_c(W)$ ,

$$(4.7) \quad \lim_{T \rightarrow \infty} \frac{1}{T^{a_I} (\log T)^{b_I - 1}} \int_{\Omega} \int_{A_I^+} f(\omega av_0/T) \xi_I(a) da d\omega = \int_{\Omega} \int_{A_I^+} f(\omega av_0^{I_0}) \tilde{\xi}_I(a) da d\omega$$

where  $v_0^{I_0}$  is the projection of  $v_0$  to the sum of the weight spaces with weights of the form  $\lambda_\iota - \sum_{\alpha \in I_0} m_\alpha \alpha$ ,  $m_\alpha \geq 0$ , and

$$\tilde{\xi}_I(a) = \left( \prod_{\alpha \in (\Sigma_\sigma^+ \cap \langle I_0 \rangle) - \langle I \rangle} \sin(\alpha(\log a))^{l_\alpha^+} \cos(\alpha(\log a))^{l_\alpha^-} \right) \cdot \exp \left( \sum_{\alpha \in \Delta_\sigma - I_0} u_\alpha \alpha(\log a) \right).$$

Since

$$\mathfrak{a}_{I_0} \cap \ker \rho = \mathfrak{a}_{I_0} \cap \ker \lambda_\iota,$$

it follows from [GOS, Theorem 5.1] that the orbit  $Gv_0^{I_0}$  supports a  $G$ -invariant measure  $\nu_{I_0}$ . Moreover, comparing the above formula with the formula in [GOS, Theorem 5.1], we conclude that in (4.7) we obtain the measure  $\nu_{I_0}$ . This measure induces the measure  $\mu_{I_0}$  supported on the Satake boundary:

$$\mu_{I_0} = \pi_*(\nu_{I_0}|_{B_1}),$$

where  $\pi$  denotes the radial projection. We have

$$c_I(\Omega) = \mu_{I_0}(\pi(\Omega v_0^{I_0})).$$

For  $c > 0$ ,  $I \subset \Delta_\sigma$ , and a bounded measurable  $\Omega \subset KM_I$ , we set

$$V_I^c = \{ \omega av_0 : \omega \in \Omega, a \in A_I^+ \text{ with } \alpha(\log a) \leq c \text{ for some } \alpha \in \Delta_\sigma - I \}.$$

**Lemma 4.8.** *For small  $c > 0$  and sufficiently large  $T > 0$ ,*

$$\text{Vol}(V_I^c \cap B_T) \ll c \cdot T^{a_I} (\log T)^{b_I - 1}.$$

*Proof.* For  $\alpha \in \Delta_\sigma$ , set

$$U_c(\alpha) = \{ a \in A_I^+ : \alpha(\log a) \leq c \}.$$

There exists  $\delta > 1$  such that  $\Omega^{-1} B_T \subset B_{\delta T}$  for all  $T > 0$ . By (4.2), this gives the estimate

$$(4.9) \quad \text{Vol}(V_I^c \cap B_T) \ll \sum_{\alpha \in \Delta_\sigma - I} \int_{a \in A_I^+ \cap U_c(\alpha) : \|av_0\| < \delta T} \xi_I(a) da.$$

Now we use the volume computation from [GOS] (see the proof of Theorem 6.4 in [GOS]) to show for every nonnegative  $f \in C_c(W)$ ,

$$\int_{A_I^+ \cap U_c(\alpha)} f(av_0/T) \xi_I(a) da \ll \left( \int_{A_I^+ \cap U_c(\alpha)} f(av) \tilde{\xi}_I(a) da \right) \cdot T^{a_I} (\log T)^{b_I - 1},$$

where  $v \in W$ ,  $\tilde{\xi}_I \in C(A^+)$ , and the map  $A_I^+ \rightarrow W : a \mapsto av$  is proper. This implies that for a compact  $L \subset A_I^+$ ,

$$\begin{aligned} \int_{A_I^+ \cap U_c(\alpha)} f(av_0/T) \xi_I(a) da &\ll \max(f) \cdot \text{Vol}(L \cap U_c(\alpha)) \cdot T^{a_I} (\log T)^{b_I-1} \\ &\ll_f c \cdot T^{a_I} (\log T)^{b_I-1}. \end{aligned}$$

Taking a function  $f$  satisfying  $\chi_{B_1} \leq f$ , we obtain

$$\begin{aligned} \int_{a \in A_I^+ \cap U_c(\alpha) : \|av_0\| < T} \xi_I(a) da &\ll \left( \int_{A_I^+ \cap U_c(\alpha)} f(av) \tilde{\xi}_I(a) da \right) \cdot T^{a_I} (\log T)^{b_I-1} \\ &\ll_f c \cdot T^{a_I} (\log T)^{b_I-1}. \end{aligned}$$

Therefore, by (4.9),

$$\text{Vol}(V_I^c \cap B_T) \ll c \cdot (\delta T)^{a_I} (\log(\delta T))^{b_I-1}.$$

This completes the proof.  $\square$

The following corollary of Theorem 1.4 will be used in the proof of Theorem 1.6:

**Corollary 4.10.** *Let  $\Delta_\sigma = I \sqcup J$  and  $B$  be a bounded subset of  $KM_I$ . Then given  $c > 0$ , there exist  $\ell > 1$  and  $\varepsilon_0 > 0$  such that for every  $(J, c)$ -regular  $g = bah \in BA_I H$  and  $0 < \varepsilon < \varepsilon_0$ ,*

$$\mathcal{O}_\varepsilon g \subset (K \cap \mathcal{O}_{\ell\varepsilon}) b (M_I \cap \mathcal{O}_{\ell\varepsilon}) (A_I \cap \mathcal{O}_{\ell\varepsilon}) a H.$$

*Proof.* Let  $b = km$  for  $k \in K$  and  $m \in M_I$ . Note that  $m \in KB \cap M_I$ , which is bounded.

The group  $M_I$  is invariant under  $\theta$  and  $\sigma$  and it has the Cartan decomposition

$$M_I = (M_I \cap K) A^I (M_I \cap H)$$

where  $A^I$  is complementary to  $A_I$ . In particular,  $m = k_0 a_0 h_0$ . By Theorem 1.4,

$$\mathcal{O}_\varepsilon g \subset (K \cap \mathcal{O}_{\ell\varepsilon}) k k_0 (M_I \cap \mathcal{O}_{\ell\varepsilon}) (A_I \cap \mathcal{O}_{\ell\varepsilon}) a_0 a H.$$

There exists  $\sigma > 1$  such that for every  $k \in K$  and small  $\varepsilon > 0$ ,  $k \mathcal{O}_\varepsilon k^{-1} \subset \mathcal{O}_{\sigma\varepsilon}$ . Hence,

$$\mathcal{O}_\varepsilon g \subset (K \cap \mathcal{O}_{\ell\varepsilon}) k (M_I \cap \mathcal{O}_{\sigma\ell\varepsilon}) k_0 a_0 h_0 (A_I \cap \mathcal{O}_{\ell\varepsilon}) a H.$$

There exists  $\eta > 1$  such that for every  $m \in KB$  and small  $\varepsilon > 0$ ,  $m^{-1} \mathcal{O}_\varepsilon m \subset \mathcal{O}_{\eta\varepsilon}$ . Hence,

$$\mathcal{O}_\varepsilon g \subset (K \cap \mathcal{O}_{\ell\varepsilon}) k m (M_I \cap \mathcal{O}_{\eta\sigma\ell\varepsilon}) (A_I \cap \mathcal{O}_{\ell\varepsilon}) a H$$

as required.  $\square$

*Proof of Theorem 1.6.* Let  $c, \varepsilon \in (0, 1)$ .

Let  $s : KM_I/(M_I \cap H) \rightarrow KM_I$  be a measurable section such that  $s(\Omega)$  is bounded and measurable. For neighborhoods  $U_1$  of  $e$  in  $K$  and  $U_2$  of  $e$  in  $M_I$ , we set

$$\begin{aligned}\Omega^+ &= U_1 s(\Omega) U_2 (M_I \cap H), \\ \Omega^- &= \bigcap_{u_1 \in U_1, u_2 \in U_2} u_1 s(\Omega) u_2 (M_I \cap H).\end{aligned}$$

One can check that as  $U_1$  and  $U_2$  shrink to  $\{e\}$ , we have

$$\Omega^+ \downarrow \bar{\Omega} \quad \text{and} \quad \Omega^- \uparrow \text{int}(\Omega).$$

Since  $\text{Vol}(\partial\Omega) = 0$ , we have  $\text{Vol}(\Omega^+ - \Omega^-) \rightarrow 0$ . Hence, it follows from (4.5) that we can choose  $U_1$  and  $U_2$  so that

$$(4.11) \quad c_I(\Omega^+) - c_I(\Omega^-) < \varepsilon.$$

Fix a set  $\tilde{\Omega} \supset \Omega$  such that  $\bar{\Omega} \subset \text{int}(\tilde{\Omega})$ , set

$$V_I = \Omega A_I^+ v_0 \quad \text{and} \quad \tilde{V}_I = \tilde{\Omega} A_I^+ v_0,$$

and define  $V_I^c$  and  $\tilde{V}_I^c$  as in Lemma 4.8. We can choose  $U_1$  and  $U_2$  so that  $\Omega^+ \subset \tilde{\Omega}$ .

We claim that there exists a neighborhood  $\mathcal{O}'$  of  $e$  in  $G$  such that

$$(4.12) \quad \mathcal{O}' \cdot S_T(\Omega) \subset S_{(1+\varepsilon)T}(\Omega^+) \cup (\tilde{V}_I^c \cap B_{(1+\varepsilon)T}).$$

By Corollary 4.10, there exists a neighborhood  $\mathcal{O}_1$  such that

$$\mathcal{O}_1^{-1} \cdot (V_I - \tilde{V}_I^c) \subset \tilde{V}_I - V_I^{c/2}.$$

This implies that

$$\mathcal{O}_1 \cdot V_I^{c/2} \subset \tilde{V}_I^c.$$

Also, by Corollary 4.10 and continuity of operator norm, there exists a neighborhood  $\mathcal{O}_2$  of  $e$  in  $G$  such that for every  $v = \omega a v_0 \in V_I - V_I^{c/2}$ ,

$$\mathcal{O}_2 v \subset (U_1 \omega U_2) A_I^+ v_0$$

and

$$\mathcal{O}_2 \cdot B_T \subset B_{(1+\varepsilon)T}.$$

Hence,

$$\mathcal{O}_2 \cdot (S_T(\Omega) - V_I^{c/2}) \subset S_{(1+\varepsilon)T}(\Omega^+).$$

Setting  $\mathcal{O}' = \mathcal{O}_1 \cap \mathcal{O}_2$ , we deduce the claim (4.12).

Similar argument shows there exists a neighborhood  $\mathcal{O}''$  of  $e$  in  $G$  such that

$$(4.13) \quad S_{(1-\varepsilon)T}(\Omega^-) \subset \left( \bigcap_{g \in \mathcal{O}''} g S_T(\Omega) \right) \cup \tilde{V}_I^c.$$

Combining (4.12) and (4.13), we deduce that for  $\mathcal{O} = \mathcal{O}' \cap \mathcal{O}''$ ,

$$(4.14) \quad \begin{aligned} \text{Vol}(\mathcal{O} \cdot \partial S_T(\Omega)) &\leq \text{Vol}(\mathcal{O} S_T(\Omega) - \cap_{g \in \mathcal{O}g} S_T(\Omega)) \\ &\leq \text{Vol}(S_{(1+\varepsilon)T}(\Omega^+)) - \text{Vol}(S_{(1-\varepsilon)T}(\Omega^-)) + \text{Vol}(\tilde{V}_I^c \cap B_{(1+\varepsilon)T}). \end{aligned}$$

By Lemma 4.8,

$$\limsup_{T \rightarrow \infty} \frac{\text{Vol}(\tilde{V}_I^c \cap B_{(1+\varepsilon)T})}{T^{a_I}(\log T)^{b_I-1}} \ll c.$$

By Proposition 4.1,

$$\begin{aligned} \lim_{T \rightarrow \infty} \frac{\text{Vol}(S_{(1+\varepsilon)T}(\Omega^+))}{T^{a_I}(\log T)^{b_I-1}} &= (1 + \varepsilon)^{a_I} c_I(\Omega^+), \\ \lim_{T \rightarrow \infty} \frac{\text{Vol}(S_{(1-\varepsilon)T}(\Omega^-))}{T^{a_I}(\log T)^{b_I-1}} &= (1 - \varepsilon)^{a_I} c_I(\Omega^-). \end{aligned}$$

Hence, it follows from (4.14) and (4.11) that

$$\begin{aligned} \limsup_{T \rightarrow \infty} \frac{\text{Vol}(\mathcal{O} \cdot \partial S_T(\Omega))}{T^{a_I}(\log T)^{b_I-1}} &\ll (1 + \varepsilon)^{a_I} c_I(\Omega^+) - (1 - \varepsilon)^{a_I} c_I(\Omega^-) + c \\ &\ll \varepsilon + c. \end{aligned}$$

Since  $\varepsilon$  and  $c$  can be taken arbitrary small, this proves that the family of sets  $S_T(\Omega)$  is well-rounded. Hence, it follows from [DRS, EM] that

$$\#(\Gamma v_0 \cap S_T(\Omega)) \sim_{T \rightarrow \infty} \text{Vol}(S_T(\Omega)).$$

This proves the theorem.  $\square$

*Proof of Theorem 1.2.* To deduce Theorem 1.2 from Theorem 1.6, we observe that

$$\mathcal{Q}_W \simeq \bigcup_{p+q=d} \text{SL}_d(\mathbb{R})/\text{SO}(p, q), \quad d = \dim W,$$

and  $\text{SL}_d(\mathbb{R})/\text{SO}(p, q)$  is an affine symmetric space. We set

$$\begin{aligned} G &= \text{SL}_d(\mathbb{R}), \\ K &= \text{SO}(d), \\ A &= \{\text{diag}(s_1, \dots, s_d) : s_i \in \mathbb{R}^+, s_1 \cdots s_d = 1\}, \\ H &= \text{SO}(p, q). \end{aligned}$$

Then we have the Cartan decomposition  $G = KAH$ . The set of simple roots is

$$\Delta_\sigma = \{\alpha_i(s) = s_i s_{i+1}^{-1} : i = 1, \dots, d-1\}.$$

Let  $\Delta_\sigma = I \sqcup J$  with  $J = \{\alpha_{i_1}, \dots, \alpha_{i_n}\}$ ,  $i_1 < \dots < i_n$ . Then

$$M_I \simeq \text{SL}_{i_1}(\mathbb{R}) \times \text{SL}_{i_2-i_1}(\mathbb{R}) \times \cdots \times \text{SL}_{d-i_n}(\mathbb{R}),$$

and  $A_I$  is the centralizer of  $M_I$  in  $A$ .

Since the set of integral quadratic forms in the question is a finite union of  $\mathrm{SL}_d(\mathbb{Z})$ -orbits, we conclude that the proof of the theorem reduces to the computation of the asymptotics of  $\#(\mathrm{SL}_d(\mathbb{Z})q_0 \cap S_T(\Omega\Omega'))$  where  $q_0 \in \mathcal{Q}_W(\mathbb{Z})$ . This shows that Theorem 1.2 is a partial case of Theorem 1.6.

It remains to compute the parameters  $a_I$  and  $b_I$ , which are determined by the volume asymptotics in Proposition 4.1. Note that the function  $\xi_I$ , defined in (4.3), decomposes as a linear combination of functions  $\exp(\chi(a))$  where  $\chi$ 's are characters of  $A_I$ . The maximal character in this decomposition is

$$\sum_{i \in J} u_i \alpha_i \quad \text{where } u_i = i(d-i) - (i-i_k)(i_{k+1}-i) \text{ for } i_k \leq i \leq i_{k+1}.$$

The highest weight of the representation of  $\mathrm{SL}_d(\mathbb{R})$  on the space of quadratic forms is

$$\sum_{i \in J} m_i \alpha_i \quad \text{where } m_i = 2(d-i)/d.$$

By [GOS, Section 6], we have

$$a_I = \max \left\{ \frac{u_i}{m_i} : i \in J \right\} = d(d-i_n)/2,$$

$$b_I = \# \left\{ i \in J : \frac{u_i}{m_i} = a_I \right\} = 1.$$

This proves the theorem. □

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