MANIN’S CONJECTURE ON RATIONAL POINTS OF BOUNDED HEIGHT AND ADELIC MIXING

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Dedicated to Prof. Gregory Margulis on the occasion of his sixtieth birthday

ABSTRACT. Let $K$ be a number field. We compute the asymptotics of the number of $K$-rational points of bounded height on a connected adjoint semisimple $K$-group $G$ for any given irreducible representation. This proves Manin’s conjecture for the wonderful compactification $X$ of $G$. We also determine the explicit asymptotic distribution of the rational points $G(K)$ on $X(\mathbb{A})$, which verifies the prediction made by Peyre. Our approach is based on the mixing property of $L^2(G(K)\backslash G(\mathbb{A}))$ which we prove with a rate of convergence.

Soit $K$ un corps de nombres. Nous déterminons le comportement asymptotique du nombre de $K$-points de hauteur bornée d’une représentation irréductible arbitraire d’un $K$-groupe $G$ semisimple, adjoint et connexe. Ceci résout la conjecture de Manin dans le cas de la compactification merveilleuse $X$ de $G$. Nous calculons également la distribution asymptotique explicite des points $G(K)$ sur $X(\mathbb{A})$, qui vérifie les prédictions de Peyre. Ce travail repose sur la propriété de mélange de $L^2(G(K)\backslash G(\mathbb{A}))$, qui est démontrée avec une estimée de vitesse.

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1. Introduction

Let $K$ be a number field and $R$ the set of all (inequivalent) normalized absolute values $x \mapsto |x|_v$ of $K$. We denote by $K_v$ the completion of $K$ with respect to $v$. The height $H(x)$ of a point $x = (x_0 : x_1 : \cdots : x_n) \in \mathbb{P}^n(K)$ is given by

$$H(x) := \prod_{v \in R} H_v(x)$$

where $H_v(x) = \max_i |x_i|_v$ for each $v \in R$. By the product formula, $H$ is a well defined function on $\mathbb{P}^n(K)$. For example, when $K = \mathbb{Q}$, we have

$$H(x) = \max_i |x_i|$$

where $(x_0, \cdots, x_n)$ is a primitive integral vector.

More generally one can consider a height function which differs from above by changing the local height $H_v$ by another norm on $K_{v}^{n+1}$ for finitely many places $v$ (see Definition 3.2).

It is easy to see that for any $T > 0$,

$$N(T) := \# \{ x \in \mathbb{P}^n(K) : H(x) < T \} < \infty.$$  

Schanuel [Sc] computed in 1964 that

$$N(T) \sim c \cdot T^{n+1} \quad \text{as } T \to \infty$$

for some explicit constant $c = c(H) > 0$.

A fundamental problem in modern algebro-arithmetic geometry is to describe the set of rational points of projective varieties in terms of their geometric invariants. One of the main conjectures in this area was made by Manin in [BM] in the late eighties. It describes the asymptotics of the numbers of rational points on projective varieties with ample anti-canonical classes (such varieties are called Fano varieties). Manin’s conjecture has been proved for flag varieties ([FMT], [Pe1]), toric varieties [BT1-2], horospherical varieties [ST], equivariant compactifications of unipotent groups (see [CT2], [ST1], [ST2]), etc. In this paper, we settle Manin’s conjecture for the wonderful compactification of a general connected semisimple adjoint group $G$ defined over a number field.

1.1. Asymptotics of rational points. Let $G$ be a connected adjoint semisimple group over $K$. Let $\iota : G \to \text{GL}_N$ be a faithful representation of $G$ defined over $K$. This defines a projective embedding over $K$:

$$\bar{\iota} : G \to \mathbb{P}(M_N) = \mathbb{P}^{N^2-1}$$

where $M_N$ denotes the space of matrices of order $N$. Fixing a height function $H = \prod_{v \in R} H_v$ on $\mathbb{P}^{N^2-1}(K)$, we obtain a height function $H_\iota$ on $G(K)$ via $\bar{\iota}$:

$$H_\iota(g) := H(\bar{\iota}(g)) = \prod_{v \in R} H_v(\iota(g)).$$  

\text{(1.1)}
Set
\[ N(H_\iota, T) := \# \{ g \in G(K) : H_\iota(g) < T \} \].

**Theorem 1.2.** Let \( G \) be a product of connected adjoint absolutely simple groups over \( K \) and \( \iota : G \to \text{GL}_N \) a faithful absolutely irreducible representation defined over \( K \). Then there exist \( a_\iota \in \mathbb{Q}^+, b_\iota \in \mathbb{N} \) and \( c = c(H_\iota) > 0 \) such that for some \( \delta > 0 \),
\[ N(H_\iota, T) = c \cdot T^{a_\iota} \cdot (\log T)^{b_\iota} \cdot (1 + O((\log T)^{-\delta})) \]
for all sufficiently large \( T \).

The proof of Theorem 1.2 is based on the uniform bounds on matrix coefficients. Recently, we developed a different approach based on Ratner’s theory of unipotent flows [GoO], which also applies to some homogeneous varieties of \( G \).

**Remark**

1. When \( G \) is absolutely simple or, more generally, when \( H_\iota \) is the product of height functions of the simple factors of \( G \), we can improve the rate of convergence in Theorem 1.2: for some \( \delta > 0 \),
\[ N(H_\iota, T) = c \cdot T^{a_\iota} \cdot P(\log T) \cdot (1 + O((\log T)^{-\delta})) \]
where \( P(x) \) is a monic polynomial of degree \( b_\iota - 1 \).

2. We note that for any connected semisimple adjoint algebraic group \( G \) defined over \( K \), there exists a finite field extension, say, \( F \), of \( K \) such that \( G \) is a direct product of connected adjoint absolutely simple groups defined over \( F \).

The constants \( a_\iota \) and \( b_\iota \) can be defined explicitly by combinatorial data coming from the root system of \( G \) and the highest weight of \( \iota \). Choose a set \( \Delta \) of simple roots in the root system \( \Phi(G, T) \) of \( G \) with respect to a maximal torus \( T \) defined over \( K \) containing a maximal \( K \)-split torus. Denote by \( 2\rho \) the sum of all positive roots in \( \Phi(G, T) \), and by \( \chi \) the highest weight of \( \iota \). Define \( u_\alpha, m_\alpha \in \mathbb{N}, \alpha \in \Delta \), by
\[ 2\rho = \sum_{\alpha \in \Delta} u_\alpha \alpha \quad \text{and} \quad \chi = \sum_{\alpha \in \Delta} m_\alpha \alpha. \]
The fact that \( m_\alpha \in \mathbb{N} \) follows since \( G \) is of adjoint type. Consider the twisted action of the Galois group \( \Gamma_K := \text{Gal}(\bar{K}/K) \) on \( \Delta \). For instance, if the \( K \)-form of \( G \) is inner, this action is just trivial. Then
\[ a_\iota = \max_{\alpha \in \Delta} \frac{u_\alpha + 1}{m_\alpha} \quad \text{and} \quad b_\iota = \# \{ \Gamma_K . \alpha : \frac{u_\alpha + 1}{m_\alpha} = a_\iota \}. \]

Note that the exponent \( a_\iota \) is independent of the field \( K \), and \( b_\iota \) depends only on the quasi-split \( K \)-form of \( G \). Therefore, by passing to a finite field extension containing the splitting field of \( G \), \( b_\iota \) also becomes independent of \( K \).
1.2. Distribution of rational points. Refining Manin’s conjecture mentioned above, Peyre made a conjecture on the asymptotic distribution of rational points of Fano varieties [Pe1]. We verify Peyre’s conjecture for “saturated” projective embeddings of semisimple adjoint groups. We call a representation $\iota : G \to \text{GL}_N$ saturated if the set

$$(1.4) \quad \{ \alpha \in \Delta : \frac{u_\alpha + 1}{m_\alpha} = a_\iota \}$$

is not contained in the root system of a proper normal subgroup of $G$. In particular, if $G$ is absolutely simple, any representation of $G$ is saturated.

For a finite subset $S$ of $R$, we define $G_S := \prod_{v \in S} G(K_v)$ and $H_{\iota,S} := \prod_{v \in S} H_v \circ \iota$.

Let $X_{\iota,S} \subset \prod_{v \in S} \mathbb{P}^{N^2-1}(K_v)$ be the closure of the image of $G_S$ under the diagonal embedding $(g_v) \mapsto (\iota(g_v))$. We identify $G_S$ with its image. It follows from the weak approximation property that $G(K)$ is dense in $X_{\iota,S}$. We compute the asymptotic distribution of $G(K)$ ordered by the height $H_\iota$.

We define a probability measure $\tilde{\mu}_{\iota,S}$ on $G_S$ by

$$\tilde{\mu}_{\iota,S} := \frac{H_{\iota,S}(g)^{-\alpha} \, dg}{\int_{G_S} H_{\iota,S}(g)^{-\alpha} \, dg},$$

where the integral is with respect to a Haar measure on $G_S$. This measure is well defined by Lemma 3.3. Let $G'_S$ denote the derived subgroup of $G_S$. Recall that $[G_S : G'_S] < \infty$.

**Theorem 1.5.** Let $G$ be a product of connected adjoint absolutely simple groups over $K$, $\iota : G \to \text{GL}_N$ a faithful absolutely irreducible saturated representation defined over $K$, and $S$ a finite subset of $R$. Then for any $f \in C(X_{\iota,S})$,

$$\lim_{T \to \infty} \frac{1}{N(H_\iota,T)} \sum_{g \in G(K) : H_\iota(g) < T} f(g) = \int_{G_S} f \, d\mu_{\iota,S}$$

where the probability measure $\mu_{\iota,S}$ is given by

$$(1.6) \quad \mu_{\iota,S} = \sum_{u \in G_S/G'_S} c_u \cdot \tilde{\mu}_{\iota,S}|_{uG'_S}$$

for some explicit positive weights $c_u$ (see (4.1)).

**Remark**

(1) If $\iota$ is not saturated, Theorem 1.2 implies that a positive proportion of $G(K)$ lies on a proper subgroup of $G$. Hence one cannot expect Theorem 1.7 to hold for non-saturated cases.
(2) The limiting distribution $\mu_{i,S}$ is not $G_S$-invariant, unless $G_S$ is compact and the height $H_{i,S}$ is $G_S$-invariant.

(3) We give examples with $\mu_{i,S} \neq \tilde{\mu}_{i,S}$ in Section 6.

(4) The space $X_{i,S}$ is a compactification of $G_S$ which is an analog of the Satake compactification defined for real groups (see, for example, [BJ]). Theorem 1.5 implies that the rational points $G(K)$ do not escape to the boundary $X_{i,S} - G_S$. It is interesting to compare this result with the distribution of the integral points $G(Z)$ of bounded height in the Satake compactification of $G(R)$ where the limiting distribution is supported on the boundary (see [Mau] or [GOS] for more details).

One can check (see Lemma 4.2) that for finite $S \subset T \subset R$, the image of the measure $\mu_{i,T}$ under the projection map $G_T \rightarrow G_S$ is equal to the measure $\mu_{i,S}$. This implies that the family of measures $\{\mu_{i,S}\}$ defines a probability measure $\mu_i$ on the space $X_i := \prod_{v \in R} X_{i,v}$. We have a global version of Theorem 1.5:

**Theorem 1.7.** Let $G$ and $i$ be as in Theorem 1.5. Then for any $f \in C(X_i)$,

$$\lim_{T \rightarrow \infty} \frac{1}{N(H_{i,T})} \sum_{g \in G(K): H_i(g) < T} f(g) = \int_{X_i} f \, d\mu_i.$$  

1.3. **Manin’s and Peyre’s conjectures.** We also state versions of Theorems 1.2 and 1.7 in terms of arithmetic geometry. Let $X$ be a smooth projective variety defined over $K$. For every line bundle on $X$ over $K$, there exists an associated height function on $X(K)$ via Weil’s height machine ([Si, Theorem B. 3.2]). For example, if $L$ is a very ample line bundle of $X$ with a $K$-embedding $\psi_L : X \rightarrow \mathbb{P}^N$, then a height function $H_L$ on $X(K)$ is defined as

$$H_L := H \circ \psi_L$$

where $H$ is a height function on $\mathbb{P}^N(K)$ defined as before. For any ample line bundle $L$ of $X$, $mL$ is very ample for some $m \in \mathbb{N}$. Then a height function $H_L$ is of the form $H_L^{1/m}$ where $H_{L'}$ is a height function associated to $L' := mL$.

We call a pair $\mathcal{L} = (L, H_L)$ a metrized line bundle. Due to the freedom of choosing a height function $H$ on $\mathbb{P}^N(K)$, $H_L$ is not uniquely determined and this is why we use the subscript $L$ rather than $L$.

Let $X$ denote the projective $K$-variety, which is the wonderful compactification of $G$ constructed by De Concini and Procesi [DP] and by De Concini and Springer [DS]. For instance, $X$ can be taken to be the Zariski closure of the image of $G$ in $\mathbb{P}(M_N)$ under an irreducible faithful representation $G \rightarrow \text{GL}_N$ whose highest weight is regular. A dominant weight $\chi$ is called regular if $\chi = \sum_{\alpha \in \Delta} m_\alpha \omega_\alpha$ with all $m_\alpha > 0$ where $\{\omega_\alpha : \alpha \in \Delta\}$ is the set of fundamental weights.

The Picard group $\text{Pic}(X)_K$ is isomorphic to the weight lattice of $G$. Under this isomorphism, the simple roots $\alpha$ correspond to the boundary divisors $D_\alpha$ such that $X - G = \cup_\alpha D_\alpha$, and the Galois action on $\text{Pic}(X)_K$ corresponds to the twisted Galois action on the weight lattice. Hence, the Picard group $\text{Pic}(X)$ is freely generated by
the line bundles corresponding to the orbits of the fundamental weights under the twisted Galois action. The closed cone \( \Lambda_{\text{eff}}(X) \) of the effective line bundles is the positive cone spanned by \( D_{\Gamma_K, \alpha}, \alpha \in \Delta \), i.e.,

\[
\Lambda_{\text{eff}}(X) = \bigoplus \mathbb{R}_{\geq 0} [D_{\Gamma_K, \alpha}]
\]

where the sum is taken over the \( \Gamma_K \)-orbits \( \Gamma_K, \alpha \) in the set \( \{ \alpha \in \Delta \} \) of simple roots and \( D_{\Gamma_K, \alpha} := \sum_{\beta \in \Gamma_K, \alpha} D_{\beta} \), and the anticanonical class \( [-K_X] \) corresponds to \( 2\rho + \sum_{\alpha \in \Delta} \alpha \). Moreover any ample line bundle class \( [L] \) of \( X \) over \( K \) corresponds to a regular dominant weight in such a way that if \( [L'] := m[L] \) corresponds to \( \chi \in X^*(T) \) for \( m \in \mathbb{N} \), the restriction of \( H_{L'} \) to \( G(K) \) coincides with a height function \( H_\chi \) with respect to the irreducible representation \( \iota \) defined over \( K \) with the highest weight \( \chi \) (cf. [STT2, Proposition 6.3]). We call an ample line bundle \( L \) saturated if the representation defined by the corresponding dominant weight is saturated. We refer to [BK, Ch 6] for a more detailed account on the wonderful compactification.

The notion of a saturated line bundle is related to the notion of a strongly saturated metrized line bundle \( \mathcal{L} \) which was introduced by Batyrev and Tschinkel in [BT3] in order to state a refined version of Manin’s conjecture. A metrized line bundle \( \mathcal{L} \) on \( X \) is called strongly saturated if for any Zariski open dense subset \( U \) of \( X \), one has

\[
\lim_{T \to \infty} \frac{\# \{ x \in U(K) : H_\mathcal{L}(x) < T \}}{\# \{ x \in X(K) : H_\mathcal{L}(x) < T \}} = 1.
\]

If \( \mathcal{L} \) is strongly saturated, then \( \mathcal{L} \) is saturated. It is also clear from the definition (1.4) that the anticanonical line bundle \( -K_X \) is always saturated.

Consider the compact space \( X(\mathbb{A}) = \prod_{v \in R} X(K_v) \). Peyre [Pe1] defined the Tama-gawa measure \( \tau_{-K_X} \) on \( X(\mathbb{A}) \) associated with the anti-canonical metrized line bundle \( -K_X = (-K_X, H_{-K_X}) \):

\[
\tau_{-K_X} := d_K^{-\dim(X)/2} \cdot \lim_{s \to 1^+} \left( s - 1 \right)^{\text{rank}(\text{Pic}(X))} \left( \prod_{v \in R - S} L_v(s, \text{Pic}(X)) \right) \cdot \prod_{v \in R} \lambda_v^{1} \cdot H_{-K_X, v}(g_v)^{-1} \, dg_v
\]

where \( S \subset R \) is a finite subset of places with bad reduction, \( \lambda_v = L_v(1, \text{Pic}(X)) \) for all \( v \in R - S \) and 1 otherwise, \( d_K \) is the discriminant of \( K \) and \( dg_v \) is a Haar measure on \( G(K_v) \). To define a measure \( \tau_\mathcal{L} \) for any ample metrized line bundle \( \mathcal{L} = (L, H_\mathcal{L}) \), we set

\[
a_L := \inf \{ a \in \mathbb{Q}^+ : a[L] + [K_X] \in \Lambda_{\text{eff}}(X) \} \quad \text{— the Nevanlinna invariant of } L,
\]

\[
b_L := \text{the codimension of the face of } \Lambda_{\text{eff}}(X) \text{ containing } a_L[L] + [K_X] \text{ in its interior}.
\]

The measure \( \tau_\mathcal{L} \) on \( G(\mathbb{A}) \subset X(\mathbb{A}) \) is defined by

\[
\tau_\mathcal{L} := d_K^{-\dim(X)/2} \cdot \lim_{s \to a_L^+} (s - a_L)^{b_L} \sum_{\chi} H_\mathcal{L}(g)^{-s} \chi(g) \, dg,
\]

where \( dg \) is a Haar measure on \( G(\mathbb{A}) \), and the sum is taken over characters from \( L^2(G(K) \setminus G(\mathbb{A})) \) such that \( \int_{G(\mathbb{A})} H_\mathcal{L}(g)^{-s} \chi(g) \, dg \neq 0 \). Since \( H_\mathcal{L} \) is invariant under
a compact open subgroup of finite adeles in $G(\mathbb{A})$, the sum in (1.8) contains only finitely many terms. The limit in (1.8) exists by Theorem 3.4. Note that $a_{-K} = 1$, $b_{-K} = \text{rank}(\text{Pic}(X))$, and

$$
\lim_{s \to 1^+} (s - 1)^{\text{rank}(\text{Pic}(X))} \int_{G(\mathbb{A})} H_{-K}(g)^{-s} \chi(g) dg = 0
$$

for all $\chi \neq 1$ (see Section 6). Hence, for $\mathcal{L} = -K$, (1.8) gives Peyre’s measure $\tau_{-K}$. An analog of Peyre’s measure for general line bundles was also introduced in [BT3], but the measure $\tau_{\mathcal{L}}$ seems to be different, in general, from the measure defined in [BT3].

The following theorem settles Manin’s conjecture (and its refinements due to Peyre) for the wonderful compactifications $X$ of semisimple adjoint groups. For a metrized ample line bundle $\mathcal{L} = (L, H_{\mathcal{L}})$ on $X$ and a subset $U$ of $X$, set

$$
N_U(\mathcal{L}, T) := \# \{ g \in U(K) : H_{\mathcal{L}}(g) < T \}.
$$

**Theorem 1.9.** Let $X$ be the wonderful compactification of a group $G$ which is a product of connected adjoint absolutely simple groups defined over $K$, and $\mathcal{L} = (L, H_{\mathcal{L}})$ a metrized ample line bundle on $X$. Then for some $\delta > 0$,

$$
N_G(\mathcal{L}, T) = c_{\mathcal{L}} \cdot T^{a_{\mathcal{L}}(\log T)^{bl-1}(1 + O((\log T)^{-\delta}))}
$$

for all sufficiently large $T$, where $c_{\mathcal{L}} > 0$ and if $L$ is saturated, $c_{\mathcal{L}} = c_{\mathcal{L}} \cdot \tau_{\mathcal{L}}(G(\mathbb{A})) > 0$. In particular, for a metrization $-K_X = (-K_X, H_{-K})$ of the anti-canonical line bundle, we have

$$
N_G(-K_X, T) = c_{-K_X} \cdot \tau_{-K_X}(G(\mathbb{A})) \cdot T^{(\log T)^{\text{rank}(\text{Pic}(X))}-1(1 + O((\log T)^{-\delta}))}.
$$

For a non-saturated line bundle $L$, the equality $c_{\mathcal{L}} = c_{\mathcal{L}} \cdot \tau_{\mathcal{L}}(G(\mathbb{A}))$ fails in general, but the variety $X$ has an asymptotic arithmetic $\mathcal{L}$-fibration in the sense of [BT3]. More precisely, there exist a connected semisimple $K$-group $H$ and a surjective $K$-homomorphism $\pi : G \to H$ such that for each $x \in H(K)$,

$$
N_{\pi^{-1}(x)}(\mathcal{L}, T) \sim c_{L} \cdot \tau_{\mathcal{L}}(\pi^{-1}(x)(\mathbb{A})) \cdot T^{a_{\mathcal{L}}(\log T)^{bl-1}}; \quad \text{and} \quad N_G(\mathcal{L}, T) \sim c_{L} \cdot \sum_{x \in H(K)} \tau_{\mathcal{L}}(\pi^{-1}(x)(\mathbb{A})) \cdot T^{a_{\mathcal{L}}(\log T)^{bl-1}} \quad \text{as } T \to \infty
$$

with $\sum_{x \in H(K)} \tau_{\mathcal{L}}(\pi^{-1}(x)(\mathbb{A})) < \infty$.

Theorem 1.9 is independently proved by Shalika, Takloo-Bighash and Tschinkel in their recent preprint [STT2] (see page 12 for comparison of our methods).

We also state the analogue of Theorem 1.7 in this setup.

**Theorem 1.10.** With the same notation as Theorem 1.9, suppose also that $L$ is saturated. Then for any $f \in C(X(\mathbb{A}))$,

$$
\lim_{T \to \infty} \frac{1}{N_G(\mathcal{L}, T)} \sum_{g \in G(\mathbb{A}) : H_{\mathcal{L}}(g) < T} f(g) = \frac{1}{\tau_{\mathcal{L}}(X(\mathbb{A}))} \int_{X(\mathbb{A})} f \, d\tau_{\mathcal{L}}.
$$
1.4. Counting and volume heuristic. To explain our strategy in counting $K$-rational points of $G$, we first recall the analogous results in counting lattice points in a simple real Lie group. Let $G \subset \text{GL}_N$ be a connected non-compact simple real Lie group and $\Gamma$ be a lattice in $G$, i.e., a discrete subgroup of finite co-volume. Fixing a norm $\| \cdot \|$ on $M_N(\mathbb{R})$, set $B_T := \{ g \in G : \| g \| \leq T \}$. By Duke-Rudnick-Sarnak [DRS] and Eskin-McMullen [EM], it is well known that

\begin{equation}
\# \Gamma \cap B_T \sim \int_{B_T} \, dg \quad \text{as } T \to \infty,
\end{equation}

where $dg$ is the Haar measure on $G$ such that $\int_{\Gamma \cap G} \, dg = 1$.

Coming back to the question of counting rational points $G(K)$, we note that $G(K)$ can indeed be embedded as a lattice in the adele group $G(\mathbb{A})$ under the diagonal map. Recall that the adele group $G(\mathbb{A})$ is the restricted topological product of $G(K_v)$'s with respect to a family of compact open subgroups, say $U_v$, of $G(K_v)$, $v \in R_f$, where $R_f$ is the set of all non-archimedean absolute values on $K$ (cf. [PR], [We]). Choosing a smooth model of $G$ over $\mathcal{O}[k^{-1}]$ for the ring $\mathcal{O}$ of integers of $K$ and for some $k \in \mathbb{Z}$, we have $U_v = G(\mathcal{O}_v)$ for almost all $v \in R_f$ where $\mathcal{O}_v$ is the valuation ring of $K_v$.

Moreover the height function $H_v = \prod_{v \in R} H_v \circ \iota$ on $G(K)$ defined in (1.1) extends to $G(\mathbb{A})$ by

\begin{equation}
H_v(g) := \prod_{v \in R} H_v(\iota(g_v)) \quad \text{for any} \ g = (g_v)_v \in G(\mathbb{A}).
\end{equation}

Since $U_v = G(\mathcal{O}_v)$ for almost all $v \in R_f$, $H_v(\iota(g_v)) = 1$ for almost all $v$, and hence $H_v$ is well defined.

Setting

$B_T := \{ g \in G(\mathbb{A}) : H_v(g) < T \}$,

note that $B_T$ is a relatively compact subset of $G(\mathbb{A})$ (Lemma 3.7) and that

$\# G(K) \cap B_T = \# G(\mathbb{A}) \cap B_T$.

In view of (1.11), one naturally asks whether the following holds:

\begin{equation}
\# G(K) \cap B_T \sim \tau(B_T) \quad \text{as } T \to \infty,
\end{equation}

where $\tau$ is the Haar measure on $G(\mathbb{A})$ such that $\tau(G(K) \setminus G(\mathbb{A})) = 1$.

It turns out that the group $G(\mathbb{A})$ is too big for (1.13) to hold in general, due to the presence of non-trivial characters in $L^2(G(K) \setminus G(\mathbb{A}))$. To define a right group for (1.13), set

$W_f := \{ w \in G(\mathbb{A}_f) : H_i(wg) = H_i(gw) = H_i(g) \quad \text{for all} \ g \in G(\mathbb{A}_f) \}$,

where $G(\mathbb{A}_f)$ is the subgroup of finite adeles. It easily follows from the definition of $H_i$ that $W_f$ is a compact open subgroup of $G(\mathbb{A}_f)$. Denoting by $\Lambda^{W_f}$ the set of all $W_f$-invariant automorphic characters appearing in $L^2(G(K) \setminus G(\mathbb{A}))$, set

\begin{equation}
G_{H_i} := \ker(\Lambda^{W_f}) = \cap \{ \ker \chi \subset G(\mathbb{A}) : \chi \in \Lambda^{W_f} \}.
\end{equation}
The subgroup $G_{H_i}$ is a finite index normal subgroup of $G(\mathbb{A})$ which clearly contains $G(K)$ (see Lemma 3.1).

**Theorem 1.15.** Let $G$ be a product of connected adjoint absolutely simple group defined over $K$ and $\iota : G \to GL_N$ be a faithful absolutely irreducible saturated representation defined over $K$. Then
\[
\# G(K) \cap B_T \sim \tau(B_T \cap G_{H_i}) \quad \text{as} \quad T \to \infty,
\]
where $\tau$ is the Haar measure on $G_{H_i}$ normalized so that $\tau(G(K) \backslash G_{H_i}) = 1$.

We remark that one cannot in general replace $G_{H_i}$ by $G(\mathbb{A})$ (see Section 6).

As in the proof of Eskin-McMullen of (1.11), our key ingredient in proving Theorem 1.15 is the mixing theorem on $L^2(G(K) \backslash G_{H_i})$.

1.5. **Adelic mixing.** Let $L^2_0(G(K) \backslash G(\mathbb{A}))$ denote the orthogonal complement to the direct sum of all one-dimensional representations occurring in $L^2(G(K) \backslash G(\mathbb{A}))$. In the case when $G$ is simply connected, $L^2_0(G(K) \backslash G(\mathbb{A}))$ coincides with the orthogonal complement $L^2_0(G(K) \backslash G(\mathbb{A}))$ to the constant functions. The terminology that a sequence $g_i$ tends to infinity as $i \to \infty$ in $G(\mathbb{A})$ means that for any compact subset $\Omega$ in $G(\mathbb{A})$, $g_i \notin \Omega$ for all sufficiently large $i$.

**Theorem 1.16 (Adelic Mixing).** Let $G$ be a connected absolutely almost simple $K$-group. Then for any $f, h \in L^2_0(G(K) \backslash G(\mathbb{A}))$,
\[
\langle f, g.h \rangle \to 0
\]
as $g \in G(\mathbb{A})$ tends to infinity.

In particular, if $f$ and $h$ are $W_f$-invariant functions of $L^2(G(K) \backslash G_{H_i})$, then as $g \to \infty$,
\[
\langle f, g.h \rangle \to \int f \, d\tau \cdot \int h \, d\tau
\]
where $\tau$ is the normalized Haar measure on $G(K) \backslash G_{H_i}$.

In fact we prove the above theorem 1.16 in a much stronger form by giving a rate of convergence (see Theorem 2.8 and 2.17). In particular we obtain the following result which is of independent interest. Set $G_{\infty} := \prod_{v \in R_\infty} G(K_v)$ where $R_\infty$ is the subset of $R$ of all archimedean valuations.

**Theorem 1.17 (Automorphic bound for $G$).** Let $G$ be connected absolutely almost simple $K$-group. Let $U_{\infty}$ be a maximal compact subgroup of $G_{\infty}$ and $W_f$ be a compact open subgroup of $G(\mathbb{A})$.

Then for any $U_{\infty}$-finite and $W_f$-invariant functions $f, h \in L^2_0(G(K) \backslash G(\mathbb{A}))$,
\[
|\langle f, g.h \rangle| \leq c_{W_f} \cdot \dim(U_{\infty}f) \cdot \dim(U_{\infty}h)^{r_0} \cdot \tilde{\xi}_G(g) \cdot \|f\|_2 \cdot \|h\|_2 \quad \text{for all} \quad g \in G(\mathbb{A}),
\]
where $c_{W_f} > 0$ and $r_0 = r_0(G_{\infty}) > 0$. Moreover if $G_{\infty}$ has no normal subgroup locally isomorphic to $Sp_{2n}(\mathbb{C})$, then $r_0 = 1$. 

Here, $\tilde{\xi}_G : G(\mathbb{A}) \to (0, 1]$ is an explicitly constructed proper function which is $L^p$-integrable for some $p = p(G) < \infty$. (see Def. 2.16).

The above bounds on matrix coefficients can be extended to smooth functions with compact support (see Theorem 2.22).

For each $v \in \mathbb{R}$, denote $G^\text{Aut}_v \subset G_v$ the automorphic dual of $G(K_v)$, i.e., the subset of unitary dual of $G(K)$ consisting of representations which are weakly contained in the representations appearing as $G(K_v)$ components of $L^2(G(K) \backslash G(\mathbb{A}))^{O_f}$ for some compact open subgroup $O_f$ of $G(\mathbb{A}_f)$.

Recall that for $G$ simply connected, it is shown by Clozel [Cl1] that the trivial representation of $G(K_v)$ is isolated in $G^\text{Aut}_v$ for each $v \in \mathbb{R}$, that is, $G$ has property $(\tau)$ (cf. [Lu]). The following corollary presents a uniform version of property $(\tau)$ of $G$ over all $v \in \mathbb{R}$.

**Corollary 1.18.** Let $G$ be a connected simply connected absolutely almost simple $K$-group. Let $\pi$ denote the quasi-regular representation of $G(\mathbb{A})$ on $L^2_0(G(K) \backslash G(\mathbb{A}))$. Let $W$ be a maximal compact subgroup of $G(\mathbb{A})$. Then there exist an explicit $p = p(G) < \infty$ such that any $W$-finite matrix coefficient of $\pi$ is $L^p(G(\mathbb{A}))$-integrable. In particular, $\pi$ is strongly $L^p$.

This corollary implies that for some $m = m(G) < \infty$, $\pi^\otimes m(G) \subset \infty \cdot L^2(G(\mathbb{A}))$ (see [HT]), and that for any non-amenable closed subgroup $H \subset G(\mathbb{A})$, the restriction of $\pi$ to $H$ does not have an almost invariant vector.

The proof of Theorem 1.17 goes roughly as follows: if $\tilde{\xi}_v$ is a uniform bound for the matrix coefficients of infinite dimensional representations in $G^\text{Aut}_v$, $\tilde{\xi}_G$ is defined to be the product $\prod_{v \in \mathbb{R}} \tilde{\xi}_v$. This can be made precise using the language of direct integral of a representation (cf. proof of Theorem 2.8). For those $v \in \mathbb{R}$ such that the $K_v$-rank of $G$ is at least 2, the uniform bounds, say $\xi_v$, of matrix coefficients of all infinite dimensional unitary representations of $G(K_v)$ were obtained in [Oh1]. For these cases, one can simply take $\tilde{\xi}_v = \xi_v$. In particular, if $K$-rank of $G$ is at least 2, we have $\tilde{\xi}_G = \prod_{v \in \mathbb{R}} \xi_v$ and $\tilde{\xi}_G$ works as a uniform bound for all unitary representation of $G(\mathbb{A})$ without $G(K_v)^+$-invariant vectors for each $v \in \mathbb{R}$ (see Theorem 2.8 for a precise statement). Moreover $\tilde{\xi}_G$ is fairly sharp in these cases. For instance, one can show that $\tilde{\xi}_G$ is optimal for $G = \text{SL}_n$ ($n \geq 3$), or $\text{Sp}_{2n}$ ($n \geq 2$) by [5.4, COU].

When there is $v \in \mathbb{R}$ with $K_v$-rank of $G$ one, finding an automorphic bound $\tilde{\xi}_v$ is essentially carried out by Clozel [Cl1]. In particular, several deep theorems in automorphic theory were used such as the Gelbart-Jacquet bound [GJ] toward Ramanujan conjecture, the results of Burger-Sarnak [BS] and Clozel-Ullmo [CU] on lifting automorphic bounds, the base changes by Rogawski [Ro] and Clozel [Cl2], and Jacquet-Langlands correspondence [JL].
1.6. Equidistribution of Hecke points. Let $K = \mathbb{Q}$. Let $S$ be a finite set of primes including the archimedean prime $\infty$. If $\Gamma$ is an $S$-arithmetic subgroup of $G_S = \prod_{p \in S} G(\mathbb{Q}_p)$ (here $\mathbb{Q}_\infty = \mathbb{R}$) and $a \in G(\mathbb{Q})$, then the Hecke operator $T_a$ on $L^2(\Gamma \backslash G_S)$ is defined by

$$T_a(f)(g) = \frac{1}{\deg(a)} \sum_{x \in \Gamma \backslash \Gamma a \Gamma} f(xg)$$

where $\deg(a) = \#\Gamma \backslash \Gamma a \Gamma$. Theorem 1.17 extends the main result in [COU] where some cases of $\mathbb{Q}$-anisotropic groups were excluded (see [EO]). In fact, the following corollary immediately follows from Theorem 1.17 and Proposition 2.6 in [COU]:

**Corollary 1.19.** Let $G$ be a connected simply connected absolutely almost simple $\mathbb{Q}$-group and $S$ a finite set of primes including $\infty$. Suppose that $G_S$ is non-compact. Let $\Gamma \subset G(\mathbb{Q})$ be an $S$-congruence subgroup of $G_S$. There exists a constant $c = c(\Gamma) > 0$ such that

$$\|T_a\| \leq c \cdot \xi_G(a) \quad \text{for any } a \in G(\mathbb{Q}).$$

This corollary in particular implies that for any sequence $a_i \in G(\mathbb{Q})$ with $\deg(a_i) \to \infty$, and for any $f \in C_c(G_S)$,

$$\lim_{i \to \infty} \frac{1}{\deg(a_i)} \sum_{x \in \Gamma a_i \Gamma} f(x) = \int_{G_S} f(g) \, d\tau_S$$

where $\tau_S$ is the normalized Haar measure on $G_S$ so that $\tau_S(\Gamma \backslash G_S) = 1$. It is interesting to note that unlike the rational points $G(\mathbb{Q})$ of bounded height (Theorem 1.5), the Hecke points are equidistributed in $G_S$ with respect to the invariant measure.

1.7. Structure of the proof and organization of the paper. Our proof of the asymptotics of the number of rational points can be divided into three steps:

- **Asymptotics of $\text{vol}(G_H \cap B_T)$ as $T \to \infty$** (recall that $G_H$ is defined in (1.14)). This reduces to the computation of certain $p$-adic integrals, which is usually done using the method of [D]. See, for example, Section 3 in [BT3] that discusses Denef’s formula [D] in the context of Manin’s conjecture. For the wonderful compactification, such computation is done in [STT2].

- **Decay of matrix coefficients in $L^2_{c0}(G(\mathbb{Q})) \backslash G(\mathbb{A})$**. This is deduced from the bound [Oh1, Oh2] for higher ranks groups and the automorphic bounds [BS, CU, Cl1, GJ, JL, Ro].

- **Using decay, deduce that $\#(G(\mathbb{Q}) \cap B_T) \sim \text{vol}(G_H \cap B_T)$ as $T \to \infty$**. A connection between decay of matrix coefficients and counting problems was first observed by Margulis [Mar] in his thesis in 1970 and exploited further by Eskin and McMullen [EM]. Here we apply this idea to the counting of rational points.
Using this strategy, we also compute the asymptotic distribution of rational points. In comparison, the method of Shalika, Takloo-Bighash, Tschinkel [STT2] is based on the study of analytic properties of the zeta function $\sum_{\gamma \in G(K) \backslash G(\mathbb{A})} H_{s}(\gamma)^{-s}$. While the decay of matrix coefficients is crucial in the proof in [STT2] as well, their approach requires more precise information about the spectral decomposition of $L^2(G(K) \backslash G(\mathbb{A}))$ which involves substantial technical difficulties, such as, dealing with the continuous spectrum. The asymptotic distribution of rational points is not discussed in [STT2].

The method outlined above applies to a wide variety of counting questions. In particular, to extend Theorem 1.9 to arbitrary bi-equivariant compactifications of $G$, one only needs to verify that the height functions satisfy the assumptions stated in Proposition 4.23.

The paper is organized as follows. We discuss the mixing property of unitary representations of $G(\mathbb{A})$ and prove Theorems 1.16 and 1.17 in section 2. In section 3, we obtain volume estimates. In section 4, we prove Theorems 1.2, 1.5, 1.7, 1.9, 1.10 and 1.15 for the saturated cases. The rate of convergence is obtained in section 5, where we also prove Theorems 1.2 and 1.9 for the unsaturated cases. Section 6 contains examples.

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2. Adelic Mixing

Let $K$ be a number field. Let $G$ be a connected absolutely almost simple group defined over $K$. We keep the same notation $R, R_f, K_v$ as in the introduction. Let $\mathcal{O}$ denote the ring of integers of $K$ and $\mathcal{O}_v$ the valuation ring of $K_v$. Set $R_{\infty} = R - R_f$. For $v \in R_f$, let $q_v$ denote the order of the residue field of $\mathcal{O}_v$. Denote by $\mathbb{A}$ the adele ring over $K$ and by $G(\mathbb{A})$ the adele group associated to $G$.

Denote by $G(\mathbb{A}_f)$ (resp. $G_{\infty}$) the subgroup of finite (resp. infinite) adeles, i.e., $((g_v)_v) \in G(\mathbb{A})$ with $g_v = e$ for all $v \in R_{\infty}$ (resp. for all $v \in R_f$). Then

$$G(\mathbb{A}) = G_{\infty} \times G(\mathbb{A}_f).$$

2.1. Definition and properties of $\xi_G$. We fix a smooth model of $G$ over $\mathcal{O}[k^{-1}]$ for some $k \in \mathbb{Z}$. There exists a finite subset $S_0 \subset R_f$ such that for any $v \in R_f - S_0$, $G$ is unramified over $K_v$ and $G(\mathcal{O}_v)$ is a hyperspecial compact subgroup (cf. [Ti2]). We set $U_v = G(\mathcal{O}_v)$ for each $v \in R_f - S_0$. Then for each $v \in R_f - S_0$, there exists the group $A_v$ of $K_v$-rational points of a maximal $K_v$-split torus of $G$ so that the following Cartan decomposition holds:

$$(2.1) \quad G(K_v) = U_v A_v^+ U_v$$

where $A_v^+$ is a closed positive Weyl chamber of $A_v$. 

For \( v \in S_0 \cup R_\infty \), there exists a good maximal compact subgroup \( U_v \) of \( G(K_v) \) such that
\[
G(K_v) = U_v A_v^+ \Omega_v U_v
\]
where \( A_v \) is the group of \( K_v \)-rational points of a maximal \( K_v \)-split torus of \( G \) and \( \Omega_v \) is a finite subset in the centralizer of \( A_v \) in \( G(K_v) \).

In particular for any \( g \in G(K_v) \), there exist unique \( a_v \in A_v^+ \) and \( d_v \in \Omega_v \) such that \( g \in U_v a_v d_v U_v \). For \( v \in R_\infty \), any maximal compact subgroup of \( G(K_v) \) is a good maximal compact subgroup and \( \Omega_v = \{ e \} \).

Let \( T \) denote the set of \( v \in R \) such that \( G(K_v) \) is compact, that is, \( U_v = G(K_v) \). It is well known that \( T \) is a finite set.

Denote by \( \Phi_v^+ \) the system of positive roots in the set of all non-multipliable roots of \( G(K_v) \) relative to \( A_v^+ \) and choose a maximal strongly orthogonal system \( S_v \) in \( \Phi_v^+ \) in the sense of [Oh2] (where an explicit construction is also given). If \( v \in R - T \) and \( K_v \neq \mathbb{C} \), then define the bi-\( U_v \)-invariant function \( \xi_v \) on \( G(K_v) \) (cf. [Oh1]): for each \( g = kadk' \in U_v A_v^+ \Omega_v U_v \),
\[
\xi_v(g) = \prod_{\alpha \in S_v} \Xi_v \left( \alpha(a) \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)^{n_\alpha}
\]
where \( \Xi_v \) is the Harish-Chandra function of \( \text{PGL}_2(K_v) \). If \( K_v = \mathbb{C} \), set
\[
\xi_v(g) = \prod_{\alpha \in S_v} \Xi_v \left( \alpha(a) \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)^{n_\alpha}
\]
where \( n_\alpha = 1/2 \) if \( \alpha \) is a long root of \( G \), when \( G \) is locally isomorphic to \( \text{Sp}_{2n}(\mathbb{C}) \), and \( n_\alpha = 1 \) for all other cases. We set \( \xi_v = 1 \) for \( v \in T \).

**Definition 2.2.** Define the function \( \xi_G \) on \( G(\mathbb{A}) \) by
\[
\xi_G(g) = \prod_{v \in R} \xi_v(g_v) \text{ for } g = (g_v)_v \in G(\mathbb{A}).
\]

Since \( 0 < \xi_v(g_v) \leq 1 \) for all \( v \in R \) and \( \xi_v(g_v) = 1 \) for almost all \( v \), \( \xi_G \) is a well defined function on \( G(\mathbb{A}) \) and \( 0 < \xi_G \leq 1 \). Set
\[
U = \prod_{v \in R} U_v.
\]

Note also that \( \xi_G \) is bi-\( U \)-invariant.

For \( v \in R - T \), denoting by \( \eta_v \) the product of all positive roots in \( S_v \), set
\[
\eta_v(kadk') := \eta_v(a)
\]
where \( kadk' \in U_v A_v^+ \Omega_v U_v \) for all \( v \) with \( K_v \neq \mathbb{C} \). As in the case of the definition of \( \xi_v \), if \( K_v = \mathbb{C} \), we set \( \eta_v = \prod_{\alpha \in S_v} \alpha^{n_\alpha} \) with the same \( n_\alpha \) defined as before. If \( v \in T \), we set \( \eta_v = 1 \).
Lemma 2.3. For any $\epsilon > 0$, there is a constant $C_\epsilon > 0$ such that for any $g = (g_v)_v \in G(\mathbb{A})$,

$$\prod_{v \in R} |\eta_v(g_v)|^{-1/2} \leq \xi_G(g) \leq C_\epsilon \cdot \prod_{v \in R} |\eta_v(g_v)|^{-1/2+\epsilon}. \tag{2.4}$$

In particular,

$$\xi_G(g) \to 0 \quad \text{as } g \to \infty \text{ in } G(\mathbb{A}).$$

Proof. For $v \in R - T$, it follows from the explicit formula for $\Xi_v$ (cf. \cite[3.7.1]{Oh1}) that for any $\epsilon > 0$, there is a constant $C_{v,\epsilon} > 0$ such that for any $g_v \in G(K_v)$,

$$|\eta_v(g_v)|^{-1/2} \leq \xi_v(g_v) \leq C_{v,\epsilon} \cdot |\eta_v(g_v)|^{-1/2+\epsilon}.$$

Moreover $C_{v,\epsilon} = 1$ for almost all $v$. This implies (2.4).

To see the second claim, first note that for any $g \in G(\mathbb{A})$,

$$\xi_G(g) \leq \xi_v(g_v) \leq C_\epsilon \cdot |\eta_v(g_v)|^{-1/2+\epsilon}. \tag{2.5}$$

Now suppose on the contrary that there exists a sequence $g_i \to \infty$ such that

$$\xi_G(g_i) \not\to 0.$$

Then by passing to a subsequence we may assume either that there is a place $v \in R - T$ such that $g_{i,v} \to \infty$ in $G(K_v)$ or that there is $v_i$ such that for any $g_{i,v_i} \notin U_{v_i}$ and $\Omega_{v_i} = \{e\}$ for all $i$. If $g_{i,v_i} \to \infty$ as $i \to \infty$, then $|\eta_v(g_{i,v_i})| \to \infty$ as $i \to \infty$ and hence $\xi_G(g_i) \to 0$ by (2.4). Therefore the first case cannot happen.

In the second case, note that since $g_{i,v_i} \notin U_{v_i}$ and $\Omega_{v_i} = \{e\}$, we have $|\eta_v(g_{i,v_i})| \geq q_{v_i}$ for each $i$. Hence by (2.5),

$$\xi_G(g_i) \leq C_\epsilon \cdot q_{v_i}^{-1/2+\epsilon}.$$

This gives a contradiction since $q_{v_i} \to \infty$. \qed 

Lemma 2.6. Let $\iota : G \to GL_N$ be a faithful absolutely irreducible representation defined over $K$ and $H_\iota$ be a height function on $G(\mathbb{A})$ associated to $\iota$. Then there exist $m \in \mathbb{N}$ and $C > 0$ such that

$$\xi_G(g) \leq C \cdot H_\iota^{-1/m}(g) \quad \text{for any } g \in G(\mathbb{A}).$$

Proof. Let $\chi$ denote the highest weight of $\iota$. Let $l \in \mathbb{N}$ be such that $\chi|_{A_\iota^+} \leq l \cdot \log_{q_v} \eta_v$ for each $v \in R$. Here $q_v = e$ if $v \in R_\infty$. Without loss of generality, we may assume

$$H_v(\iota(a)) = q_v^{\chi(a)} \quad \text{for each } a \in A_v^+ \text{ and } v \in R_f.$$

Since $|\eta_v(a_v)| = q_v^{\log_{q_v} |\eta_v(a_v)|}$ for $a_v \in A_v^+$, by Lemma 2.3, there exist some $c_1, c_2 > 0$ such that for any $g = (g_v)_v \in G(\mathbb{A})$,

$$\xi_G^\iota(g) \leq c_1 \cdot \prod_v |\eta_v(g_v)|^{-1} \leq c_2 \cdot \prod_v H_v^{-1}(\iota(g_v)) = c_2 \cdot H_\iota^{-1}(g).$$

\qed
Even though we do not need the following fact in this paper, it is of independent interest:

**Proposition 2.7.** There exists $0 < p = p(G) < \infty$ such that $\xi_G \in L^p(G(\mathbb{A}))$.

**Proof.** Choose any absolutely irreducible representation of $G$ defined over $K$, for instance, the adjoint representation, and let $H_\iota$ be a height function on $G(\mathbb{A})$ associated to $\iota$ (see (1.12)). By Theorem 3.4, the height zeta function

$$Z(s) := \int_{G(\mathbb{A})} H_\iota(g)^{-s} d\tau(g)$$

converges for $\text{Re}(s) > a_\iota$ where $a_\iota$ is defined as in (1.3). Since $\xi_G^m(g) \leq C \cdot H_\iota^{-1}(g)$ for some $m \in \mathbb{N}$ by Lemma 2.6, $\xi_G$ is $L^p$-integrable for any $p > ma_\iota$. \hfill \Box

2.2. Uniform bound for matrix coefficients of $G(\mathbb{A})$. Let $W_f \subset G(\mathbb{A}_f)$ be a compact open subgroup. Write $W_v = W_f \cap G(K_v)$ for each $v \in \mathcal{R}$. Then $W_v = U_v$ for almost all $v \in \mathcal{R}$. For each $v \in \mathcal{R}$, by [Be], there exists $d_{W_v} < \infty$ such that for any irreducible unitary representation $\rho$ of $G(K_v)$, the dimension of $W_v$-invariant vectors of $\rho$ is at most $d_{W_v}$. Moreover $d_{W_v}$ can be taken to 1 whenever $W_v$ is a hyper-special compact subgroup. Hence the following number is well-defined:

$$d_{W_f} := \prod_{v \in \mathcal{R}_f} d_{W_v} < \infty.$$

Set $U_f := \prod_{v \in \mathcal{R}_f} U_v$ and $U_\infty := \prod_{v \in \mathcal{R}_\infty} U_v$.

**Theorem 2.8.** Let $G$ be a connected absolutely almost simple $K$-group with $K$-rank at least 2. Let $W_f$ be a compact open subgroup of $G(\mathbb{A}_f)$. Let $\pi$ be any unitary representation of $G(\mathbb{A})$ without $G(K_v)^+$-invariant vector for every $v \in \mathcal{R}$. Then for any $U_\infty$-finite and $W_f$-invariant unit vectors $x$ and $y$,

$$|\langle \pi(g)x,y \rangle| \leq d_{W_f} \cdot c_{W_f} \cdot (\dim(U_\infty x) \cdot \dim(U_\infty y))^{(r+1)/2} \cdot \xi_G(g) \quad \text{for all } g \in G(\mathbb{A})$$

where $c_{W_f} := d_{W_f} \cdot \prod_v [U_v : U_v \cap W_v] \cdot (\max_{d \in \Omega_v} [U_v : d U_v d^{-1}])$ and $d_v, r \geq 1$ depend only on $G$. Moreover if $G(K_v) \not\cong \text{Sp}_{2n}(\mathbb{C})$ locally for any $v \in \mathcal{R}_\infty$, $d_0 = 1$ and $r = 1$.

The proof of above theorem is based on theorems in [Oh1]. More precisely, recall:

**Theorem 2.10** (Theorem 1.1-2, [Oh1]). Suppose that the $K_v$-rank of $G$ is at least 2. Let $\pi_v$ be a unitary representation of $G(K_v)$ without $G(K_v)^+$-invariant vectors. Then for any $U_v$-finite unit vectors $x$ and $y$,

$$|\langle \pi_v(g)x,y \rangle| \leq d_v \cdot c_v \cdot (\dim(U_v x) \cdot \dim(U_v y))^{r_v/2} \cdot \xi_v(g) \quad \text{for any } g \in G(K_v)$$

where $c_v = \max_{d \in \Omega_v} [U_v : d U_v d^{-1}]$ and $d_v, r_v \geq 1$ depend only on $G(K_v)$. Moreover whenever $G(K_v) \not\cong \text{Sp}_{2n}(\mathbb{C})$ locally, $d_v = 1$ and $r_v = 1$. 
In the case when \(G(K_v) \cong \text{Sp}_{2n}(\mathbb{C})\) locally, the above theorem was stated only for \(U_v\)-invariant vectors in [Oh1]. However using the remark following Prop. 2.7 in [Oh1], the proof can be modified for the above claim.

**Proof of Theorem 2.8** For \(g = (g_v)_v \in G(\mathbb{A})\), choose a finite subset \(S_g\) of places containing

\[
\{ v \in R_f : g_v \notin U_v \} \cup R_{\infty}.
\]

Note that for \(v \in R - S_g\), we have \(g_v \in U_v\) and hence \(\xi_v(g_v) = 1\). Therefore for \(g = (g_v)_v \in G(\mathbb{A})\),

\[
\xi(g) = \prod_{v \in S_g} \xi_v(g_v).
\]

Let \(G_g = \prod_{v \in S_g} G(K_v)\) and \(W_g = \prod_{v \in S_g - R_f} W_v\). As a \(G_g\) representation, \(\pi\) has a Hilbert integral decomposition:

\[
\pi = \int_{Z_g} \oplus m_z \rho_z \, d\nu(z)
\]

where \(Z_g\) is the unitary dual of \(G_g\) and \(\rho_z\) is irreducible, \(m_z\) is a multiplicity for each \(z \in Z_g\) and \(\nu\) is a measure on \(Z_g\) (see [Di] or [Section 2.3, Zi]). We may assume that for all \(z\), \(\rho_z\) has no \(G(K_v)^+\)-invariant vector (see [Prop. 2.3.2, Zi]).

If we write \(L_z = \oplus m_z \rho_z\), \(x = \int x_z d\nu(z)\) and \(y = \int y_z d\nu(z)\) with

\[
x_z = \sum_{i} x_{zi} \quad \text{and} \quad y_z = \sum_{i} y_{zi} \in L_z,
\]

we have

\[
\langle x, y \rangle = \int_{Z_g} \sum_{i} \langle x_{zi}, y_{zi} \rangle \, d\nu(z).
\]

It follows from the definition of a Hilbert direct integral that

\[\dim \langle U_\infty x_{zi} \rangle \leq \dim \langle U_\infty x_z \rangle \leq \dim \langle U_\infty x \rangle,\]

\(x_{zi}\) is \(W_g\)-invariant for almost all \(z\) and all \(i\), and similarly for \(y\). Without loss of generality, we assume the above holds for all \(z\). We claim that

\[|\langle \rho_z(g)x_{zi}, y_{zi} \rangle| \leq c_{W_f} \cdot d_0 \cdot \xi_G(g) \cdot (\dim(U_\infty x) \cdot \dim(U_\infty y))^{(r+1)/2} \cdot \|x_{zi}\| \cdot \|y_{zi}\|\]

where \(r = \max_v r_v\) and \(c_0 = d_{W_f} \prod_v (c_v \cdot [U_v : U_v \cap W_v]) < \infty, d_0 = \prod_v d_v < \infty\) with \(c_v, d_v, r_v\) as in Theorem 2.10. Since \(G_g\) is a type (I) group [Be], we may write \(\rho_z = \bigotimes_{v \in S_g} \rho_{z(v)}\) where \(\rho_{z(v)}\) is an irreducible representation of \(G(K_v)\) without \(G(K_v)^+\)-invariant vectors. Since the finite linear combinations of pure tensor vectors are dense, it suffices to prove (2.11) assuming \(x_{zi}\) and \(y_{zi}\) are finite sums of pure tensors. Hence we can write

\[
x_{zi} = \sum_j \bigotimes_{v \in S_g} x_{zi} j(v) \quad \text{and} \quad y_{zi} = \sum_k \bigotimes_{v \in S_g} y_{zi} k(v).
\]
where for each $v \in S_g$, $x_{zi}(v)$ (resp. $y_{zik}(v)$) are mutually orthogonal and the number of summands for $x_{zi}$ (resp. $y_{zik}$) is at most $\dim(U_{\infty}x) \cdot d_{W_j}$ (resp. $\dim(U_{\infty}y) \cdot d_{W_j}$). Hence by Cauchy-Schwartz inequality, for $x_{zi} = \prod_{v \in S_g} x_{zi}(v)$ and $y_{zik} = \prod_{v \in S_g} y_{zik}(v)$

$$\sum_j \|x_{zi}\| \leq (\dim(U_{\infty}x) \cdot d_{W_j})^{1/2}\|x_{zi}\|; \text{ and } \sum_k \|y_{zik}\| \leq (\dim(U_{\infty}y) \cdot d_{W_j})^{1/2}\|y_{zik}\|.$$

Since for $v \in R_f$

$$\dim(U_vx) \leq |U_v : W_v \cap U_v| \text{ and } \dim(U_vy) \leq |U_v : W_v \cap U_v|,$$

by Theorem 2.10, we have for $c_0 = \prod_v c_v$,

$$c_0 \cdot d_0 \cdot \prod_{v \in S_g} \xi_v(g_v) \cdot (\dim(U_{\infty}x) \cdot \dim(U_{\infty}y))^{r/2} \cdot (\prod_{v \in R_f} |U_v : W_v \cap U_v|) \cdot \left(\sum_{j,k} \|x_{zi}\| \cdot \|y_{zik}\|\right)$$

$$\leq c_0 \cdot d_0 \cdot \xi_G(g) \cdot (\dim(U_{\infty}x) \cdot \dim(U_{\infty}y))^{(r+1)/2} \cdot (\prod_{v \in R_f} |U_v : W_v \cap U_v|) \cdot d_{W_f} (\|x_{zi}\| \cdot \|y_{zik}\|)$$

$$= c_{W_f} \cdot d_0 \cdot \xi_G(g) \cdot (\dim(U_{\infty}x) \cdot \dim(U_{\infty}y))^{(r+1)/2} \cdot (\|x_{zi}\| \cdot \|y_{zik}\|)$$

proving (2.11). Therefore again by Cauchy-Schwartz inequality,

$$c_{W_f} \cdot d_0 \cdot \xi_G(g) \cdot (\dim(U_{\infty}x) \cdot \dim(U_{\infty}y))^{(r+1)/2} \cdot (\|x_{zi}\| \cdot \|y_{zik}\|)$$

By integrating over $Z_g$, we obtain (2.9).

Since $U$-finite vectors form a dense subset by Peter-Weyl theorem, the above implies an adelic version of Howe-Moore theorem [HM] on the vanishing of matrix coefficients:

**Corollary 2.14.** Let $G$ and $\pi$ be as in Theorem 2.8. Then for any vectors $x$ and $y$,

$$\langle \pi(g)x, y \rangle \to 0 \text{ as } g \to \infty \text{ in } G(A).$$

### 2.3. Automorphic bound for $G(A)$.

If $G$ has $K$-rank at most one, the analogue of Theorem 2.8 does not hold in general. However if we look at those infinite dimensional representations occurring in $L^2(G(K) \backslash G(A))$, we still obtain a similar upper bound.

We now consider the unitary representation of $G(A)$ on the space $L^2(G(K) \backslash G(A))$ by right translations. Let $L^2_{ad}(G(K) \backslash G(A))$ denote the orthogonal complement to the direct sum of all one-dimensional representations occurring in $L^2(G(K) \backslash G(A))$. If $G$ is simply connected, it follows from the strong approximation property that
Conjecture 2.15. Let $G$ be a connected absolutely almost simple $K$-group. Let $W_f$ be a compact open subgroup of $G(\mathbb{A}_f)$. Then for any $U_\infty \times W_f$-invariant unit vectors $f, h \in L^2_0(G(K) \backslash G(\mathbb{A}))$,

$$|\langle f, g, h \rangle| \leq c_{W_f} \cdot \xi_G(g) \quad \text{for all } g \in G(\mathbb{A})$$

where $c_{W_f} > 0$ is a constant depending only on $G$ and $W_f$.

The above holds for groups of $K$-rank at least 2 by Theorem 2.8. For $G = \text{PGL}_2$, Conjecture 2.15 is essentially equivalent to the Ramanujan conjecture. We will prove a weaker statement of Conjecture 2.15 where the function $\xi_G$ is replaced by a function $\tilde{\xi}_G$ with slower decay such that $\xi_G \leq \tilde{\xi}_G \leq \varepsilon_G^{1/2}$.

Definition 2.16. Set $R_1 := \{v \in R : \text{rank}_{K_v}(G) = 1\}$. Define

$$\tilde{\xi}_G(g) := \prod_{v \in R_1} \xi_v(g_v)^{1/2} \prod_{v \in R - R_1} \xi_v(g_v)$$

where $g = (g_v)_v \in G(\mathbb{A})$.

Theorem 2.17 (Automorphic bounds). Let $G$ be a connected absolutely almost simple $K$-group. Let $W_f$ be a compact open subgroup of $G(\mathbb{A}_f)$. Then for any $U_\infty$-finite and $W_f$-invariant unit vectors $x, y \in L^2_0(G(K) \backslash G(\mathbb{A}))$,

$$|\langle x, g, y \rangle| \leq c_{W_f} \cdot (\dim(U_\infty x) \cdot \dim(U_\infty y))^{(r + 1)/2} \cdot \tilde{\xi}_G(g) \quad \text{for all } g \in G(\mathbb{A})$$

where $c_{W_f} > 0, r = r(G) \geq 1$. Moreover $r = 1$ provided for any $v \in R$, $G(K_v) \not\cong \text{Sp}_{2n}(\mathbb{C})$ ($n \geq 2$) locally.

Recall that for unitary representations $\rho_1$ and $\rho_2$ of $G(K_v)$, $\rho_1$ is said to be weakly contained in $\rho_2$ if every diagonal matrix coefficients of $\rho_1$ can be approximated uniformly on compact subsets by convex combinations of diagonal matrix coefficients of $\rho_2$. For each $v \in R$, denote by $\hat{G}_v$ the unitary dual of $G(K_v)$ and by $\hat{G}_v^\text{Aut} \subset \hat{G}_v$ the automorphic dual of $G(K_v)$, i.e., the subset of unitary dual of $G(K_v)$ consisting of representations which are weakly contained in the representations appearing as $G(K_v)$ components of $L^2(G(K) \backslash G(\mathbb{A}))^{O_f}$ for some compact open subgroup $O_f$ of $G(\mathbb{A}_f)$.

Theorem 2.18 (Burger-Sarnak [BS], Clozel-Ullmo [CU]). Let $G$ be a connected absolutely almost simple $K$-group. Let $H \subset G$ be a connected semisimple $K$-subgroup. Then for any $v \in R$ and for any $\rho_v \in \hat{G}_v^\text{Aut}$, any irreducible representation $H(K_v)$ weakly contained in $\rho_v|_{H(K_v)}$ is contained in $\hat{H}_v^\text{Aut}$.

Lemma 2.19. For any $v \in R$ such that $G(K_v)$ is non-compact, $L^2_0(G(K) \backslash G(\mathbb{A}))$ has no non-zero $G(K_v)^+$-invariant function.
Proof. (cf. proof of Lemma 3.8 in [GaO].) Let $\mathcal{L}_v$ denote the set of $f \in L^2_{00}(G(K) \backslash G(\mathbb{A}))$ fixed by $G(K_v)^\circ$. We need to show that $\mathcal{L}_v = \{0\}$. Consider the subgroup $G^{(v)}$ of $G(\mathbb{A})$ consisting of elements whose $v$-component is trivial, and consider continuous functions in $G^{(v)}$ with compact support of the form $\prod_{w \in R - \{v\}} f_w$ where $f_w \in C_c(G(K_w))$ for all $w \in R - \{v\}$ and $f_w|_{U_w} = 1$ for almost all $w$. By considering the convolutions with these functions, we obtain a dense family of the continuous functions of $\mathcal{L}_v$. Hence it suffices to show that any continuous function $f \in \mathcal{L}_v$ is trivial. Let $f \in \mathcal{L}_v$ be continuous. Let $G$ be the simply connected cover of $G$ and denote by $pr : \hat{G} \to G$ the covering map. Consider the projection map

$$\hat{G}(K) \backslash \hat{G}(\mathbb{A}) \to G(K) \backslash G(\mathbb{A}).$$

Let $\hat{f}$ be the pull back of $f$. Since the image of $\hat{G}(K_v)$ is $G(K_v)^\circ$ under the map $pr$, the function $\hat{f}$ is left $\hat{G}(K)$-invariant and right $G(K_v)$-invariant. On the other hand, the strong approximation property implies that $\hat{G}(K)\hat{G}(K_v)$ is dense in $\hat{G}(\mathbb{A})$ (cf. [Theorem 7.12, PR]). Therefore $\hat{f}$ is constant. It follows that $f$ is a sum of characters of $G(\mathbb{A})$, and hence 0 since $f \in L^2_{00}(G(K) \backslash G(\mathbb{A}))$. \qed

Proof of Theorem 2.17. The case when $K$-rank is at least 2 follows from Theorem 2.8 and Lemma 2.19. Suppose first that $G$ has $K$-rank one. Then there is a $K$-embedding of a $K$-group $H$ where $H$ is either SL$_2$ or PGL$_2$. Using a Gelbart-Jacquet [GJ] estimate toward the Ramanujan conjecture for SL$_2$ or PGL$_2$ and Theorem 2.18, we obtain for $v \in R_1$, any infinite dimensional $\rho_v \in \hat{G}_v^{Aut}$, and $U_v$-finite vectors $x_v, y_v$, (cf. Theorem 3.4 [COU])

$$(2.20) \quad |\langle \rho_v(g)(x_v), y_v \rangle| \leq c_v \cdot \xi_v(g)^{1/2} \cdot (\dim(U_v x_v) \cdot \dim(U_v y_v))^{1/2}$$

for any $g \in G(K_v)$. Combining this with Theorem 2.10, we can derive the desired bound by the same argument as in the proof of Theorem 2.8.

Now suppose $G$ is $K$-anisotropic. If $R_1 \neq \emptyset$, it follows from the classification theorem by Tits [Ti1] that $G$ is of Dynkin type $A$. Applying [Theorem 1.1, Cl], we deduce that there exists a $K$-embedding of $K$-subgroup $H$ of type $A$ such that $H$ has $K_v$-rank one whenever $v \in R_1$. Let $v \in R_1$. Then up to isogeny, we have either that $H = PGL_1(D)$ for a quaternion algebra $D$ over $K$ and $H = PGL_2$ over $K_v$, or $H = PGU(D, *)$ for a division algebra $D$ of prime degree $d$ over a quadratic extension $F$ of $K$ with a second kind involution $*$, and $H = PGU(n-1, 1)$ over $K_v$ (with $n \geq 3$). In the former case we use the Jacquet-Langlands correspondence [JL] to transfer the Gelbart-Jacquet automorphic bound of $PGL_2$ to $H(K_v)$ via Theorem 2.18. In the second case which is hardest, it is explained in [Cl1] that by the base changes obtained by Rogawski [Ro] and Clozel [Cl2], we can use the bound of $PGL_n(F_w)$ to get a bound for $H(K_v)$ where $w$ is a place of $F$ lying above $v$ and $F_w$ is a quadratic extension of $K_v$. This gives us (2.20) for $v \in R_1$ again in view of Theorem 2.18 and Theorem 2.10. Combining with Theorem 2.10 for those places $v \in R - (R_1 \cup T)$ as in the proof of Theorem 2.8, we obtain the desired bound. This finishes the proof. \qed
Let $X_1, \cdots, X_m$ be an orthonormal basis of the Lie algebra $\text{Lie}(U_\infty)$ with respect to an Ad-invariant scalar product. Then the elliptic operator

$$D := 1 - \sum_{i=1}^{m} X_i^2$$

lies in the center of the universal enveloping algebra of $\text{Lie}(U_\infty)$. We say a function $f$ on $G(K) \backslash G(A)$ is smooth if $f$ is $W_f$-invariant for some compact open subgroup of $G(A_f)$ and smooth for the action of $G_\infty$.

In Theorem 2.17, we can relax $U_\infty$-finite conditions to smooth conditions provided we replace the $L^2$-norms by $L^2$-Sobolev norms:

**Theorem 2.22.** Let $G$ be a connected absolutely almost simple $K$-group. Let $W_f$ be a compact open subgroup of $G(A_f)$. Then for any $W_f$-invariant smooth functions $f,h \in L^2_{00}(G(K) \backslash G(A))$ with compact support,

$$|\langle f, g, h \rangle| \leq c_{W_f} \cdot \tilde{\xi}_G(g) \cdot \|D^l(f)\| \cdot \|D^l(h)\| \text{ for all } g \in G(A)$$

where $c_{W_f} > 0$ and $l$ is any sufficiently large integer.

**Proof.** Deducing this from Theorem 2.17 is quite standard in view of the results of Harish-Chandra explained in [Ch 4, Wa]. We give a sketch of the proof. Denote by $\pi$ the representation $L^2_{00}(G(K) \backslash G(A))$ with compact support, $\pi = \bigoplus_{\nu \in \hat{U}_\infty} \pi_\nu$ where $\pi_\nu$ is the $\nu$-isotypic component of $\pi$. Then $\pi_\nu$ acts as a scalar, say, $c_\nu$ on $\pi_\nu$. We write $f = \sum_{\nu \in \hat{U}_\infty} f_\nu$ and $h = \sum_{\nu \in \hat{U}_\infty} h_\nu$. One has $\|f_\nu\| = c_\nu^{-2} \|\mathcal{D}^l f_\nu\|$ and similarly for $h$. Then

$$|\langle f, g, h \rangle| \leq \sum_{(\nu_1, \nu_2) \in \hat{U}_\infty \times \hat{U}_\infty} |\langle f_{\nu_1}, g, h_{\nu_2} \rangle|$$

Using Theorem 2.17, we then obtain

$$|\langle f, g, h \rangle|$$

$$\leq c_{W_f} \cdot \tilde{\xi}_G(g) \left( \sum_{\nu \in \hat{U}_\infty} \|f_\nu\| \dim(U_\infty f_\nu)^{(r+1)/2} \right) \left( \sum_{\nu \in \hat{U}_\infty} \|h_\nu\| \dim(U_\infty h_\nu)^{(r+1)/2} \right)$$

$$\leq c_{W_f} \cdot \tilde{\xi}_G(g) \cdot \|D^l(f)\| \cdot \|D^l(h)\| \cdot \sum_{\nu \in \hat{U}_\infty} c_\nu^{-2l} \dim(\nu)^{r+1}$$

Now if $l \in \mathbb{N}$ is sufficiently large, then $\sum_{\nu} c_\nu^{-2l} \dim(\nu)^{r+1} < \infty$ [Wa]. This proves the claim. \hfill \Box

If $G$ is a connected semisimple $K$-group, we say that a sequence $\{g_i \in G(A)\}$ tends to infinity strongly if for any non-trivial connected simple normal $K$-subgroup $H$ of $G$, $\pi(g_i)$ tends to $\infty$ as $i \to \infty$, where $\pi : G(A) \to G(A)/H(A)$ denotes the canonical projection.
Theorem 2.23 (Mixing for $L^2(G(K) \setminus G(\mathbb{A}))$). Let $G$ be a product of connected absolutely almost simple $K$-groups. Then for any $f, h \in L^2_{00}(G(K) \setminus G(\mathbb{A}))$,
\[ \langle f, g, h \rangle \rightarrow 0 \]
as $g \in G(\mathbb{A})$ tends to infinity strongly.

Proof. Write $G = G_1 \times \cdots \times G_m$ where each $G_i$ is a connected absolutely almost simple $K$-group. By Theorem 2.17 and Peter-Weyl theorem, for each $1 \leq i \leq m$, and for any $f_i, h_i \in L^2_{00}(G_i(K) \setminus G_i(\mathbb{A}))$,
\[ \langle f_i, g, h_i \rangle \rightarrow 0 \]
as $g_i \rightarrow \infty$ in $G_i(\mathbb{A})$.

Consider $\otimes_{i=1}^m L^2(G_i(K) \setminus G_i(\mathbb{A}))$ as a subset of $L^2(G(K) \setminus G(\mathbb{A}))$. Then the finite sums of the functions of the form $h = \otimes_{i=1}^m h_i \in L^2(G(K) \setminus G(\mathbb{A}))$, $h_j \in L^2(G_j(K) \setminus G_j(\mathbb{A}))$, such that for at least one $j$, $h_j \in L^2_{00}(G_j(K) \setminus G_j(\mathbb{A}))$ form a dense subset of the space $L^2_{00}(G(K) \setminus G(\mathbb{A}))$. Hence it suffices to prove the claim for $f = \otimes_{i=1}^m f_i$ and $h = \otimes_{i=1}^m h_i$ of such type. Suppose $h_j \in L^2_{00}(G_j(K) \setminus G_j(\mathbb{A}))$ for some $1 \leq j \leq m$. If $g = (g_1, \cdots, g_m)$ with $g_i \in G_i(\mathbb{A})$, then
\[ |\langle f, g, h \rangle| = \prod_{i=1}^m |\langle f_i, g_i, h_i \rangle| \leq |\langle f_j, g_j, h_j \rangle| \cdot \left( \prod_{i \neq j} \|f_i\| \cdot \|h_i\| \right). \]
If $f'_j$ denotes the projection of $f_j$ to $L^2_{00}(G_j(K) \setminus G_j(\mathbb{A}))$, then
\[ \langle f_j, g_j, h_j \rangle = \langle f'_j, g_j, h_j \rangle. \]
Since $g \rightarrow \infty$ strongly and hence $g_j \rightarrow \infty$, we obtain $\langle f'_j, g_j, h_j \rangle \rightarrow 0$ by (2.24). This proves the claim. $\square$

3. Volume asymptotics

Let $G$ be a connected adjoint semisimple algebraic group over $K$. As in the previous section, let $L^2_{00}(G(K) \setminus G(\mathbb{A}))$ be the orthogonal complement to the direct sum of all one dimensional representations in $L^2(G(K) \setminus G(\mathbb{A}))$. Denote by $\Lambda$ an orthogonal basis for $L^2_{00}(G(K) \setminus G(\mathbb{A})) \perp \times \times \times$ consisting of continuous unitary characters of $G(\mathbb{A})$. Then
\[ L^2(G(K) \setminus G(\mathbb{A})) = L^2_{00}(G(K) \setminus G(\mathbb{A})) \oplus \sum_{\chi \in \Lambda} \mathbb{C} \chi. \]

For a compact open subgroup $W_f$ of $G(\mathbb{A}_f)$, set
\[ \Lambda^{W_f} := \{ \chi \in \Lambda : u.\chi = \chi \text{ for any } u \in W_f \}; \]
\[ \ker(\Lambda^{W_f}) := \cap \ker(\chi) : \chi \in \Lambda^{W_f}. \]
Note that $\ker(\Lambda^{W_f})$ is a normal subgroup of $G(\mathbb{A})$. Moreover:

Lemma 3.1. We have
(1) $G(K)G_\infty W_f \subset \ker(\Lambda^{W_f})$. 

(2) \( \#\Lambda^W_f = [G(\mathbb{A}) : \ker(\Lambda^W_f)] < \infty. \)

(3) For any \( g \in G(\mathbb{A}), \)
\[
\sum_{\chi \in \Lambda^W_f} \chi(g) = \begin{cases} 
\#\Lambda^W_f & \text{if } g \in \ker(\Lambda^W_f), \\
0 & \text{otherwise}.
\end{cases}
\]

Proof. Note that an element of \( \Lambda \) is precisely a continuous homomorphism \( \chi : G(\mathbb{A}) \rightarrow S^1 \) which contains \( G(\mathbb{K}) \) in the kernel, where \( S^1 \) is the unit circle. Since \( G_\infty^0 \) is a connected semisimple group, \( G_\infty^0 \subset \ker(\chi) \) for any \( \chi \in \Lambda \). Hence
\[
G(\mathbb{K})G_\infty^0W_f \subset \ker(\Lambda^W_f).
\]
Since \( G_\infty^0 \) has a finite index in \( G_\infty \), it follows from [Theorem 5.1, PR] that there exist finitely many \( u_1, \ldots, u_h \in G(\mathbb{A}) \) such that
\[
G(\mathbb{A}) = \bigcup_{i=1}^h G(\mathbb{K})u_iG_\infty^0W_f.
\]
It follows \( [G(\mathbb{A}) : \ker(\Lambda^W_f)] < \infty. \) Clearly the quotient \( G(\mathbb{A})/\ker(\Lambda^W_f) \) is a finite abelian group whose dual is isomorphic to \( \Lambda^W_f \). Hence (2) and (3) easily follow from the duality of finite abelian groups. □

Let \( \iota : G \rightarrow GL_N \) be an absolutely irreducible faithful representation defined over \( \mathbb{K} \). We give a definition of a height function on \( G(\mathbb{A}) \) associated to \( \iota \) which is slightly more general than those considered in the introduction (see (1.12)). It is this class of the functions for which we prove our main theorems.

**Definition 3.2.** A height function \( H_\iota \) on \( G(\mathbb{A}) \) is defined by the product \( \prod_{v \in R} H_{\iota,v} \) where \( H_{\iota,v} \) is a function on \( G(\mathbb{K}_v) \) for \( v \in R \) satisfying the following:

1. there exists a finite subset \( S \subset R \) such that
\[
H_{\iota,v}(g) = \max_{ij} |\iota(g)_{ij}|_v \quad \text{for all } v \in R - S;
\]

2. for \( v \in S \), there exists \( C > 0 \) such that
\[
C^{-1} \cdot \max_{ij} |\iota(g)_{ij}|_v \leq H_{\iota,v}(g) \leq C \cdot \max_{ij} |\iota(g)_{ij}|_v;
\]

3. for any \( v \in S \cap R_\infty \), there exists \( b > 0 \) such that for any small \( \epsilon > 0, \)
\[
(1 - b \cdot \epsilon)H_{\iota,v}(x) \leq H_{\iota,v}(g x h) \leq (1 + b \cdot \epsilon)H_{\iota,v}(x)
\]
for any \( x \in G(\mathbb{K}_v) \) and any \( g, h \) in the \( \epsilon \)-neighborhood of \( e \) in \( G(\mathbb{K}_v) \) with respect to a Riemannian metric;

4. for any \( v \in S \cap R_f \), \( H_{\iota,v} \) is bi-invariant under a compact open subgroup of \( G(\mathbb{K}_v) \).

Fix a height function \( H_\iota \) on \( G(\mathbb{A}) \). Note that the height function \( H_\iota \) is bi-invariant under a compact open subgroup, say \( W_f \), of \( G(\mathbb{A}_f) \). For \( T > 0 \), set
\[
B_T := \{ g \in G(\mathbb{A}) : H_\iota(g) < T \}.\]
For a finite subset $S$ of $R$, let $G^S$ denote the subgroup of $G(\mathbb{A})$ consisting of $(g_v)$, with $g_v = e$ for all $v \in S$, and set $G_S := \prod_{v \in S} G(K_v)$. Note that $G^0 = G(\mathbb{A})$ and $G(\mathbb{A}) = G_S G^S$. Let $\tau_S$ and $\tau^S$ denote Haar measures on $G_S$ and $G^S$ respectively.

The following lemma easily follows from the decomposition of a Haar measures $dg_v$ in terms of Cartan decomposition.

**Lemma 3.3.** There exists $\epsilon > 0$ such that for every finite set $S \subset R$,

$$\tau_S(B_T \cap G_S) = O(T^{a_\lambda - \epsilon})$$

where the implied constant depends on $S$. In particular, for any $v \in R$,

$$\int_{G(K_v)} H_{i,v}(g_v)^{-a_v} dg_v < \infty.$$

Given an automorphic character $\chi$, we consider the following integral

$$Z^S(s,\chi) := \int_{G^S} H_{i}(g)^{-s} \chi(g) d\tau^S(g).$$

The following follows from Theorem 7.1 in [STT2] and the properties of the Hecke $L$-functions:

**Theorem 3.4.** Let $S$ be a finite subset of $R$ and $a_\lambda$, $b_\lambda$ as in (1.3). Then $Z^S(s,\chi)$ converges absolutely when $\text{Re}(s) > a_\lambda$, and there exists $\epsilon > 0$ such that $Z^S(s,\chi)$ has a meromorphic continuation to $\text{Re}(s) > a_\lambda - \epsilon$ with unique poles at $s = a_\lambda$ of order at most $b_\lambda$. The order of the pole is exactly $b_\lambda$ for $\chi = 1$. Moreover, for some constants $\kappa \in \mathbb{R}$ and $k > 0$,

$$\left| \frac{1}{s^{b_\lambda}} Z^S(s,\chi) \right| \leq k \cdot |1 + \text{Im}(s)|^\kappa$$

for $\text{Re}(s) > a_\lambda - \epsilon$.

For $g \in G(\mathbb{A})$, define

$$\delta_S(g) := \sum_{\chi \in \Lambda^{w_f}} c_{S,\chi} \chi(g) \quad \text{with} \quad c_{S,\chi} = \lim_{s \to a_\lambda} (s - a_\lambda)^b Z^S(s,\chi).$$

It will be a consequence of Theorem 3.9 that for all $g$, $\delta_S(g) > 0$.

We use the following version of Ikehara Tauberian theorem to deduce the volume asymptotics from Theorem 3.4.

**Theorem 3.6.** Fix $a > 0$ and $\delta > 0$. Let $\alpha(t)$ be a non-negative non-decreasing function on $(\delta, \infty)$ such that

$$f(s) := \int_{\delta}^{\infty} t^{-s} d\alpha$$

converges for $\text{Re}(s) > a$. Suppose that

- $f(s)$ has a meromorphic continuation to the half plane $\text{Re}(s) > a - \epsilon > 0$ and has a unique pole at $s = a$ with order $b$;
• For some $\kappa \in \mathbb{R}$ and $k > 0$, 
\[
\left| \frac{f(s)(s-a)^b}{g^b} \right| \leq k \cdot |1 + \text{Im}(s)|^{\kappa}
\]
for $\text{Re}(s) > a - \epsilon$.

Then for some $\delta > 0$,
\[
\int_\delta^T d\alpha = \alpha(T) - \alpha(\delta) = \frac{c}{a(b-1)!} \cdot T^a P(\log T) + O(T^{a-\delta}) \quad \text{as } T \to \infty
\]
where $c = \lim_{s \to a} (s-a)^b f(s)$ and $P(x)$ is a monic polynomial of degree $b-1$.

**Proof.** This can be proven by repeating the same argument as in the appendix of [CT1] simply replacing the sum $\sum_n n^{-s} \alpha_n$ by the integral $\int_\delta^T t^{-s} d\alpha(t)$.

**Lemma 3.7.** • We have
\[
\delta_0 := \inf_{g \in G(\mathbb{A})} H_i(g) > 0.
\]

• For each $T > 0$, $B_T$ is a relatively compact subset of $G(\mathbb{A})$. In particular, the height function $H_i : G(\mathbb{A}) \to [\delta_0, \infty)$ is proper.

**Proof.** By Definition 3.2, there exists a finite subset $R_0$, such that for all $v \in R - R_0$, $H_v(\iota(g)) \geq 1$ for any $g \in G(K_v)$. Let $0 < \delta \leq 1$ be such that $H_v(\iota(g)) \geq \delta$ for $v \in R_0$ and $\delta_0 = \delta^{\#R_0}$. Then $H_i(g) \geq \delta_0$ for all $g \in G(\mathbb{A})$. Note that
\[
B_T \subset G(\mathbb{A}) \cap \prod_v \{g_v \in G(K_v) : H_v(\iota(g_v)) \leq \delta_0^{-1} T\}.
\]
Since for almost all $v \in R$, $H_v(\iota(g_v)) \geq q_v$ whenever $g_v \notin G(O_v)$, it follows that for some finite subset $R_1 \subset R$, we have
\[
B_T \subset \{ (g_v)_v \in G(\mathbb{A}) : H_v(\iota(g_v)) \leq \delta_0^{-1} T \quad \text{for } v \in R_1, \quad g_v \in G(O_v) \text{ otherwise} \}
\]
and hence $B_T$ is a relatively compact subset of $G(\mathbb{A})$.

**Theorem 3.9.** Let $a_i \in \mathbb{Q}^+$ and $b_i \in \mathbb{N}$ be as in (1.3). Then for any finite subset $S \subset R$ and $g \in G^S$,
\[
(3.10) \quad \tau^S(B_T \cap g \ker(\Lambda^{W_f}) \cap G^S) = \frac{\delta_0^{-1}}{\# \Lambda^{W_f} \cdot a_i(b_i - 1)!} \cdot T^{a_i} P(\log T) + O(T^{a_i - \delta})
\]
where the leading term is positive, $P(x)$ is a monic polynomial of degree $b_i - 1$, and $\delta > 0$.

**Proof.** By the above lemma, $B_T$ is a relatively compact subset of $G(\mathbb{A})$ and hence $\tau^S(B_T \cap G^S) < \infty$ for each $T \geq 1$ and for any finite $S$. Let $\delta_0$ be as in Lemma 3.7,
\[
\alpha(t) = \tau^S(B_t \cap g \ker(\Lambda^{W_f}) \cap G^S) \quad \text{for } t \in [\delta_0, \infty),
\]
and 
\[ f(s) = \int_{\delta_0}^{\infty} t^{-s} \, d\alpha. \]

Then by Lemma 3.1(3),

\[
\begin{align*}
  f(s) &= \int_{g \ker(\Lambda^W_f) \cap G^S} H_s(h)^{-s} \, d\tau^S(h) \\
  &= (\#\Lambda^W_f)^{-1} \sum_{\chi \in \Lambda^W_f} \int_{G^S} H_s(h)^{-s} \chi(g^{-1}h) \, d\tau^S(h) \\
  &= (\#\Lambda^W_f)^{-1} \sum_{\chi \in \Lambda^W_f} Z^S(s, \chi(g^{-1})).
\end{align*}
\]

Hence, (3.10) follows from Theorems 3.4 and 3.6.

Now we show that the leading term in (3.10) is nonzero. For any \( g \in G(A) \), there exists \( c > 0 \) such that \( g B_T \subset B(cT) \). Since \( [G^S : (\ker(\Lambda^W_f) \cap G^S)] < \infty \), there exists \( c > 0 \) such that

\[
\tau^S(B_T \cap G^S) \sim [G^S : (\ker(\Lambda^W_f) \cap G^S)] \cdot \tau^S(B_T \cap g \ker(\Lambda^W_f) \cap G^S).
\]

Since \( Z^S(s, 1) \) has a pole of order exactly \( b_i \) at \( s = a_i \),

\[
\lim_{T \to \infty} \frac{\tau^S(B_T \cap G^S)}{T^{a_i} (\log T)^{b_i-1}} > 0.
\]

Hence,

\[
\lim_{T \to \infty} \frac{\tau^S(B_T \cap g \ker(\Lambda^W_f) \cap G^S)}{T^{a_i} (\log T)^{b_i-1}} > 0,
\]

and \( \delta^S(g) > 0 \). \(\square\)

**Proposition 3.11.** For \( g \in G(A) \) and a co-finite subgroup \( V_f \) of \( W_f \),

\[ \tau(B_T \cap g \ker(\Lambda^W_f)) \sim [\ker(\Lambda^W_f) : \ker(\Lambda^V_f)] \cdot \tau(B_T \cap g \ker(\Lambda^V_f)). \]

**Proof.** Note that if \( \chi \in \Lambda^V_f - \Lambda^W_f \) then \( \chi(w) \neq 1 \) for some \( w \in W_f \). Since \( H_s \) is \( W_f \)-invariant,

\[
Z^\theta(s, \chi) = \int_{G(A)} H_i^{-s}(wg) \chi(wg) \, d\tau(g) = \chi(w) Z^\theta(s, \chi),
\]

and hence \( c_{\theta, \chi} = 0 \). Therefore,

\[
\sum_{\chi \in \Lambda^W_f} c_{\theta, \chi} \chi(g) = \sum_{\chi \in \Lambda^V_f} c_{\theta, \chi} \chi(g),
\]

and the claim follows from Theorem 3.9 and Lemma 3.1(2). \(\square\)
4. Equidistribution for saturated cases

Let $G$ be a product of connected adjoint absolutely simple groups defined over $K$ and $\iota$ be a faithful absolutely irreducible representation of $G$. Recall the compact spaces $X_\iota$ and $X(\mathbb{A})$ defined in Theorems 1.7 and 1.10 respectively. Since the arguments for both spaces are essentially identical, we consider the space $X_\iota$. Without loss of generality, we may consider $G(K_v)$ as a subset of $X_{\iota,v}$ and $G(\mathbb{A})$ as a subset of $X_\iota$. Fix a height function $H_\iota = \prod_{v \in R} H_{\iota,v}$ on the associated adele group $G(\mathbb{A})$ relative to $\iota$ as in Definition 3.2.

Setting $H_{\iota,S} = \prod_{v \in S} H_{\iota,v}$, we define

$$m_{\iota,S} := \int_{G_S} \delta_S(g) H_{\iota,S}(g)^{-a_\iota} d\tau_S(g)$$

where $\delta_S$ is given in (3.5). By Lemma 3.3, $m_{\iota,S} < \infty$. We also define a probability measure on $G_S$:

$$\mu_{\iota,S} := m_{\iota,S}^{-1} \cdot \delta_S(g) \cdot H_{\iota,S}(g)^{-a_\iota} d\tau_S(g).$$

It gives a probability measure on $X_{\iota,S} = \prod_{v \in S} X_{\iota,v} \supset \overline{\iota(G_S)}$. Note that $\mu_{\iota,S}$ is given by (1.6) with

$$c_u = \delta_S(u) \cdot m_{\iota,S}^{-1} \cdot \left( \int_{G_S} H_{\iota,S}(g)^{-a_\iota} d\tau_S(g) \right).$$

**Lemma 4.2.** For finite sets $S \subset T$ in $R$, the projection $G_T \rightarrow G_S$ maps the measure $\mu_{\iota,T}$ to the measure $\mu_{\iota,S}$.

**Proof.** It follows from (3.5) that

$$c_{S,\chi} = c_{T,\chi} \left( \int_{G_{T-S}} H_{\iota,T-S}(h)^{-a_\iota} \chi(h) d\tau_{T-S}(h) \right).$$

Then

$$\int_{G_{T-S}} \delta_T(gh) H_{\iota,T}(gh)^{-a_\iota} d\tau_{T-S}(h)$$

$$= \int_{G_{T-S}} \sum_{\chi \in \Lambda^W_T} c_{T,\chi} \chi(gh) H_{\iota,T}(gh)^{-a_\iota} d\tau_{T-S}(h)$$

$$= \sum_{\chi \in \Lambda^W_T} c_{T,\chi} \left( \int_{G_{T-S}} H_{\iota,T-S}(h)^{-a_\iota} \chi(h) d\tau_{T-S}(h) \right) \chi(g) H_{\iota,S}(g)^{-a_\iota}$$

$$= \left( \sum_{\chi \in \Lambda^W_T} c_{S,\chi} \chi(g) \right) H_{\iota,S}(g)^{-a_\iota} = \delta_S(g) H_{\iota,S}(g)^{-a_\iota}.$$

This implies the claim. \qed
Lemma 4.2 implies that there exists a unique probability measure \( \mu_i \) on \( X_i \) such that the image of \( \mu_i \) under the projection map \( X_i \to X_i, S \) is equal to \( \mu_i, S \).

This section is devoted to a proof of the following:

**Theorem 4.3.** Suppose that \( i \) is saturated. Then for any \( f \in C(X_i) \),

\[
\lim_{T \to \infty} \frac{1}{\tau(B_T \cap G_{H_i})} \sum_{g \in G(K) : H_i(g) < T} f(g) = \int_{X_i} f d\mu_i,
\]

where \( \tau \) is the Haar measure on \( G_{H_i} \) normalized so that \( \tau(G(K) \setminus G_{H_i}) = 1 \).

Note that Theorem 4.3 implies Theorem 1.15 (by taking \( f = 1 \)) and Theorems 1.5 and 1.7. The proof of Theorem 1.10 is similar. Combining Theorem 4.3 with Theorem 3.9 and Lemma 4.11, we deduce Theorems 1.2 and 1.9 for the saturated case. The rate of convergence as well as nonsaturated case are discussed in the next section.

For a compact open subgroup \( W_f \) of \( G(A_f) \) under which \( H_i \) is bi-invariant, we consider the subgroup \( \ker(\Lambda_{W_f}) \) of \( G(A_f) \) where \( \Lambda_{W_f} \) is the set of all automorphic characters in \( L^2(G(K) \setminus G(A)) \) which are \( W_f \)-invariant, and let \( \tau_{W_f} \) denote the normalized Haar measure on \( \ker(\Lambda_{W_f}) \) so that

\[
\tau_{W_f}(G(K) \setminus \ker(\Lambda_{W_f})) = 1.
\]

In the following, we fix a compact open subgroup \( W_f \) of \( G(A_f) \) under which \( H_i \) is bi-invariant and set \( G_{H_i} = \ker(\Lambda_{W_f}) \) and \( \tau = \tau_{W_f} \).

**Lemma 4.4.** For any \( f \in C(X_i) \),

\[
\lim_{T \to \infty} \frac{1}{\tau(B_T \cap G_{H_i})} \int_{B_T \cap G_{H_i}} f d\tau = \int_{X_i} f d\mu_i.
\]

**Proof.** To derive a formula for the Haar measure \( \tau \) on \( G_{H_i} \), we observe that by the weak approximation, \( G_{H_i} \) projects onto \( G_S \). Hence,

\[
G_{H_i} = \bigcup_{g \in G_S/(G_{H_i} \cap G_S)} g(G_{H_i} \cap G_S)s_g(G_{H_i} \cap G_S)
\]

where

\[
g \mapsto s_g : G_S \to G_S
\]

is map which factors through the finite group \( G_S/(G_S \cap G_{H_i}) \) and \( \chi(g) = \chi(s_g^{-1}) \) for every \( \chi \in \Lambda_{W_f} \). Then one can choose a Haar measure \( \tau^S \) on \( G_S \) such that

\[
\int_{G_{H_i}} f d\tau = \int_{g \in G_S} \int_{h \in G_S \cap G_{H_i}} f(gs gh) d\tau^S(h) d\tau^S(g).
\]

for any \( f \in C_c(G_{H_i}) \).
First, we prove the lemma for a function \( f \in C(X) \) that factors through \( X_{i,S} \) for some finite \( S \subset R \), i.e.,
\[
f((x_v)_{v \in R}) := f_S((x_v)_{v \in S}) \quad \text{for some } f_S \in C(X_{i,S}).
\]

It follows from (4.5) that
\[
\int_{B_T \cap G_{H_i}} f \, d\tau = \int_{G_S} f_S(g) \tau^S(B_{T,H_{i,s}(g)^{-1}} \cap s_g G_{H_i} \cap G^S) \, d\tau_S(h).
\]

By Theorem 3.9, there exists a constant \( c > 0 \) such that for any \( g \in G_S \) with \( H_{i,s}(g) \leq T/2 \),
\[
\tau^S(B_{T,H_{i,s}(g)^{-1}} \cap s_g G_{H_i} \cap G^S) \leq c H_{i,s}(g)^{-a_i} T^{a_i} (\log TH_{i,s}(g)^{-1})^{b_i-1}.
\]

With \( \delta_0 \) as in Lemma 3.7, there exists \( d > 0 \) such that for any \( g \in G_S \) satisfying \( T/2 \leq H_{i,s}(g) \leq T\delta_0^{-1} \),
\[
\tau^S(B_{T,H_{i,s}(g)^{-1}} \cap s_g G_{H_i} \cap G^S) \leq \tau^S(B_2 \cap s_g G_{H_i} \cap G^S) \leq d \cdot H_{i,s}(g)^{-a_i} T^{a_i}.
\]

Also, for \( H_{i,s}(g) > T\delta_0^{-1} \), the above volume is zero.

Setting
\[
y_T(g) := \frac{\tau^S(B_{T,H_{i,s}(g)^{-1}} \cap s_g G_{H_i} \cap G^S)}{\tau^S(B_T \cap G_{H_i} \cap G^S)},
\]
we deduce that for some constant \( C > 0 \),
\[
y_T(g) \leq C \cdot H_{i,s}(g)^{-a_i} \quad \text{for any } g \in G_S.
\]

In particular, \( y_T \) is in \( L^1(G_S) \) by Lemma 3.3. Hence, by Theorem 3.9,
\[
y_T(g) \to \delta(s_{g^{-1}})^{-a} H_{i,s}(g)^{-a} \delta(e)^{-1} \quad \text{as } T \to \infty.
\]

Since \( \delta(s_{g^{-1}})^{-1} = \delta(g) \), we apply the dominated convergence theorem to (4.6) and deduce that
\[
\int_{B_T \cap G_{H_i}} f \, d\tau \sim \mu_{i,s}(f) \cdot m_{i,s} \cdot \delta(e)^{-1} \cdot \tau^S(B_T \cap G_{H_i} \cap G^S) \quad \text{as } T \to \infty.
\]

Note that \( \mu_{i,s}(f) = \mu_i(f) \). Taking \( f = 1 \), we also get
\[
\tau(B_T \cap G_{H_i}) \sim m_{i,s} \cdot \delta(e)^{-1} \cdot \tau^S(B_T \cap G_{H_i} \cap G^S) \quad \text{as } T \to \infty.
\]

Therefore, for any \( f \) that factors through \( X_{i,S} \),
\[
\int_{B_T \cap G_{H_i}} f \, d\tau \sim \mu_i(f) \cdot \tau(B_T \cap G_{H_i}) \quad \text{as } T \to \infty.
\]

Let \( f \in C(X_i) \). Fix any \( \epsilon > 0 \). We can find a finite subset \( S \subset R \) and continuous functions \( f^+, f^- \) that factor through \( X_{i,S} \) such that
\[
f^- \leq f \leq f^+ \quad \text{and} \quad \|f^+ - f^-\|_\infty < \epsilon.
\]
By (4.9),
\[ \int_{B_T \cap G_{H_i}} f^\pm \, d\tau \sim \mu_i(f^\pm) \cdot \tau(B_T \cap G_{H_i}) \quad \text{as } T \to \infty. \]
Since \( \mu(f^+ - f^-) \leq \epsilon \) and \( \epsilon > 0 \) is arbitrary, it is easy to deduce that
\[ \int_{B_T \cap G_{H_i}} f \, d\tau \sim \mu_i(f) \cdot \tau(B_T \cap G_{H_i}) \quad \text{as } T \to \infty. \]
This finishes the proof. \( \square \)

**Corollary 4.10.** Let \( V_f \) be a compact open subgroup of \( G(A) \) under which \( H_i \) is bi-invariant. Then

1. \( \tau(B_T \cap G_{H_i}) \sim_{T \to \infty} \tau(V_f(B_T \cap \ker(\Lambda_{V_f}))) \);
2. for any \( f \in C(X_i) \),
\[ \int_{B_T \cap G_{H_i}} f \, d\tau \sim_{T \to \infty} \int_{B_T \cap \ker(\Lambda_{V_f})} f \, d\tau_{V_f}. \]

**Proof.** It suffices to prove the claim for the case when \( V_f \subset W_f \). Since the restriction of \( \tau \) to \( \ker(\Lambda_{V_f}) \) is equal to \( [\ker(\Lambda_{W_f}) : \ker(\Lambda_{V_f})] \cdot \tau_{V_f} \), the first claim follows from Proposition 3.11. The second claim follows from the claim (1) and Lemma 4.4. \( \square \)

**Lemma 4.11.** In the notation of Theorem 1.9, let \( L \) and \( L' \) be metrizations of the line bundle \( L \) and \( (B_T', W_f') \), and \( (B_T', W_f') \) be defined as above with respect to \( L \) and \( L' \) respectively. Then
\[ \lim_{T \to \infty} \frac{\tau(B_T \cap \ker(\Lambda_{W_f}))}{\tau(B_T' \cap \ker(\Lambda_{W_f}'))} = \frac{[G(A) : \ker(\Lambda_{W_f})] \cdot \tau_L(G(A))}{[G(A) : \ker(\Lambda_{W_f}')] \cdot \tau_{L'}(G(A))} \]
where \( \tau \) is a Haar measure on \( G(A) \).

**Proof.** Let \( V_f = W_f \cap W_f' \). By Proposition 3.11, it suffices to show that
(4.12) \[ \lim_{T \to \infty} \frac{\tau(B_T \cap \ker(\Lambda_{W_f}))}{\tau(B_T' \cap \ker(\Lambda_{W_f}'))} = \frac{\tau_L(G(A))}{\tau_{L'}(G(A))}. \]

Let \( S \) be a finite set such that \( H_{L,v} = H_{L',v} \) for every \( v \in R - S \). If we set \( H_{L,S} = \prod_{v \in S} H_{L,v} \), then it follows from (1.8) and Theorem 3.4 that
(4.13) \[ \frac{\tau_L(G(A))}{\tau_{L'}(G(A))} = \frac{\int_{G_S} H_{L,S}(g)^{-a_l} \delta_S(g) \, d\tau_S}{\int_{G_S} H_{L',S}(g)^{-a_l} \delta_S(g) \, d\tau_S}. \]

Theorem 3.9 with \( S = \emptyset \) and (4.13) imply that the both sides of (4.12) stay the same when \( H_{L,S} \) and \( H_{L',S} \) are replaced by constant multiples. Hence, we can assume that
(4.14) \[ H_{L,S}(e) = H_{L',S}(e) = 1. \]
As in the proof of Lemma 4.4, we obtain
\[
\tau(B_T \cap \ker(\Lambda^{V_j})) = \int_{g \in G_S} \tau^S(B_{TH_{\ell,S}(g)} \cap s_g \ker(\Lambda^{V_j}) \cap G^S) \, d\tau_S(g)
\sim \left( \int_{g \in G_S} H_{\ell,S}(g)^{-a_s} \delta_S(g) \, d\tau_S(g) \right) \cdot \tau^S(B_T \cap \ker(\Lambda^{V_j}) \cap G^S).
\]

Similarly,
\[
\tau(B'_T \cap \ker(\Lambda^{V_j})) \sim \left( \int_{g \in G_S} H'_{\ell,S}(g)^{-a_s} \delta_S(g) \, d\tau_S(g) \right) \cdot \tau^S(B'_T \cap \ker(\Lambda^{V_j}) \cap G^S).
\]
Since by (4.14),
\[
B_T \cap G^S = B'_T \cap G^S,
\]
this finishes the proof.

For a fixed \( f \in C(X_i) \), we define a function \( F_T \) on \( G_{H_i} \times G_{H_i} \), by
\[
F_T(g, h) = \sum_{\gamma \in G(K)} f(g^{-1} \gamma h) \cdot \chi_{B_T}(g^{-1} \gamma h)
\]
Clearly \( F_T \) is well defined as a function on \( Y \times Y \) where \( Y = G(K) \backslash G_{H_i} \).
Note that
\[
F_T(e, e) = \sum_{\gamma \in G(K) : H_i \gamma H_i \leq T} f(\gamma).
\]

**Proposition 4.15 (Weak-convergence).** Suppose that \( \iota \) is saturated. Let \( f \in C(X_i) \).

For \( i = 1, 2 \), let \( \alpha_i \in C(Y) \) be a \( W_f \)-invariant function and \( \int_Y \alpha_i \, d\tau = 1 \). If \( \alpha(x, y) := \alpha_1(x)\alpha_2(y) \), then
\[
\lim_{T \to \infty} \frac{1}{\tau(B_T \cap G_{H_i})} \int_{Y \times Y} F_T \cdot \alpha \, d(\tau \times \tau) = \lim_{T \to \infty} \frac{1}{\tau(B_T \cap G_{H_i})} \int_{B_T \cap G_{H_i}} f \, d\tau.
\]

**Proof.** Observe that
\[
\langle F_T, \alpha \rangle_{Y \times Y} = \int_{x \in Y} \int_{y \in Y} \left( \sum_{\gamma \in G(K)} f(x^{-1} \gamma y) \chi_{B_T}(x^{-1} \gamma y) \right) \alpha_1(x)\alpha_2(y) \, d\tau(y) \, d\tau(x)
\]
\[
= \int_{x \in Y} \int_{b \in G_{H_i}} f(x^{-1} h) \chi_{B_T}(x^{-1} h) \alpha_1(x)\alpha_2(h) \, d\tau(h) \, d\tau(x)
\]
\[
= \int_{g \in G_{H_i}} f(g) \chi_{B_T}(g) \left( \int_{x \in Y} \alpha_1(x)\alpha_2(xg) \, d\tau(x) \right) \, d\tau(g)
\]
\[
= \int_{g \in B_T \cap G_{H_i}} f(g) \langle \alpha_1, g \cdot \alpha_2 \rangle \, d\tau(g)
\]
Write $G = G_1 \cdots G_m$ as a product of connected absolutely simple $K$-groups. Since the height function $H_i$ is proper, we have that $g \to \infty$ strongly in $G(\mathbb{A})$ if and only if $H_i(g_i) \to \infty$ for each $i = 1, \ldots, m$ where $g = g_1 \cdots g_m$, $g_i \in G_i(\mathbb{A})$.

For $C > 0$, define
\[
B^C := \{g_1 \cdots g_m \in G(\mathbb{A}) : H_i(g_i) > C \text{ for each } i = 1, \ldots, m\}.
\]

Note that $\alpha_1 - 1, \alpha_2 - 1 \in L^2_{\alpha}(Y)$. Hence, by Theorem 4.18, for any given $\epsilon > 0$, there exists $C > 0$ such that
\[
|\langle \alpha_1, g \cdot \alpha_2 \rangle - 1| = |\langle \alpha_1 - 1, g \cdot (\alpha_2 - 1) \rangle| < \epsilon \quad \text{for all } g \in B^C.
\]

Hence
\[
\left| \int_{g \in B_T \cap G_{H_i}} f(g) \langle \alpha_1, g \cdot \alpha_2 \rangle \, d\tau(g) - \int_{g \in B_T \cap G_{H_i}} f(g) \, d\tau(g) \right| < \max f \cdot (\|\alpha_1\| \cdot \|\alpha_2\| + 1) \cdot \tau((B_T - B^C) \cap G_{H_i}) + \max f \cdot \tau(B_T \cap B^C \cap G_{H_i}) \cdot \epsilon.
\]

Using that $\iota$ is saturated, we deduce that
\[
\lim_{T \to \infty} \frac{\tau((B_T - B^C) \cap G_{H_i})}{\tau(B_T \cap G_{H_i})} = 0.
\]

Since $\epsilon > 0$ is arbitrary, by (4.16), this proves the claim. \hfill \Box

By Lemma 4.4, the following theorem implies Theorem 4.3:

**Theorem 4.18.** Suppose that $\iota$ is saturated. For any $f \in C(X_i)$, we have
\[
\lim_{T \to \infty} \frac{1}{\tau(B_T \cap G_{H_i})} \sum_{g \in G(K) : H_i(g) < T} f(g) = \lim_{T \to \infty} \frac{1}{\tau(B_T \cap G_{H_i})} \int_{B_T \cap G_{H_i}} f \, d\tau.
\]

**Proof.** Recall that $G_{H_i} = \ker(\Lambda^{W_f})$ and $\tau = \tau_{W_f}$. It suffices to prove our theorem for non-negative functions $f \in C(X_i)$. Fix $\epsilon > 0$. Let $W_\infty$ be a symmetric neighborhood of $e$ in $G_\infty^\times$ such that
\[
W_\infty B_T W_\infty \subset B_{(1+\epsilon)T} \quad \text{and} \quad B_{(1-\epsilon)T} \subset \cap_{g,h \in W_\infty} hB_T g \quad \text{for all } T > 1.
\]

By uniform continuity, there exists a cofinite subgroup $V_f$ of $W_f$ such that
\[
f(g^{-1}xh) - \epsilon \leq f(x) \leq f(g^{-1}xh) + \epsilon \quad \text{for all } x \in X \text{ and } g, h \in V_f.
\]

Replacing $W_\infty$ by a smaller one if necessary, we may assume that (4.19) holds for any $g, h \in W$ where $W := W_\infty \times V_f$. It follows that for any $T > \delta_0$ and for any $g, h \in W$,
\[
F_{(1-\epsilon)T}(g, h) \leq F_T(e, e) \leq F_{(1+\epsilon)T}(g, h)
\]
where
\[
F_T^+(g, h) = \sum_{\gamma \in G(K)} (f(g^{-1}\gamma h) \pm \epsilon) \cdot \chi_{B_T}(g^{-1}\gamma h).
\]
Set \( Y = G(K) \setminus \ker(\Lambda^T) \). Now let \( \psi \in C_c(Y) \) be a non-negative \( V_f \)-invariant function such that \( \text{supp}(\psi) \subset G(K) \setminus G(K)W \) and \( \int_Y \psi \, d\tau_V = 1 \). By integrating (4.20) over \( Y \times Y \) against the function \( \alpha(x, y) = \psi(x) \cdot \psi(y) \), we obtain
\[
\langle F_{(1-\epsilon)T}, \alpha \rangle \leq F_T(e, e) \leq \langle F_{(1+\epsilon)T}, \alpha \rangle.
\]

Note that Theorem 3.9 implies the following: there exist constants \( a_\epsilon \geq 1 \) and \( b_\epsilon \leq 1 \) tending to 1 as \( \epsilon \to 0 \) such that for all sufficiently small \( \epsilon > 0 \),
\[
(4.21) \quad b_\epsilon \leq \liminf_T \frac{\tau_V(B_{(1-\epsilon)T} \cap \ker(\Lambda^T))}{\tau_V(B_T \cap \ker(\Lambda^T))} \leq \limsup_T \frac{\tau_V(B_{(1+\epsilon)T} \cap \ker(\Lambda^T))}{\tau_V(B_T \cap \ker(\Lambda^T))} \leq a_\epsilon.
\]

Hence by applying Proposition 4.15,
\[
\limsup_T \frac{F_T(e, e)}{\tau_V(B_T \cap \ker(\Lambda^T))} \leq \limsup_T \frac{\langle F_{(1-\epsilon)T}, \alpha \rangle}{\tau_V(B_T \cap \ker(\Lambda^T))} \leq \limsup_T \frac{\langle F_{(1+\epsilon)T}, \alpha \rangle}{\tau_V(B_T \cap \ker(\Lambda^T))} \leq a_\epsilon \limsup_T \left( \frac{\int_{B_T \cap \ker(\Lambda^T)} (f + \epsilon) \, d\tau_V}{\tau_V(B_T \cap \ker(\Lambda^T))} \right) + \epsilon
\]
by Corollary 4.10, and similarly,
\[
b_\epsilon \left( \liminf_T \frac{\int_{B_T \cap \ker(\Lambda^T)} f \, d\tau_V}{\tau(B_T \cap \ker(\Lambda^T))} - \epsilon \right) \leq \liminf_T \frac{F_T(e, e)}{\tau_V(B_T \cap \ker(\Lambda^T))}.
\]
Taking \( \epsilon \to 0 \), this implies by Corollary 4.10 that
\[
\lim_T \frac{F_T(e, e)}{\tau(B_T \cap \ker(\Lambda^T))} = \lim_T \frac{\int_{B_T \cap G_H} f \, d\tau}{\tau(B_T \cap G_H)}.
\]

To derive the asymptotic formula for the number of \( K \)-rational points, it suffices to take \( f = 1 \) in Theorem 4.18. In this case the above computation simplifies significantly, and it applies to general families of balls \( B_T \), which we presently introduce.

For an increasing sequence \( \{B_T\} \) of relatively compact subsets of \( G(\mathbb{A}) \) and a compact open subgroup \( W_f \subset G(\mathbb{A}_f) \), we call \( \{B_T\} \) \( W_f \)-well rounded if the following holds:

1. \( W_f B_T W_f = B_T \) for any \( T > 1 \).
(2) for any small $\epsilon > 0$, there exists a neighborhood $W_\epsilon \subset G_\infty$ of $e$ such that

$$W_\epsilon B_T W_\epsilon \subset B_{(1+\epsilon)T} \quad \text{and} \quad B_{(1-\epsilon)T} \subset \cap_{g,h \in W_\epsilon} gB_T h$$

for all $T > 1$;

(3) $\tau_{W_f}(B_T \cap \ker(\Lambda^{W_f})) \to \infty$ as $T \to \infty$ and there exist constants $a_\epsilon \geq 1$ and $b_\epsilon \leq 1$ tending to 1 as $\epsilon \to 0$ such that for all sufficiently small $\epsilon > 0$,

$$b_\epsilon \leq \liminf_T \frac{\tau_{W_f}(B_{(1-\epsilon)T} \cap \ker(\Lambda^{W_f}))}{\tau_{W_f}(B_T \cap \ker(\Lambda^{W_f}))} \leq \limsup_T \frac{\tau_{W_f}(B_{(1+\epsilon)T} \cap \ker(\Lambda^{W_f}))}{\tau_{W_f}(B_T \cap \ker(\Lambda^{W_f}))} \leq a_\epsilon.$$

The proof of Theorem 4.18 gives

**Proposition 4.22.** Suppose that $G$ is absolutely almost simple. Let $W_f$ be a compact open subgroup of $G(\mathbb{A}_f)$. Then for any $W_f$-well rounded sequence $\{B_T\}$ of relatively compact subsets of $G(\mathbb{A})$,

$$\#G(K) \cap B_T \sim_{T \to \infty} \tau_{W_f}(B_T \cap \ker(\Lambda^{W_f})).$$

Proposition 4.22 can be used, in particular, to compute the asymptotics of integral points. Let $S$ be a finite subset of $R$ containing $R_\infty$ and

$$B_T = \{g = (g_v) \in G(\mathbb{A}) : H_s(g) < T, g_v \in G(\mathcal{O}_v) \text{ for } v \notin S\}.$$

Then

$$G(K) \cap B_T = \{g \in G(\mathcal{O}_S) : H_s(g) < T\}$$

where $\mathcal{O}_S$ denotes the ring of $S$-integers. Clearly, the sequence $\{B_T\}$ satisfies properties (1) and (2) for a suitable open subgroup $W_f$. Verification of (3) reduces to the computation of the asymptotics of the volume $\tau_{W_f}(B_T \cap \ker(\Lambda^{W_f}))$ as $T \to \infty$. We refer to [Ba] where some results on asymptotics of the number of $S$-integral points were obtained.

We also state a version of Proposition 4.22 with error term, which follows from the proof of Proposition 5.5 below. A proper function $H : G(\mathbb{A}) \to \mathbb{R}^+$ is called $W_f$-well rounded if the following properties hold:

(1) $H$ is bi-$W_f$-invariant;

(2) there exists $b > 0$ such that for any small $\epsilon > 0$,

$$(1 - b\epsilon)H(x) \leq H(gxh) \leq (1 + b\epsilon)H(x) \quad \text{for any } g, h \in W_\epsilon \text{ and } x \in G(\mathbb{A}),$$

where $W_\epsilon$ denotes the Riemannian ball at $e$ of radius $\epsilon$ in $G_\infty$;

(3) there exist $a = a(H) > 0$ and $b = b(H) \geq 1$ such that the associated zeta function

$$Z(s) := \sum_{\chi \in \Lambda^{W_f}} \int_{G(\mathbb{A})} H(g)^{-s} \chi(g) d\tau(g)$$

has a meromorphic continuation to Re$(s) > a - \epsilon$ with the unique pole at $s = a$ of order $b$ and of positive residue, and for some constants $\kappa \in \mathbb{R}$ and
\[ k > 0, \quad \left| \frac{(s - a)^b \mathcal{Z}(s)}{s^b} \right| \leq k \cdot |1 + \text{Im}(s)|^\kappa \]

for \( \text{Re}(s) > a - \epsilon \).

**Proposition 4.23.** Suppose that \( G \) is absolutely almost simple. Let \( W_f \) be a compact open subgroup of \( G(\mathbb{A}_f) \). Then for any \( W_f \)-well rounded function \( H : G(\mathbb{A}) \rightarrow \mathbb{R}^+ \),

\[ \# \{ g \in G(K) : H(g) < T \} \sim_{T \to \infty} T^{a(H)} P(\log T) + T^{a(H) - \delta} \]

where \( P(x) \) is a polynomial of degree \( b(H) - 1 \) and \( \delta > 0 \).

5. Arithmetic Fibrations

In this section we prove Theorems 1.2 and 1.9 for a general case, that is, without the saturation assumption on \( \iota \). Let \( G, \iota \) and \( H_i \) be as in the beginning of Section 4. Fix a compact open subgroup \( W_f \) of \( G(\mathbb{A}_f) \) under which \( H_i \) is bi-invariant.

Let \( M \) be the smallest connected normal \( K \)-subgroup of \( G \) whose root system contains the set

\[ \{ \alpha \in \Delta : \frac{u_\alpha + 1}{m_\alpha} = a_i \}. \]

Note that \( \iota \) is saturated if and only if \( M = G \). There exists a connected normal \( K \)-subgroup \( N \) of \( G \) so that \( G = MN \) and \( M \cap N = \{ e \} \). Let \( \pi : G \rightarrow N \) be the canonical projection. Note that any element of \( G(\mathbb{A}) \) can be uniquely written as \( g_1 g_2 \) with \( g_1 \in M(\mathbb{A}) \) and \( g_2 \in N(\mathbb{A}) \).

As is well known, the restriction \( \iota|_M \) of \( \iota \) is a direct sum of finitely many copies of an irreducible representation, say, \( \iota' \), of \( M \) with the highest weight given by the restriction of that of \( \iota \). The definition of \( M \) implies that \( \iota' \) is saturated, \( a'_\iota = a_\iota \) and \( b'_\iota = b_\iota \). For \( x \in N(K) \), the function \( g \mapsto H_i(gx) \) defines a height function on \( M(\mathbb{A}) \) as in the definition 3.2. Hence Theorem 1.2 for the saturated cases implies that for each \( x \in N(K) \),

\[ N_{\pi^{-1}(x)}(H_i, T) = \# \{ g \in M(K) : H_i(gx) < T \} \sim c_x \cdot T^{a_i(\log T)^{b_i - 1}} \]

for some \( c_x > 0 \).

Noting that \( N(H_i, T) = \sum_{x \in N(K)} N_{\pi^{-1}(x)}(H_i, T) \), we prove the following:

**Theorem 5.2.** Under the assumptions of Theorem 1.2, we have for some \( \delta > 0 \),

\[ N(H_i, T) = c(H_i) \cdot T^{a_i(\log T)^{b_i - 1}} (1 + O((\log T)^{-\delta})) \]

where \( c(H_i) := \sum_{x \in N(K)} c_x < \infty \).

The rest of this section is devoted to the proof of the above theorem.

**Lemma 5.4.** Let \( G_1 \) and \( G_2 \) be normal algebraic \( K \)-subgroups of \( G \) with \( G = G_1 G_2 \) and \( G_1 \cap G_2 = \{ e \} \). There exists \( \kappa > 1 \) such that for any \( g_1 \in G_1(\mathbb{A}) \) and \( g_2 \in G_2(\mathbb{A}) \),

\[ \kappa^{-1} \cdot H_i(g_1) H_i(g_2) \leq H_i(g_1 g_2) \leq \kappa \cdot H_i(g_1) H_i(g_2). \]
Proof. Let $\chi$ denote the highest weight of $\iota$. Then there exists a finite subset $S \subset R$ such that for any $v \in R - S$,
\[ G(K_v) = U_v A_v^+ U_v \quad \text{and} \quad H_v(\iota(g)) = \chi(a) \quad \text{for} \quad g = u_1 a u_2 \in G(K_v) \]
where $U_v$ and $A_v^+$ are defined as in Section 2. In particular, it follows that for each $v \in R - S$, and for any $g_1 \in G_1(K_v)$ and $g_2 \in G_2(K_v),$
\[ H_v(\iota(g_1 g_2)) = H_v(\iota(g_1)) H_v(\iota(g_2)). \]
On the other hand, for $v \in S$, $H_{s,v}$ is equivalent to $\chi$ in the sense that there exists $\kappa_v > 1$ such that
\[ \kappa_v^{-1} \cdot \chi(a) \leq H_{s,v}(g) \leq \kappa_v \cdot \chi(a) \quad \text{for} \quad g = u_1 a u_2 \in U_v A_v^+ \Omega_v U_v = G(K_v). \]
This implies the lemma. \hfill \qed

A key ingredient in deducing Theorem 5.2 is the following stronger version of (5.1):

**Proposition 5.5.** There exist $\beta, \delta, d > 0$ such that for each $x \in N(K)$ and for any $T \geq \beta \cdot H_1(x),$
\[ |N_{x^{-1}}(H_1, T) - c_x \cdot T^{a_1} (\log T)^{b_1 - 1}| \leq d \cdot d_x \cdot T^{a_1} (\log T)^{b_1 - 1 - \delta} \]
where $d_x = H_1(x)^{-a_1} (\log H_1(x))^{b_1 - 1}$. Moreover $c_x = O(H_1(x)^{-a_1}).$

**Proof.** Let $\delta_0 > 0$ be such that $H_1(g) \geq \delta_0$ for $g \in G(A)$ (Lemma 3.7). For each $x \in N(A)$, we define a function $H^x_1$ on $M(A)$ by
\[ H^x_1(g) := H_1(g x), \quad g \in M(A). \]
It is easy to see that $H^x_1$ is a height function as in the definition 3.2 with respect to the representation $\iota'. Set$
\[ B^x_T = \{ g \in M(A) : H^x_1(g) < T \}. \]
Since $x$ commutes with $M(A)$, the group $M_{H^x}$ is independent of $x$, and we denote it by $M_{H^x}$. Let $Y = M(K) \backslash M_{H^x}$ and $\tau$ be the invariant probability measure on $Y$. For each $x \in N(K)$, set
\[ F^x_T(g, h) := \sum_{\gamma \in M(K)} \chi_B^x(g^{-1} \gamma h), \quad g, h \in M(A). \]
We may consider $F^x_T$ as a function on $Y \times Y$. Write $M = M_1 \cdots M_r$ as a product of connected absolutely simple $K$-groups. For a collection of smooth $(W_f \cap \mathcal{M}_i(A))$-invariant functions $\psi_i \in C_c(M_i(K) \backslash M_i(A) \cap M_{H^x})$, $1 \leq i \leq r$, define $\psi \in C_c(Y)$ and $\alpha \in C_c(Y \times Y)$ by
\[ \psi(z_1, \cdots, z_r) := \prod_{i=1}^r \psi_i(z_i) \quad \text{and} \quad \alpha(y_1, y_2) := \psi(y_1) \psi(y_2). \]
Assume that \( \int \psi_i \, d\tau_i = 1 \) for each \( i \) where \( \tau_i \) is the invariant probability measure on \( M_i(K) \setminus M_i(\mathbb{A}) \cap M_H \). We claim that for a sufficiently large \( l \in \mathbb{N} \), independent of \( x \), we have for any \( x \in \mathbb{N}(K) \),

\[
\langle F_T^x, \alpha \rangle_{Y \times Y} = c_x \cdot T^a (\log T)^{b-1} + O(d_x \cdot C'_\psi \cdot T^a (\log T)^{b-1-\delta}).
\]

where \( C'_\psi = \max(1, \max_i \| D^l \psi_i \|^{2r}) \) for some large \( l \) and \( D \) is the elliptic operator defined in (2.21).

As in the proof of Proposition 4.15, we derive that

\[
\langle F_T^x, \alpha \rangle = \int_{g \in B_T^x \cap M_H} \langle \psi, g \psi \rangle \, d\tau(g)
\]

Note that

\[
| \langle \psi, g \psi \rangle - 1 | = | \prod_{i=1}^n (\psi_i, g_i \psi_i) - 1 |
\]

\[
= | \sum_{i=1}^n (\prod_{j=1}^{i-1} \psi_j, g_j \psi_j) (\psi_i, g_i \psi_i) - 1 |
\]

\[
\leq r \cdot C'_\psi \cdot \max_i | \psi_i, g_i \psi_i | - 1 |
\]

\[
= r \cdot C'_\psi \cdot \max_i | \psi_i - 1, g_i \psi_i | - 1 |
\]

where \( C'_\psi = \max(1, \max_i \| \psi_i \|^{2r-2}) \). Since \( \psi_i - 1 \in L^2(\psi_i, g_i \psi_i) \) for each \( i \), we deduce from Theorem 2.22 that

\[
(5.6)
\]

\[
| \langle F_T^x, \alpha \rangle - \tau(B_T^x \cap M_H) | \leq 2r \cdot \left( \prod_i c_{W_i \cap M_i(\mathbb{A})} \cdot C'_\psi \cdot \int_{g=g_1 \cdots g_r \in B_T^x \cap M_H} \max_i \xi_{M_i}(g_i) \, d\tau(g)
\]

where \( C'_\psi = \max(1, \max_i \| D^l \psi_i \|^{2r}) \) for some large \( l \).

Since \( \xi_{M_i} \leq \xi_{M_i}^{1/2} \), it follows from Lemma 2.6 that there exist \( m \in \mathbb{N} \) and \( C_1 > 0 \) such that for any \( 1 \leq i \leq r \),

\[
\tilde{\xi}_{M_i}(g_i) < C_1 \cdot H_i(g_i)^{-1/m} \quad \text{for any} \ g_i \in M_i(\mathbb{A}).
\]

Define a function on \( M(\mathbb{A}) \) by

\[
\tilde{H}(g_1 \cdots g_r) := \min_{i} H_i(g_i), \quad g_i \in M_i(\mathbb{A}).
\]

Let \( \kappa \) be as in Lemma 5.4 for \( G_1 = M \) and \( G_2 = \mathbb{N} \) so that \( B_T^x \subset B_{\kappa T \cdot H_i}^{-1} \). It then follows from (5.6) that

\[
(5.7) \quad | \langle F_T^x, \alpha \rangle - \tau(B_T^x \cap M_H) | < C_2 \cdot C'_\psi \cdot \int_{B_{\kappa T \cdot H_i}^{-1} \cap M_H} \tilde{H}(g)^{-1/m} \, d\tau(g)
\]

for a constant \( C_2 > 0 \) independent of \( x \).
Since $\iota'$ is saturated, for every proper normal $K$-subgroup $L$ of $M$,
\[
\tau_L(B_T \cap L_{H_i}) \ll (\log T)^{-1} \tau(B_T \cap M_{H_i})
\]
where $\tau_L$ is a Haar measure on $L(\mathbb{A})$.
For each $C > 1$, set
\[
B^C = \{ g \in M(\mathbb{A}) : \tilde{H}(g) > C \}.
\]
Note that
\[
(B_T - B^C) \cap M_{H_i} \subset \bigcup_{i=1}^{r} \Omega_i
\]
where $\Omega_i = \{ g = g_1 \cdots g_r \in M_{H_i} : H_i(g_i) \leq C, \ H_i(g) < T \}$. Now denoting by $L^{(i)}$ the subgroup of $M$ generated by $M_{1}, \ldots, M_{i-1}, M_{i+1}, \ldots, M_{r}$, let $\kappa_i > 1$ be a constant as in Lemma 5.4 for $G_1 = M_{i}$ and $G_2 = L^{(i)}$. Then for any $C \gg 1$,
\[
\tau(\Omega_i) \leq \int_{H_i(g_i) < C} \tau_{L^{(i)}}(B_{\kappa_i \delta_0} H_i \cap L^{(i)}_{H_i}) d\tau_{M_i}(g_i)
\]
\[
\ll C^{\kappa_i} (\log C)^{b_i - 1} (\log T)^{-1} \tau(B_{\kappa_i} H_i \cap M_{H_i})
\]
where $\kappa_0 = \max_i (\kappa_i \delta_0^{-1})$.
Hence for any $C \gg 1$ and $T \gg C$,
\[
\tau( (B_T - B^C) \cap M_{H_i} ) \ll C^{\kappa_i} (\log C)^{b_i - 1} (\log T)^{-1} \tau(B_{\kappa_0} H_i \cap M_{H_i})
\]
Therefore
\[
\int_{B_T \cap M_{H_i}} \tilde{H}^{-1/m} d\tau = \int_{B_T \cap B^C \cap M_{H_i}} \tilde{H}^{-1/m} d\tau + \int_{(B_T - B^C) \cap M_{H_i}} \tilde{H}^{-1/m} d\tau
\]
\[
\ll (C^{-1/m} + \delta_0^{-1/m} \cdot C^{\kappa_i} (\log C)^{b_i - 1} (\log T)^{-1}) \cdot \tau(B_{\kappa_0} H_i \cap M_{H_i})
\]
\[
\ll (\log T)^{-\delta} \cdot \tau(B_{\kappa_0} H_i \cap M_{H_i}) \quad \text{for } C = (\log T)^{1/(2\kappa_i)}
\]
for some $\delta > 0$. We now deduce from (5.7) and (5.8) that
\[
\langle F_T^+, \alpha \rangle = \tau(B_T \cap M_{H_i}) + O \left( C^{\alpha_i} (\log T)^{-\delta} \cdot \tau(B_{\kappa_0} H_i \cap M_{H_i}) \right)
\]
for some $\delta > 0$.
Let $S \subset R$ be as in the proof of Lemma 5.4. As in (4.6), we get
\[
\tau(B_T \cap M_{H_i}) = \int_{g \in B_{\kappa_0} H_i \cap M_{H_i}} \tau_S(B_{\kappa_0} H_i \cap M_{H_i} \cap M^S) \tau_S(g)
\]
By Theorem 3.9,
\[
\tau_S(B_T \cap s_g M_{H_i} \cap M^S) = c_0 \delta (s_g^{-1}) T^{a_i} (\log T)^{b_i - 1} + O(T^{a_i} (\log T)^{b_i - 2})
\]
for some $c_0 > 0$ and $T \gg 1$. Note that $\delta_S(s_g^{-1}) = \delta_S(g) = \delta_S(gx)$ and it is bounded. We deduce that when $H_i(gx) \ll T/\delta_0$, 
$$
\tau^S(B_{\kappa T H_i^{-1}(gx)} \cap M_{H_i} \cap M^S) \leq c \cdot \delta_S(g)(T \cdot H_i^{-1}(gx))^{a_i}(\log T)^{b_i-1} + O((T \cdot H_i^{-1}(gx))^{a_i}(\log H_i(gx))^{b_i-1}(\log T)^{b_i-2}).
$$

for $c = c(S, W_f, \kappa) > 0$. To estimate the integral over the domain with $H_i(gx) \gg T/\delta_0$, it suffices to note that by Lemma 3.3, 
$$
\tau_S(B_{T H_i^{-1}(x)} \cap M_S) \ll (TH_i^{-1}(x))^{a_\epsilon - \epsilon}.
$$

Since by Lemmas 3.3 and 5.4, 
$$
\int_{g \in M_S} \delta_S(g) H_i(gx)^{-a_i} (\log H_i(gx))^{b_i-1} d\tau_S(g) \ll (\log H_i(x))^{b_i-1} H_i(x)^{-a_i},
$$

it follows from the above estimates that for $T \gg H_i(x)$,
$$
\tau(B_T^{c_i} \cap M_{H_i}) = c_x T^{a_i}(\log T)^{b_i-1} + O(d_x T^{a_i}(\log T)^{b_i-2}),
$$

where

$$
(5.11) \quad c_x = c \cdot \int_{g \in M_S} \delta(gx) H_i(gx)^{-a_i} d\tau_S(g) \ll H_i(x)^{-a_i}.
$$

Hence combining (5.9) and (5.10), we have for $T \gg H_i(x)$,
$$
\langle F_T^\varphi, \alpha \rangle = c_x \cdot T^{a_i}(\log T)^{b_i-1} + O(d_x \cdot C_\varphi T^{a_i}(\log T)^{b_i-1-\delta}).
$$

Denote by $\tau_\infty$ and $\tau_f$ Haar measures on $G_\infty$ and $G(\mathbb{A}_f)$ respectively so that $\tau = \tau_\infty \times \tau_f$. Let $\phi_\epsilon$ be a smooth symmetric nonnegative function on $M_\infty$, which is a product $\prod_{i=1}^r \phi_i,\epsilon$ of smooth functions on the simple factors of $M_\infty$, $\int_{M_\infty} \phi_\epsilon d\tau_\infty = 1$ and $\text{supp}(\phi_\epsilon)$ is contained in the Riemannian ball at $e$ in $M_\infty$ of radius $\epsilon$, and for some $\rho > 0$, max$_i \|D^i \phi_i,\epsilon\|^{2r} \ll \epsilon^{-\rho}$ (see, for example, Lemma 4.4 in [GaO]). By the definition of $H_i$ in 3.2, there exists $b > 0$ such that 
$$
\text{supp}(\phi_\epsilon) \cdot B_T^\varphi \cdot \text{supp}(\phi_\epsilon) \subset B_T^{(1+2\rho)T}
$$

for every $T > 1$ and $x \in N(K)$.

Define 
$$
\psi_\epsilon(g) = \frac{1}{\tau_f(W_f)} \sum_{\gamma \in M(K)} \phi_\epsilon(\gamma g_\infty) \cdot \chi_W(\gamma g_f), \quad g = g_\infty g_f \in M_\infty M(\mathbb{A}_f).
$$

Define $\alpha_\epsilon(y_1, y_2) = \psi_\epsilon(y_1) \psi_\epsilon(y_2)$ for $(y_1, y_2) \in Y \times Y$. Then
$$
N_{\pi^{-1}(x)}(H_i, T) \leq \langle F_T^{(1+2\rho)T}, \alpha_\epsilon \rangle
$$
$$
= c_x T^{a_i}(\log T)^{b_i-1} + O(c_x \cdot \epsilon \cdot T^{a_i}(\log T)^{b_i-1} + d_x \cdot \epsilon^{-\rho} T^{a_i}(\log T)^{b_i-1-\delta}).
$$

Setting $\epsilon = (\log T)^{-\delta/(\rho+1)}$, we derive the upper estimate for $N_{\pi^{-1}(x)}(H_i, T)$. The lower estimate is proved similarly. \qed
**Proof of Theorem 5.2** According to the choice of $\mathbf{N}$, for any simple root $\alpha \in \Delta$ whose restriction to $\mathbf{N}$ is a root, we have

$$\frac{u_\alpha + 1}{m_\alpha} < a_\iota.$$  

Hence it follows from Theorem 3.9 that $N_{\mathbf{N}}(H_\iota, T) = O(T^{u_\iota - \varepsilon})$ for some $\varepsilon > 0$. Since $c_x \ll H_\iota(x)^{-a_\iota}$ by (5.11) and $d_x = H_\iota(x)^{-a_\iota}(\log H_\iota(x))^{b_\iota - 1}$, it follows that

$$C(H_\iota) := \sum_{x \in \mathbf{N}(K)} c_x < \infty \text{ and } \sum_{x \in \mathbf{N}(K)} d_x < \infty.$$  

Let $\delta_0 > 0$ be as in (3.8). Applying Lemma 5.4 for $\mathbf{M}$ and $\mathbf{N}$ with $\kappa$ therein, we have

$$\sum_{x \in \mathbf{N}(K) : H_\iota(x) > \beta^{-1}T} N_{\pi-1(x)}(H_\iota, T)$$  

$$= \# \{xy \in \mathbf{N}(K)\mathbf{M}(K) : H_\iota(x) > \beta^{-1}T, \ H_\iota(xy) < T \}$$  

$$\leq N_{\mathbf{M}}(H_\iota, \kappa \beta^{-1}) \cdot N_{\mathbf{N}}(H_\iota, \kappa T \delta_0^{-1})$$  

$$= O(T^{u_\iota - \varepsilon}).$$

Now applying Proposition 5.5 with $\beta, \delta$ therein, since $\sum_{x \in \mathbf{N}(K)} d_x < \infty$,

$$\sum_{x \in \mathbf{N}(K) : H_\iota(x) \leq \beta^{-1}T} N_{\pi-1(x)}(H_\iota, T)$$

$$= \left( \sum_{x \in \mathbf{N}(K) : H_\iota(x) \leq \beta^{-1}T} c_x \right) T^{u_\iota} (\log T)^{b_\iota - 1} + O(T^{u_\iota} (\log T)^{b_\iota - 1 - \delta}).$$

Therefore as $T \to \infty$,

$$N(H_\iota, T) = \sum_{x \in \mathbf{N}(K) : H_\iota(x) \leq \beta^{-1}T} N_{\pi-1(x)}(H_\iota, T) + O(T^{u_\iota - \varepsilon})$$

$$= \left( \sum_{x \in \mathbf{N}(K) : H_\iota(x) \leq \beta^{-1}T} c_x \right) T^{u_\iota} (\log T)^{b_\iota - 1} (1 + O((\log T)^{-\delta})).$$

Since $\sum_{x \in \mathbf{N}(K) : H_\iota(x) \leq \beta^{-1}T} c_x = C(H_\iota) + O(T^{\varepsilon})$, we have

$$N(H_\iota, T) = C(H_\iota) \cdot T^{u_\iota} (\log T)^{b_\iota - 1} (1 + O((\log T)^{-\delta}))$$

finishing the proof.
6. Examples

Let $G$ be a connected semisimple adjoint algebraic group defined over a number field $K$. For simplicity, we assume that $G$ is split over $K$. Let $\iota: G \to \text{GL}_N$ be an absolutely irreducible representation defined over $K$ with the highest weight $\lambda_\iota$. We define $a_\iota$ and $b_\iota$ as in (1.3) and set

$$\Delta_\iota = \{ \alpha \in \Delta : \frac{u_\alpha + 1}{m_\alpha} = a_\iota \}.$$

For $\alpha \in \Delta$, we denote by $\check{\alpha}$ the corresponding coroot. Given a height function $H_\iota$ as in Definition 3.2, we denote by $W_f$ the compact open subgroup of $G(\mathbb{A}_f)$ that leaves $H_\iota$ bi-invariant and by $\Lambda^{W_f}$ the finite set of $W_f$-invariant automorphic characters of $G(\mathbb{A})$. It follows from Theorem 7.1 in [STT2] that if for a finite subset $S \subset \mathbb{R}$ and an automorphic character $\chi$, $c_{S,\chi} = \lim_{s \to a_\iota^+} (s - a_\iota)^b_\iota \int_{G^S} H_\iota(g)^{-s} \chi(g) \, dg \neq 0$,

then

$$(6.1) \quad \chi(\check{\alpha}) = 1 \quad \text{for all } \alpha \in \Delta_\iota,$$

and conversely if (6.1) holds, then $c_{S,\chi} \neq 0$ for all sufficiently large $S \subset \mathbb{R}$.

We consider several examples:

(i) Suppose that $\lambda_\iota$ is a multiple of $2\rho + \sum_{\alpha \in \Delta} \alpha$. In particular, this holds for $\lambda_\iota$ corresponding to the anticanonical class and for all rank 1 groups. In this case, $\Delta_\iota = \Delta$. If a character $\chi \in \Lambda^{W_f}$ satisfies (6.1) then it follows from the Cartan decomposition (2.1) that $\chi(G^S) = 1$ for sufficiently large $S$ and by the weak approximation, $\chi = 1$. This shows that $c_{S,\chi} = 0$ for every finite $S \subset \mathbb{R}$ and every $\chi \in \Lambda^{W_f} - \{1\}$. Hence, Theorem 3.9 implies that for a Haar measure $\tau$ on $G(\mathbb{A})$,

$$\tau(B_T) \sim [G(\mathbb{A}) : G_{H,\iota}] \cdot \tau(B_T \cap G_{H,\iota}).$$

In this case, Theorem 1.15 can be stated as

$$(6.2) \quad \#G(K) \cap B_T \sim \tau(B_T)$$

where $\tau$ is normalized so that $\tau(G(K) \backslash G(\mathbb{A})) = 1$. Also, (1.6) together with (4.1) imply that the asymptotic distribution of rational points is given by

$$(6.3) \quad \mu_\iota = \prod_{v \in R} \frac{H_{i,v}(g_v)^{-a_v} \, dg_v}{H_{i,v}(g_v)^{-a_v} \, dg_v}.$$

(ii) Suppose that $K = \mathbb{Q}$ and $W_f = \prod_{v \in R_f} G(\mathbb{Z}_p)$ (with respect to the canonical model over $\mathbb{Z}$). Then according to Remark in Section 2 in [GaO],

$$G(\mathbb{A}) = G(\mathbb{Q})G(\mathbb{R})^g W_f.$$

Hence, $\Lambda^{W_f} = \{1\}$, (6.2) holds, and the measure $\mu_\iota$ is given by (6.3).
(iii) According to [PR, §8.2], there exists a lattice $L \subset K^N$ such that $G$ has class number 1 with respect to $L$, i.e.,

$$G(\mathbb{A}) = G(K)G_\infty \left( \prod_{v \in R_f} \text{Stab}_{G(K_v)}(L \otimes \mathcal{O}_v) \right).$$

We take the height function $H = \prod_{v \in R} H_v$ where $H_v$ is the maximum norm with respect to $L$ for $v \in R_f$ and $H_v$ is a norm invariant under a maximal compact subgroup $U_v$ of $G(K_v)$ for $v \in R_\infty$. Then

$$W_f = \prod_{v \in R_f} \text{Stab}_{G(K_v)}(L \otimes \mathcal{O}_v).$$

This implies that for any $\chi \in \Lambda^W$ which is not $U_v$-invariant for some $v \in R_\infty$ and for any finite $S \subset R_f$, we have $c_S \chi = 0$. On the other hand, using that $G(K_v) = U_v G(K_v)^\circ$ for $v \in R_\infty$ (see Ch. III, [PR]), we deduce from (6.4) that if $\chi \in \Lambda^W$ is $U_v$-invariant for all $v \in R_\infty$, then $\chi = 1$. Hence, $c_S \chi = 0$ for all finite $S \subset R_f$ and $\chi \in \Lambda^W - \{1\}$. This implies that (6.2) holds and

$$\mu_{t,R_f} = \prod_{v \in R_f} \frac{H_{t,v}(g_v)^{-a_v} dg_v}{\int_{G(K_v)} H_{t,v}(g_v)^{-a_v} dg_v},$$

(iv) (cf. Example 8.8, [STT2]) Let $G = \text{PGL}_4$ and $t$ be the adjoint representation. By [PR, §8.2], there exists a lattice $L \subset K^{15}$ such that $G$ has class number 2 with respect to $L$. We take the height function $H = \prod_{v \in R} H_v$ where $H_v$ is the maximum norm with respect to $L$ for $v \in R_f$. The group $W_f$ is given by (6.5). By [PR, §8.2], $G(K)G_\infty W_f$ is a normal subgroup of index 2 in $G(\mathbb{A})$. If we additionally assume that the number field $K$ is totally complex, then $G_\infty$ is connected and, hence, $\Lambda^W = \{1, \chi\}$ for some automorphic character $\chi$ of order 2. Every automorphic character of $G(\mathbb{A})$ is of the form $\eta \circ \det$ where $\eta$ is a Hecke character such that $\eta^2 = 1$. Since the map $\det : \text{PGL}_4(K_v) \rightarrow K_v^*/(K_v^*)^4$ is surjective for every $v \in R$, it follows that $\chi = \eta \circ \det$ with $\eta^2 = 1$. In this case, the roots and coroots are given by

$$\alpha_i(\text{diag}(a_1, \ldots, a_4)) = a_i a_{i+1}^{-1}, \quad \check{\alpha}_i(t) = \text{diag}(t, \ldots, t, 1, \ldots, 1)$$

for $i = 1, 2, 3$, and

$$\lambda_i = \alpha_1 + \alpha_2 + \alpha_3, \quad 2\rho = 3\alpha_1 + 4\alpha_2 + 3\alpha_3.$$ 

Hence, $a_i = 5, b_i = 1, \Delta_i = \{\alpha_2\}$. Then (6.1) is equivalent to $\eta^2 = 1$, and we deduce that $c_S \chi \neq 0$ for sufficiently large finite $S \subset R$. Since the function $\delta_S = c_{S,1} + c_{S,\chi} \chi$ restricted to $G_S$ is not constant for sufficiently large $S \subset R$,
we conclude that the weights $c_u$ defined in (4.1) are not constant and

$$
\mu_{\iota,S} \neq \prod_{v \in S} \int_{G(K_v)} H_{\iota,v}(g_v)^{-a_\iota} \, dg_v
$$

in Theorem 1.5.

We also note that in this case, Theorem 3.9 implies that for a Haar measure $\tau$ on $G(A)$ and an automorphic character $\chi$ such that $c_{\emptyset,1} \neq 0$, we have

$$
\lim_{T \to \infty} \tau(B_T \cap \ker(\chi)) = c_{\emptyset,1} \cdot \frac{1}{2} (c_{\emptyset,1} + c_{\emptyset,\chi}) \neq \frac{1}{2}.
$$

In particular, it might happen that in Theorem 1.15, $\tau(B_T)$ is not asymptotic to $[G(\mathbb{A}) : G_{H_1}] \cdot \tau(B_T \cap G_{H_1})$ as $T \to \infty$.

References


RATIONAL POINTS OF BOUNDED HEIGHT


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