MANIN'S CONJECTURE ON RATIONAL POINTS OF BOUNDED HEIGHT AND ADELIC MIXING

ALEX GORODNIK, FRANÇOIS MAUCOURANT, HEE OH

Dedicated to Prof. Gregory Margulis on the occasion of his sixtieth birthday

Abstract. Let K be a number field. We compute the asymptotics of the number of K-rational points of bounded height on a connected adjoint semisimple K-group $\mathbf{G}$ for any given irreducible representation. This proves Manin's conjecture for the wonderful compactification X of $\mathbf{G}$. We also determine the explicit asymptotic distribution of the rational points $\mathbf{G}(K)$ on $X(\mathbb{A})$, which verifies the prediction made by Peyre. Our approach is based on the mixing property of $L^2(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$ which we prove with a rate of convergence.

Soit K un corps de nombres. Nous déterminons le comportement asymptotique du nombre de K-points de hauteur bornée d'une représentation irréductible arbitraire d'un K-groupe $\mathbf{G}$ semisimple, adjoint et connexe. Ceci résout la conjecture de Manin dans le cas de la compactification merveilleuse X de $\mathbf{G}$. Nous calculons également la distribution asymptotique explicite des points $\mathbf{G}(K)$ sur $X(\mathbb{A})$, qui vérifie les prédictions de Peyre. Ce travail repose sur la propriété de mélange de $L^2(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$, qui est démontrée avec une estimée de vitesse.

Contents

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1. Introduction

Let K be a number field and R the set of all (inequivalent) normalized absolute values $x \mapsto |x|_v$ of K. We denote by K_v the completion of K with respect to v. The height $H(x)$ of a point $x = (x_0 : x_1 : \cdots : x_n) \in \mathbb{P}^n(K)$ is given by

$$H(x) := \prod_{v \in R} H_v(x)$$

where $H_v(x) = \max_i |x_i|_v$ for each $v \in R$. By the product formula, H is a well defined function on $\mathbb{P}^n(K)$. For example, when $K = \mathbb{Q}$, we have

$$H(x) = \max_i |x_i|$$

where $(x_0, \cdots, x_n)$ is a primitive integral vector.

More generally one can consider a height function which differs from above by changing the local height H_v by another norm on K_v^{n+1} for finitely many places v (see Definition 3.2).

It is easy to see that for any $T > 0$,

$$N(T) := \#\{x \in \mathbb{P}^n(K) : H(x) < T\} < \infty.$$

Schanuel [Sc] computed in 1964 that

$$N(T) \sim c \cdot T^{n+1} \quad \text{as } T \to \infty$$

for some explicit constant $c = c(H) > 0$.

A fundamental problem in modern algebro-arithmetic geometry is to describe the set of rational points of projective varieties in terms of their geometric invariants. One of the main conjectures in this area was made by Manin in [BM] in the late eighties. It describes the asymptotics of the numbers of rational points on projective varieties with ample anti-canonical classes (such varieties are called Fano varieties). Manin's conjecture has been proved for flag varieties ([FMT], [Pe1]), toric varieties [BT1-2], horospherical varieties [ST], equivariant compactifications of unipotent groups (see [CT2], [ST1], [ST2]), etc. In this paper, we settle Manin's conjecture for the wonderful compactification of a general connected semisimple adjoint group $\mathbf{G}$ defined over a number field.

1.1. Asymptotics of rational points.

Let $\mathbf{G}$ be a connected adjoint semisimple group over K. Let $\iota : \mathbf{G} \to \mathrm{GL}_N$ be a faithful representation of $\mathbf{G}$ defined over K. This defines a projective embedding over K:

$$\bar{\iota} : \mathbf{G} \to \mathbb{P}(\mathrm{M}_N) = \mathbb{P}^{N^2-1}$$

where M_N denotes the space of matrices of order N. Fixing a height function $H = \prod_{v \in R} H_v$ on $\mathbb{P}^{N^2-1}(K)$, we obtain a height function H_ι on $\mathbf{G}(K)$ via $\bar{\iota}$:

$$H_\iota(g) := H(\bar{\iota}(g)) = \prod_{v \in R} H_v(\iota(g)). \tag{1.1}$$

Set

$$N(H_\iota, T) := \#\{g \in \mathbf{G}(K) \ : \ H_\iota(g) \ < \ T\}.$$

Theorem 1.2. *Let* $\mathbf{G}$ *be a product of connected adjoint absolutely simple groups over* K *and* $\iota : \mathbf{G} \to \mathrm{GL}_N$ *a faithful absolutely irreducible representation defined over* K*. Then there exist* $a_\iota \in \mathbb{Q}^+$, $b_\iota \in \mathbb{N}$ *and* $c = c(H_\iota) > 0$ *such that for some* $\delta > 0$,

$$N(H_\iota, T) = c \cdot T^{a_\iota} (\log T)^{b_\iota - 1} \cdot (1 + O((\log T)^{-\delta}))$$

for all sufficiently large T*.*

The proof of Theorem 1.2 is based on the uniform bounds on matrix coefficients. Recently, we developed a different approach based on Ratner's theory of unipotent flows [GoO], which also applies to some homogeneous varieties of $\mathbf{G}$.

Remark

(1) When $\mathbf{G}$ is absolutely simple or, more generally, when H_ι is the product of height functions of the simple factors of $\mathbf{G}$, we can improve the rate of convergence in Theorem 1.2: for some $\delta > 0$,

$$N(H_\iota, T) = c \cdot T^{a_\iota} P(\log T) \cdot (1 + O(T^{-\delta}))$$

where $P(x)$ is a monic polynomial of degree $b_\iota - 1$.

(2) We note that for any connected semisimple adjoint algebraic group $\mathbf{G}$ defined over K, there exists a finite field extension, say, F, of K such that $\mathbf{G}$ is a direct product of connected adjoint absolutely simple groups defined over F.

The constants a_ι and b_ι can be defined explicitly by combinatorial data coming from the root system of $\mathbf{G}$ and the highest weight of ι. Choose a set Δ of simple roots in the root system $\Phi(\mathbf{G}, \mathbf{T})$ of $\mathbf{G}$ with respect to a maximal torus $\mathbf{T}$ defined over K containing a maximal K-split torus. Denote by 2ρ the sum of all positive roots in $\Phi(\mathbf{G}, \mathbf{T})$, and by χ the highest weight of ι. Define $u_\alpha, m_\alpha \in \mathbb{N}$, $\alpha \in \Delta$, by

$$2\rho = \sum_{\alpha \in \Delta} u_\alpha \alpha \quad \text{and} \quad \chi = \sum_{\alpha \in \Delta} m_\alpha \alpha.$$

The fact that $m_\alpha \in \mathbb{N}$ follows since $\mathbf{G}$ is of adjoint type. Consider the twisted action of the Galois group $\Gamma_K := \mathrm{Gal}(\bar{K}/K)$ on Δ. For instance, if the K-form of $\mathbf{G}$ is inner, this action is just trivial. Then

$$a_\iota = \max_{\alpha \in \Delta} \frac{u_\alpha + 1}{m_\alpha} \quad \text{and} \quad b_\iota = \#\{\Gamma_K.\alpha : \frac{u_\alpha + 1}{m_\alpha} = a_\iota\}. \tag{1.3}$$

Note that the exponent a_ι is independent of the field K, and b_ι depends only on the quasi-split K-form of $\mathbf{G}$. Therefore, by passing to a finite field extension containing the splitting field of $\mathbf{G}$, b_ι also becomes independent of K.

1.2. **Distribution of rational points.** Refining Manin's conjecture mentioned above, Peyre made a conjecture on the asymptotic distribution of rational points of Fano varieties [Pe1]. We verify Peyre's conjecture for "saturated" projective embeddings of semisimple adjoint groups. We call a representation $\iota : \mathbf{G} \to \mathrm{GL}_N$ *saturated* if the set

$$\{\alpha \in \Delta : \frac{u_\alpha + 1}{m_\alpha} = a_\iota\} \tag{1.4}$$

is not contained in the root system of a proper normal subgroup of $\mathbf{G}$. In particular, if $\mathbf{G}$ is absolutely simple, any representation of $\mathbf{G}$ is saturated.

For a finite subset S of R, we define

$$\mathbf{G}_S := \prod_{v\in S} \mathbf{G}(K_v) \quad \text{and} \quad H_{\iota,S} := \prod_{v \in S} H_v \circ \iota.$$

Let $X_{\iota,S} \subset \prod_{v\in S} \mathbb{P}^{N^2-1}(K_v)$ be the closure of the image of $\mathbf{G}_S$ under the diagonal embedding $(g_v) \mapsto (\bar\iota(g_v))$. We identify $\mathbf{G}_S$ with its image. It follows from the weak approximation property that $\mathbf{G}(K)$ is dense in $X_{\iota,S}$. We compute the asymptotic distribution of $\mathbf{G}(K)$ ordered by the height H_ι.

We define a probability measure $\tilde\mu_{\iota,S}$ on $\mathbf{G}_S$ by

$$\tilde\mu_{\iota,S} := \frac{H_{\iota,S}(g)^{-a_\iota} dg}{\int_{\mathbf{G}_S} H_{\iota,S}(g)^{-a_\iota} dg},$$

where the integral is with respect to a Haar measure on $\mathbf{G}_S$. This measure is well defined by Lemma 3.3. Let $\mathbf{G}'_S$ denote the derived subgroup of $\mathbf{G}_S$. Recall that $[\mathbf{G}_S : \mathbf{G}'_S] < \infty$.

Theorem 1.5. *Let $\mathbf{G}$ be a product of connected adjoint absolutely simple groups over K, $\iota : \mathbf{G} \to \mathrm{GL}_N$ a faithful absolutely irreducible saturated representation defined over K, and S a finite subset of R. Then for any $f \in C(X_{\iota,S})$,*

$$\lim_{T\to\infty} \frac{1}{N(H_\iota, T)} \sum_{g \in \mathbf{G}(K): H_\iota(g) < T} f(g) = \int_{\mathbf{G}_S} f \, d\mu_{\iota,S}$$

where the probability measure $\mu_{\iota,S}$ is given by

$$\mu_{\iota,S} = \sum_{u \in \mathbf{G}_S/\mathbf{G}'_S} c_u \cdot \tilde\mu_{\iota,S}|_{u\mathbf{G}'_S} \tag{1.6}$$

for some explicit positive weights c_u (see (4.1)).

Remark

(1) If ι is not saturated, Theorem 1.2 implies that a positive proportion of $\mathbf{G}(K)$ lies on a proper subgroup of $\mathbf{G}$. Hence one cannot expect Theorem 1.7 to hold for non-saturated cases.

(2) The limiting distribution $\mu_{\iota,S}$ is not $\mathbf{G}_S$-invariant, unless $\mathbf{G}_S$ is compact and the height $H_{\iota,S}$ is $\mathbf{G}_S$-invariant.
(3) We give examples with $\mu_{\iota,S} \neq \tilde{\mu}_{\iota,S}$ in Section 6.
(4) The space $X_{\iota,S}$ is a compactification of $\mathbf{G}_S$ which is an analog of the Satake compactification defined for real groups (see, for example, [BJ]). Theorem 1.5 implies that the rational points $\mathbf{G}(K)$ do not escape to the boundary $X_{\iota,S} - \mathbf{G}_S$. It is interesting to compare this result with the distribution of the integral points $\mathbf{G}(\mathbb{Z})$ of bounded height in the Satake compactification of $\mathbf{G}(\mathbb{R})$ where the limiting distribution is supported on the boundary (see [Mau] or [GOS] for more details).

One can check (see Lemma 4.2) that for finite $S \subset T \subset R$, the image of the measure $\mu_{\iota,T}$ under the projection map $\mathbf{G}_T \to \mathbf{G}_S$ is equal to the measure $\mu_{\iota,S}$. This implies that the family of measures $\{\mu_{\iota,S}\}$ defines a probability measure μ_ι on the space $X_\iota := \prod_{v\in R} X_{\iota,v}$. We have a global version of Theorem 1.5:

Theorem 1.7. *Let $\mathbf{G}$ and ι be as in Theorem 1.5. Then for any $f \in C(X_\iota)$,*

$$\lim_{T\to\infty} \frac{1}{N(H_\iota, T)} \sum_{g\in\mathbf{G}(K):H_\iota(g)<T} f(g) = \int_{X_\iota} f\, d\mu_\iota.$$

1.3. **Manin's and Peyre's conjectures.** We also state versions of Theorems 1.2 and 1.7 in terms of arithmetic geometry. Let X be a smooth projective variety defined over K. For every line bundle on X over K, there exists an associated height function on $X(K)$ via Weil's height machine ([Si, Theorem B. 3.2]). For example, if L is a very ample line bundle of X with a K-embedding $\psi_L : X \to \mathbb{P}^N$, then a height function $H_\mathcal{L}$ on $X(K)$ is defined as

$$H_\mathcal{L} := H \circ \psi_L$$

where H is a height function on $\mathbb{P}^N(K)$ defined as before. For any ample line bundle L of X, mL is very ample for some $m \in \mathbb{N}$. Then a height function $H_\mathcal{L}$ is of the form $H_{\mathcal{L}'}^{1/m}$ where $H_{\mathcal{L}'}$ is a height function associated to $L' := mL$.

We call a pair $\mathcal{L} = (L, H_\mathcal{L})$ a metrized line bundle. Due to the freedom of choosing a height function H on $\mathbb{P}^N(K)$, $H_\mathcal{L}$ is not uniquely determined and this is why we use the subscript $\mathcal{L}$ rather than L.

Let X denote the projective K-variety, which is the wonderful compactification of $\mathbf{G}$ constructed by De Concini and Procesi [DP] and by De Concini and Springer [DS]. For instance, X can be taken to be the Zariski closure of the image of $\mathbf{G}$ in $\mathbb{P}(\mathrm{M}_N)$ under an irreducible faithful representation $\mathbf{G} \to \mathrm{GL}_N$ whose highest weight is regular. A dominant weight χ is called regular if $\chi = \sum_{\alpha\in\Delta} m_\alpha \omega_\alpha$ with all $m_\alpha > 0$ where $\{\omega_\alpha : \alpha \in \Delta\}$ is the set of fundamental weights.

The Picard group $\mathrm{Pic}(X)_{\bar{K}}$ is isomorphic to the weight lattice of $\mathbf{G}$. Under this isomorphism, the simple roots α correspond to the boundary divisors D_α such that $X - \mathbf{G} = \cup_\alpha D_\alpha$, and the Galois action on $\mathrm{Pic}(X)_{\bar{K}}$ corresponds to the twisted Galois action on the weight lattice. Hence, the Picard group $\mathrm{Pic}(X)$ is freely generated by the line bundles corresponding to the orbits of the fundamental weights under the twisted Galois action. The closed cone $\Lambda_{\text{eff}}(X)$ of the effective line bundles is the positive cone spanned by $D_{\Gamma_K.\alpha}$, $\alpha \in \Delta$, i.e.,

$$\Lambda_{\text{eff}}(X) = \oplus\, \mathbb{R}_{\geq 0}\, [D_{\Gamma_K.\alpha}]$$

where the sum is taken over the Γ_K-orbits $\Gamma_K.\alpha$ in the set $\{\alpha \in \Delta\}$ of simple roots and $D_{\Gamma_K.\alpha} := \sum_{\beta\in\Gamma_K.\alpha} D_\beta$, and the anticanonical class $[-K_X]$ corresponds to $2\rho + \sum_{\alpha\in\Delta}\alpha$. Moreover any ample line bundle class $[L]$ of X over K corresponds to a regular dominant weight in such a way that if $[L'] := m[L]$ corresponds to $\chi \in X^*(\mathbf{T})$ for $m \in \mathbb{N}$, the restriction of $H_{\mathcal{L}'}$ to $\mathbf{G}(K)$ coincides with a height function H_ι with respect to the irreducible representation ι defined over K with the highest weight χ (cf. [STT2, Proposition 6.3]). We call an ample line bundle L *saturated* if the representation defined by the corresponding dominant weight is saturated. We refer to [BK, Ch 6] for a more detailed account on the wonderful compactification.

The notion of a saturated line bundle is related to the notion of a strongly saturated metrized line bundle $\mathcal{L}$ which was introduced by Batyrev and Tschinkel in [BT3] in order to state a refined version of Manin's conjecture. A metrized line bundle $\mathcal{L}$ on X is called *strongly saturated* if for any Zariski open dense subset U of X, one has

$$\lim_{T\to\infty} \frac{\#\{x \in U(K)\ :\ H_{\mathcal{L}}(x)\ <\ T\}}{\#\{x \in X(K)\ :\ H_{\mathcal{L}}(x)\ <\ T\}} = 1.$$

If $\mathcal{L}$ is strongly saturated, then L is saturated. It is also clear from the definition (1.4) that the anticanonical line bundle $-K_X$ is always saturated.

Consider the compact space $X(\mathbb{A}) = \prod_{v\in R} X(K_v)$. Peyre [Pe1] defined the Tamagawa measure $\tau_{-\mathcal{K}_X}$ on $X(\mathbb{A})$ associated with the anti-canonical metrized line bundle $-\mathcal{K}_X = (-K_X, H_{-\mathcal{K}_X})$:

$$\tau_{-\mathcal{K}_X} := d_K^{-\dim(X)/2} \cdot \lim_{s\to 1^+} (s-1)^{\operatorname{rank}(\operatorname{Pic}(X))} \left(\prod_{v\in R-S} L_v(s, \operatorname{Pic}(X)) \right) \cdot \prod_{v\in R} \lambda_v^{-1} \cdot H_{-\mathcal{K}_X, v}(g_v)^{-1} dg_v$$

where $S \subset R$ is a finite subset of places with bad reduction, $\lambda_v = L_v(1, \operatorname{Pic}(X))$ for all $v \in R - S$ and 1 otherwise, d_K is the discriminant of K and dg_v is a Haar measure on $\mathbf{G}(K_v)$. To define a measure $\tau_{\mathcal{L}}$ for any ample metrized line bundle $\mathcal{L} = (L, H_{\mathcal{L}})$, we set

$a_L := \inf\{a \in \mathbb{Q}^+ : a[L] + [K_X] \in \Lambda_{\text{eff}}(X)\}$ — the Nevanlinna invariant of L,

$b_L :=$ the codimension of the face of $\Lambda_{\text{eff}}(X)$ containing $a_L[L] + [K_X]$ in its interior.

The measure $\tau_{\mathcal{L}}$ on $\mathbf{G}(\mathbb{A}) \subset X(\mathbb{A})$ is defined by

$$\tau_{\mathcal{L}} := d_K^{-\dim(X)/2} \cdot \lim_{s\to a_L^+} (s - a_L)^{b_L} \sum_{\chi} H_{\mathcal{L}}(g)^{-s} \chi(g) dg, \tag{1.8}$$

where dg is a Haar measure on $\mathbf{G}(\mathbb{A})$, and the sum is taken over characters from $L^2(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$ such that $\int_{\mathbf{G}(\mathbb{A})} H_{\mathcal{L}}(g)^{-s}\chi(g)dg \neq 0$. Since $H_{\mathcal{L}}$ is invariant under a compact open subgroup of finite adeles in $\mathbf{G}(\mathbb{A})$, the sum in (1.8) contains only finitely many terms. The limit in (1.8) exists by Theorem 3.4. Note that $a_{-K_X} = 1$, $b_{-K_X} = \mathrm{rank}(\mathrm{Pic}(X))$, and

$$\lim_{s\to 1^+} (s-1)^{\mathrm{rank}(\mathrm{Pic}(X))} \int_{\mathbf{G}(\mathbb{A})} H_{-\mathcal{K}_X}(g)^{-s}\chi(g)dg = 0$$

for all $\chi \neq 1$ (see Section 6). Hence, for $\mathcal{L} = -\mathcal{K}_X$, (1.8) gives Peyre's measure $\tau_{-\mathcal{K}_X}$. An analog of Peyre's measure for general line bundles was also introduced in [BT3], but the measure $\tau_{\mathcal{L}}$ seems to be different, in general, from the measure defined in [BT3].

The following theorem settles Manin's conjecture (and its refinements due to Peyre) for the wonderful compactifications X of semisimple adjoint groups. For a metrized ample line bundle $\mathcal{L} = (L, H_{\mathcal{L}})$ on X and a subset U of X, set

$$N_U(\mathcal{L}, T) := \#\{g \in U(K) \ : \ H_{\mathcal{L}}(g) \ < \ T\}.$$

Theorem 1.9. *Let X be the wonderful compactification of a group $\mathbf{G}$ which is a product of connected adjoint absolutely simple groups defined over K, and $\mathcal{L} = (L, H_{\mathcal{L}})$ a metrized ample line bundle on X. Then for some $\delta > 0$,*

$$N_{\mathbf{G}}(\mathcal{L}, T) = c_{\mathcal{L}} \cdot T^{a_L} (\log T)^{b_L - 1} (1 + O((\log T)^{-\delta}))$$

for all sufficiently large T, where $c_{\mathcal{L}} > 0$ and if L is saturated, $c_{\mathcal{L}} = c_L \cdot \tau_{\mathcal{L}}(\mathbf{G}(\mathbb{A})) > 0$. In particular, for a metrization $-\mathcal{K}_X = (-K_X, H_{-\mathcal{K}_X})$ of the anti-canonical line bundle, we have

$$N_{\mathbf{G}}(-\mathcal{K}_X, T) = c_{-K_X} \cdot \tau_{-\mathcal{K}_X}(\mathbf{G}(\mathbb{A})) \cdot T (\log T)^{\mathrm{rank}(\mathrm{Pic}(X)) - 1} (1 + O((\log T)^{-\delta})).$$

For a non-saturated line bundle L, the equality $c_{\mathcal{L}} = c_L \cdot \tau_{\mathcal{L}}(\mathbf{G}(\mathbb{A}))$ fails in general, but the variety X has an *asymptotic arithmetic $\mathcal{L}$-fibration* in the sense of [BT3]. More precisely, there exist a connected semisimple K-group $\mathbf{H}$ and a surjective K-homomorphism $\pi : \mathbf{G} \to \mathbf{H}$ such that for each $x \in \mathbf{H}(K)$,

$$N_{\pi^{-1}(x)}(\mathcal{L}, T) \sim c_L \cdot \tau_{\mathcal{L}}(\pi^{-1}(x)(\mathbb{A})) \cdot T^{a_L} (\log T)^{b_L - 1}; \quad \text{and}$$

$$N_{\mathbf{G}}(\mathcal{L}, T) \sim c_L \cdot \sum_{x \in \mathbf{H}(K)} \tau_{\mathcal{L}}(\pi^{-1}(x)(\mathbb{A})) \cdot T^{a_L} (\log T)^{b_L - 1} \quad \text{as } T \to \infty$$

with $\sum_{x \in \mathbf{H}(K)} \tau_{\mathcal{L}}(\pi^{-1}(x)(\mathbb{A})) < \infty$.

Theorem 1.9 is independently proved by Shalika, Takloo-Bighash and Tschinkel in their recent preprint [STT2] (see page 12 for comparison of our methods).

We also state the analogue of Theorem 1.7 in this setup.

Theorem 1.10. *With the same notation as Theorem 1.9, suppose also that L is saturated. Then for any $f \in C(X(\mathbb{A}))$,*

$$\lim_{T\to\infty} \frac{1}{N_{\mathbf{G}}(\mathcal{L}, T)} \sum_{g \in \mathbf{G}(K) : H_{\mathcal{L}}(g) < T} f(g) = \frac{1}{\tau_{\mathcal{L}}(X(\mathbb{A}))} \int_{X(\mathbb{A})} f \, d\tau_{\mathcal{L}}.$$

1.4. **Counting and volume heuristic.** To explain our strategy in counting K-rational points of $\mathbf{G}$, we first recall the analogous results in counting lattice points in a simple real Lie group. Let $G \subset \mathrm{GL}_N$ be a connected non-compact simple real Lie group and Γ be a lattice in G, i.e., a discrete subgroup of finite co-volume. Fixing a norm $\|\cdot\|$ on $\mathrm{M}_N(\mathbb{R})$, set $B_T := \{g \in G : \|g\| \le T\}$. By Duke-Rudnick-Sarnak [DRS] and Eskin-McMullen [EM], it is well known that

$$\#\Gamma \cap B_T \sim \int_{B_T} dg \quad \text{as } T \to \infty, \tag{1.11}$$

where dg is the Haar measure on G such that $\int_{\Gamma\backslash G} dg = 1$.

Coming back to the question of counting rational points $\mathbf{G}(K)$, we note that $\mathbf{G}(K)$ can indeed be embedded as a lattice in the adele group $\mathbf{G}(\mathbb{A})$ under the diagonal map. Recall that the adele group $\mathbf{G}(\mathbb{A})$ is the restricted topological product of $\mathbf{G}(K_v)$'s with respect to a family of compact open subgroups, say U_v, of $\mathbf{G}(K_v)$, $v \in R_f$, where R_f is the set of all non-archimedean absolute values on K (cf. [PR], [We]). Choosing a smooth model of $\mathbf{G}$ over $\mathcal{O}[k^{-1}]$ for the ring $\mathcal{O}$ of integers of K and for some $k \in \mathbb{Z}$, we have $U_v = \mathbf{G}(\mathcal{O}_v)$ for almost all $v \in R_f$ where $\mathcal{O}_v$ is the valuation ring of K_v. Moreover the height function $H_\iota = \prod_{v\in R} H_v \circ \iota$ on $\mathbf{G}(K)$ defined in (1.1) extends to $\mathbf{G}(\mathbb{A})$ by

$$H_\iota(g) := \prod_{v\in R} H_v(\iota(g_v)) \quad \text{for any } g = (g_v)_v \in \mathbf{G}(\mathbb{A}). \tag{1.12}$$

Since $U_v = \mathbf{G}(\mathcal{O}_v)$ for almost all $v \in R_f$, $H_v(\iota(g_v)) = 1$ for almost all v, and hence H_ι is well defined.

Setting

$$B_T := \{g \in \mathbf{G}(\mathbb{A}) : H_\iota(g) < T\},$$

note that B_T is a relatively compact subset of $\mathbf{G}(\mathbb{A})$ (Lemma 3.7) and that

$$N(H_\iota, T) = \#\mathbf{G}(K) \cap B_T.$$

In view of (1.11), one naturally asks whether the following holds:

$$\#\mathbf{G}(K) \cap B_T \sim \tau(B_T) \quad \text{as } T \to \infty, \tag{1.13}$$

where τ is the Haar measure on $\mathbf{G}(\mathbb{A})$ such that $\tau(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A})) = 1$.

It turns out that the group $\mathbf{G}(\mathbb{A})$ is too big for (1.13) to hold in general, due to the presence of non-trivial characters in $L^2(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$. To define a right group for (1.13), set

$$W_f := \{w \in \mathbf{G}(\mathbb{A}_f) : H_\iota(wg) = H_\iota(gw) = H_\iota(g) \quad \text{for all } g \in \mathbf{G}(\mathbb{A})\},$$

where $\mathbf{G}(\mathbb{A}_f)$ is the subgroup of finite adeles. It easily follows from the definition of H_ι that W_f is a compact open subgroup of $\mathbf{G}(\mathbb{A}_f)$. Denoting by Λ^{W_f} the set of all W_f-invariant automorphic characters appearing in $L^2(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$, set

$$\mathbf{G}_{H_\iota} := \ker(\Lambda^{W_f}) = \cap\{\ker\chi \subset \mathbf{G}(\mathbb{A}) : \chi \in \Lambda^{W_f}\}. \tag{1.14}$$

The subgroup $\mathbf{G}_{H_\iota}$ is a finite index normal subgroup of $\mathbf{G}(\mathbb{A})$ which clearly contains $\mathbf{G}(K)$ (see Lemma 3.1).

Theorem 1.15. *Let* $\mathbf{G}$ *be a product of connected adjoint absolutely simple group defined over* K *and* $\iota : \mathbf{G} \to \mathrm{GL}_N$ *be a faithful absolutely irreducible saturated representation defined over* K*. Then*

$$\#\mathbf{G}(K) \cap B_T \sim \tau(B_T \cap \mathbf{G}_{H_\iota}) \quad \text{as } T \to \infty,$$

where τ *is the Haar measure on* $\mathbf{G}_{H_\iota}$ *normalized so that* $\tau(\mathbf{G}(K)\backslash\mathbf{G}_{H_\iota}) = 1$*.*

We remark that one cannot in general replace $\mathbf{G}_{H_\iota}$ by $\mathbf{G}(\mathbb{A})$ (see Section 6).

As in the proof of Eskin-McMullen of (1.11), our key ingredient in proving Theorem 1.15 is the mixing theorem on $L^2(\mathbf{G}(K)\backslash\mathbf{G}_{H_\iota})$.

1.5. **Adelic mixing.** Let $L^2_{00}(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$ denote the orthogonal complement to the direct sum of all one-dimensional representations occurring in $L^2(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$. In the case when $\mathbf{G}$ is simply connected, $L^2_{00}(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$ coincides with the orthogonal complement $L^2_0(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$ to the constant functions. The terminology that a sequence g_i tends to infinity as $i \to \infty$ in $\mathbf{G}(\mathbb{A})$ means that for any compact subset Ω in $\mathbf{G}(\mathbb{A})$, $g_i \notin \Omega$ for all sufficiently large i.

Theorem 1.16 (Adelic Mixing)**.** *Let* $\mathbf{G}$ *be a connected absolutely almost simple* K*-group. Then for any* $f, h \in L^2_{00}(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$*,*

$$\langle f, g.h\rangle \to 0$$

as $g \in \mathbf{G}(\mathbb{A})$ *tends to infinity.*

In particular, if f and h are W_f-invariant functions of $L^2(\mathbf{G}(K)\backslash\mathbf{G}_{H_\iota})$, then as $g \to \infty$,

$$\langle f, g.h\rangle \to \int f\, d\tau \cdot \int h\, d\tau$$

where τ is the normalized Haar measure on $\mathbf{G}(K)\backslash\mathbf{G}_{H_\iota}$.

In fact we prove the above theorem 1.16 in a much stronger form by giving a rate of convergence (see Theorem 2.8 and 2.17). In particular we obtain the following result which is of independent interest. Set $\mathbf{G}_\infty := \prod_{v\in R_\infty} \mathbf{G}(K_v)$ where R_∞ is the subset of R of all archimedean valuations.

Theorem 1.17 (Automorphic bound for $\mathbf{G}$)**.** *Let* $\mathbf{G}$ *be connected absolutely almost simple* K*-group. Let* U_∞ *be a maximal compact subgroup of* $\mathbf{G}_\infty$ *and* W_f *be a compact open subgroup of* $\mathbf{G}(\mathbb{A}_f)$*.*

Then for any U_∞*-finite and* W_f*-invariant functions* $f, h \in L^2_{00}(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$*,*

$$|\langle f, g.h\rangle| \le c_{W_f} \cdot (\dim\langle U_\infty f\rangle \cdot \dim\langle U_\infty h\rangle)^{r_0} \cdot \tilde{\xi}_{\mathbf{G}}(g) \cdot \|f\|_2 \cdot \|h\|_2 \quad \text{for all } g \in \mathbf{G}(\mathbb{A}),$$

where $c_{W_f} > 0$ *and* $r_0 = r_0(\mathbf{G}_\infty) > 0$*. Moreover if* $\mathbf{G}_\infty$ *has no normal subgroup locally isomorphic to* $\mathrm{Sp}_{2n}(\mathbb{C})$*, then* $r_0 = 1$*.*

Here, $\tilde{\xi}_{\mathbf{G}} : \mathbf{G}(\mathbb{A}) \to (0,1]$ *is an explicitly constructed proper function which is* L^p*-integrable for some* $p = p(\mathbf{G}) < \infty$*. (see Def. 2.16).*

The above bounds on matrix coefficients can be extended to smooth functions with compact support (see Theorem 2.22).

For each $v \in R$, denote $\hat{\mathbf{G}}_v^{\mathrm{Aut}} \subset \hat{\mathbf{G}}_v$ the automorphic dual of $\mathbf{G}(K_v)$, i.e., the subset of unitary dual of $\mathbf{G}(K_v)$ consisting of representations which are weakly contained in the representations appearing as $\mathbf{G}(K_v)$ components of $L^2(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))^{O_f}$ for some compact open subgroup O_f of $\mathbf{G}(\mathbb{A}_f)$.

Recall that for $\mathbf{G}$ simply connected, it is shown by Clozel [Cl1] that the trivial representation of $\mathbf{G}(K_v)$ is isolated in $\hat{\mathbf{G}}_v^{\mathrm{Aut}}$ for each $v \in R$, that is, $\mathbf{G}$ has property (τ) (cf. [Lu]). The following corollary presents a uniform version of property (τ) of $\mathbf{G}$ over all $v \in R$.

Corollary 1.18. *Let* $\mathbf{G}$ *be a connected simply connected absolutely almost simple* K*-group. Let* π *denote the quasi-regular representation of* $\mathbf{G}(\mathbb{A})$ *on* $L^2_0(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$*. Let* W *be a maximal compact subgroup of* $\mathbf{G}(\mathbb{A})$*. Then there exist an explicit* $p = p(\mathbf{G}) < \infty$ *such that any* W*-finite matrix coefficient of* π *is* $L^p(\mathbf{G}(\mathbb{A}))$*-integrable. In particular,* π *is strongly* L^p*.*

This corollary implies that for some $m = m(\mathbf{G}) < \infty$,

$$\pi^{\otimes m(\mathbf{G})} \subset \infty \cdot L^2(\mathbf{G}(\mathbb{A}))$$

(see [HT]), and that for any non-amenable closed subgroup $H \subset \mathbf{G}(\mathbb{A})$, the restriction of π to H does not have an almost invariant vector.

The proof of Theorem 1.17 goes roughly as follows: if $\tilde{\xi}_v$ is a uniform bound for the matrix coefficients of infinite dimensional representations in $\hat{\mathbf{G}}_v^{\mathrm{Aut}}$, $\tilde{\xi}_{\mathbf{G}}$ is defined to be the product $\prod_{v\in R} \tilde{\xi}_v$. This can be made precise using the language of direct integral of a representation (cf. proof of Theorem 2.8). For those $v \in R$ such that the K_v-rank of $\mathbf{G}$ is at least 2, the uniform bounds, say ξ_v, of matrix coefficients of *all* infinite dimensional unitary representations of $\mathbf{G}(K_v)$ were obtained in [Oh1]. For these cases, one can simply take $\tilde{\xi}_v = \xi_v$. In particular, if K-rank of $\mathbf{G}$ is at least 2, we have $\tilde{\xi}_{\mathbf{G}} = \prod_{v\in R} \xi_v$ and $\tilde{\xi}_{\mathbf{G}}$ works as a uniform bound for all unitary representation of $\mathbf{G}(\mathbb{A})$ without $\mathbf{G}(K_v)^+$-invariant vectors for each $v \in R$ (see Theorem 2.8 for a precise statement). Moreover $\tilde{\xi}_{\mathbf{G}}$ is fairly sharp in these cases. For instance, one can show that $\tilde{\xi}_{\mathbf{G}}$ is optimal for $\mathbf{G} = \mathrm{SL}_n$ $(n \geq 3)$, or Sp_{2n} $(n \geq 2)$ by [5.4, COU].

When there is $v \in R$ with K_v-rank of $\mathbf{G}$ one, finding an automorphic bound $\tilde{\xi}_v$ is essentially carried out by Clozel [Cl1]. In particular, several deep theorems in automorphic theory were used such as the Gelbart-Jacquet bound [GJ] toward Ramanujan conjecture, the results of Burger-Sarnak [BS] and Clozel-Ullmo [CU] on lifting automorphic bounds, the base changes by Rogawski [Ro] and Clozel [Cl2], and Jacquet-Langlands correspondence [JL].

1.6. **Equidistribution of Hecke points.** Let $K = \mathbb{Q}$. Let S be a finite set of primes including the archimedean prime ∞. If Γ is an S-arithmetic subgroup of $\mathbf{G}_S = \prod_{p\in S} \mathbf{G}(\mathbb{Q}_p)$ (here $\mathbb{Q}_\infty = \mathbb{R}$) and $a \in \mathbf{G}(\mathbb{Q})$, then the Hecke operator T_a on $L^2(\Gamma\backslash\mathbf{G}_S)$ is defined by

$$\mathrm{T}_a(f)(g) = \frac{1}{\deg(a)} \sum_{x\in\Gamma\backslash\Gamma a\Gamma} f(xg)$$

where $\deg(a) = \#\Gamma\backslash\Gamma a\Gamma$. Theorem 1.17 extends the main result in [COU] where some cases of $\mathbb{Q}$-anisotropic groups were excluded (see [EO]). In fact, the following corollary immediately follows from Theorem 1.17 and Proposition 2.6 in [COU]:

Corollary 1.19. *Let* $\mathbf{G}$ *be a connected simply connected absolutely almost simple* $\mathbb{Q}$-*group and* S *a finite set of primes including* ∞. *Suppose that* $\mathbf{G}_S$ *is non-compact. Let* $\Gamma \subset \mathbf{G}(\mathbb{Q})$ *be an* S-*congruence subgroup of* $\mathbf{G}_S$. *There exists a constant* $c = c(\Gamma) > 0$ *such that*

$$\| \mathrm{T}_a \| \leq c \cdot \tilde{\xi}_{\mathbf{G}}(a) \quad \textit{for any } a \in \mathbf{G}(\mathbb{Q}).$$

This corollary in particular implies that for any sequence $a_i \in \mathbf{G}(\mathbb{Q})$ with $\deg(a_i) \to \infty$, and for any $f \in C_c(\mathbf{G}_S)$,

$$\lim_{i\to\infty} \frac{1}{\deg(a_i)} \sum_{x\in\Gamma a_i\Gamma} f(x) = \int_{\mathbf{G}_S} f(g)\, d\tau_S$$

where τ_S is the normalized Haar measure on $\mathbf{G}_S$ so that $\tau_S(\Gamma\backslash\mathbf{G}_S) = 1$. It is interesting to note that unlike the rational points $\mathbf{G}(\mathbb{Q})$ of bounded height (Theorem 1.5), the Hecke points are equidistributed in $\mathbf{G}_S$ with respect to the invariant measure.

1.7. **Structure of the proof and organization of the paper.** Our proof of the asymptotics of the number of rational points can be divided into three steps:

- *Asymptotics of* $\mathrm{vol}(\mathbf{G}_{H_\iota} \cap B_T)$ *as* $T \to \infty$ (recall that $\mathbf{G}_{H_\iota}$ is defined in (1.14)). This reduces to the computation of certain p-adic integrals, which is usually done using the method of [D]. See, for example, Section 3 in [BT3] that discusses Denef's formula [D] in the context of Manin's conjectue. For the wonderful compactification, such computation is done in [STT2].
- *Decay of matrix coefficients in* $L^2_{00}(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$.
  This is deduced from the bound [Oh1, Oh2] for higher ranks groups and the automorphic bounds [BS, CU, Cl1, GJ, JL, Ro].
- *Using decay, deduce that* $\#(\mathbf{G}(K) \cap B_T) \sim \mathrm{vol}(\mathbf{G}_{H_\iota} \cap B_T)$ *as* $T \to \infty$.
  A connection between decay of matrix coefficients and counting problems was first observed by Margulis [Mar] in his thesis in 1970 and exploited further by Eskin and McMullen [EM]. Here we apply this idea to the counting of rational points.

Using this strategy, we also compute the asymptotic distribution of rational points.

In comparison, the method of Shalika, Takloo-Bighash, Tschinkel [STT2] is based on the study of analytic properties of the zeta function $\sum_{\gamma\in\mathbf{G}(K)} H_\iota(\gamma)^{-s}$. While the decay of matrix coefficients is crucial in the proof in [STT2] as well, their approach requires more precise information about the spectral decomposition of $L^2(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$ which involves substantial technical difficulties, such as, dealing with the continuous spectrum. The asymptotic distribution of rational points is not discussed in [STT2].

The method outlined above applies to a wide variety of counting questions. In particular, to extend Theorem 1.9 to arbitrary bi-equivariant compactifications of $\mathbf{G}$, one only needs to verify that the height functions satisfy the assumptions stated in Proposition 4.23.

The paper is organized as follows. We discuss the mixing property of unitary representations of $\mathbf{G}(\mathbb{A})$ and prove Theorems 1.16 and 1.17 in section 2. In section 3, we obtain volume estimates. In section 4, we prove Theorems 1.2, 1.5, 1.7, 1.9, 1.10 and 1.15 for the saturated cases. The rate of convergence is obtained in section 5, where we also prove Theorems 1.2 and 1.9 for the unsaturated cases. Section 6 contains examples.

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2. Adelic Mixing

Let K be a number field. Let $\mathbf{G}$ be a connected absolutely almost simple group defined over K. We keep the same notation R, R_f, K_v as in the introduction. Let $\mathcal{O}$ denote the ring of integers of K and $\mathcal{O}_v$ the valuation ring of K_v. Set $R_\infty = R - R_f$. For $v \in R_f$, let q_v denote the order of the residue field of $\mathcal{O}_v$. Denote by $\mathbb{A}$ the adele ring over K and by $\mathbf{G}(\mathbb{A})$ the adele group associated to $\mathbf{G}$.

Denote by $\mathbf{G}(\mathbb{A}_f)$ (resp. $\mathbf{G}_\infty$) the subgroup of finite (resp. infinite) adeles, i.e., $((g_v)_v) \in \mathbf{G}(\mathbb{A})$ with $g_v = e$ for all $v \in R_\infty$ (resp. for all $v \in R_f$). Then

$$\mathbf{G}(\mathbb{A}) = \mathbf{G}_\infty \times \mathbf{G}(\mathbb{A}_f).$$

2.1. **Definition and properties of $\xi_{\mathbf{G}}$.** We fix a smooth model of $\mathbf{G}$ over $\mathcal{O}[k^{-1}]$ for some $k \in \mathbb{Z}$. There exists a finite subset $S_0 \subset R_f$ such that for any $v \in R_f - S_0$, $\mathbf{G}$ is unramified over K_v and $\mathbf{G}(\mathcal{O}_v)$ is a hyperspecial compact subgroup (cf. [Ti2]). We set $U_v = \mathbf{G}(\mathcal{O}_v)$ for each $v \in R_f - S_0$. Then for each $v \in R_f - S_0$, there exists the group A_v of K_v-rational points of a maximal K_v-split torus of $\mathbf{G}$ so that the following Cartan decomposition holds:

$$\mathbf{G}(K_v) = U_v A_v^+ U_v \tag{2.1}$$

where A_v^+ is a closed positive Weyl chamber of A_v.

For $v \in S_0 \cup R_\infty$, there exists a good maximal compact subgroup U_v of $\mathbf{G}(K_v)$ such that

$$\mathbf{G}(K_v) = U_v A_v^+ \Omega_v U_v$$

where A_v is the group of K_v-rational points of a maximal K_v-split torus of $\mathbf{G}$ and Ω_v is a finite subset in the centralizer of A_v in $\mathbf{G}(K_v)$.

In particular for any $g \in \mathbf{G}(K_v)$, there exist unique $a_v \in A_v^+$ and $d_v \in \Omega_v$ such that $g \in U_v a_v d_v U_v$. For $v \in R_\infty$, any maximal compact subgroup of $\mathbf{G}(K_v)$ is a good maximal compact subgroup and $\Omega_v = \{e\}$.

Let $\mathcal{T}$ denote the set of $v \in R$ such that $\mathbf{G}(K_v)$ is compact, that is, $U_v = \mathbf{G}(K_v)$. It is well known that $\mathcal{T}$ is a finite set.

Denote by Φ_v^+ the system of positive roots in the set of all non-multipliable roots of $\mathbf{G}(K_v)$ relative to A_v^+ and choose a maximal strongly orthogonal system $\mathcal{S}_v$ in Φ_v^+ in the sense of [Oh2] (where an explicit construction is also given). If $v \in R - \mathcal{T}$ and $K_v \neq \mathbb{C}$, then define the bi-U_v-invariant function ξ_v on $\mathbf{G}(K_v)$ (cf. [Oh1]): for each $g = kadk' \in U_v A_v^+ \Omega_v U_v$,

$$\xi_v(g) = \prod_{\alpha \in \mathcal{S}_v} \Xi_v \begin{pmatrix} \alpha(a) & 0 \\ 0 & 1 \end{pmatrix}$$

where Ξ_v is the Harish-Chandra function of $\mathrm{PGL}_2(K_v)$. If $K_v = \mathbb{C}$, set

$$\xi_v(g) = \prod_{\alpha \in \mathcal{S}_v} \Xi_v \begin{pmatrix} \alpha(a) & 0 \\ 0 & 1 \end{pmatrix}^{n_\alpha}$$

where $n_\alpha = 1/2$ if α is a long root of $\mathbf{G}$, when $\mathbf{G}$ is locally isomorphic to $\mathrm{Sp}_{2n}(\mathbb{C})$, and $n_\alpha = 1$ for all other cases. We set $\xi_v = 1$ for $v \in \mathcal{T}$.

Definition 2.2. *Define the function $\xi_{\mathbf{G}}$ on $\mathbf{G}(\mathbb{A})$ by*

$$\xi_{\mathbf{G}}(g) = \prod_{v \in R} \xi_v(g_v) \quad \text{for } g = (g_v)_v \in \mathbf{G}(\mathbb{A}).$$

Since $0 < \xi_v(g_v) \leq 1$ for all $v \in R$ and $\xi_v(g_v) = 1$ for almost all v, $\xi_{\mathbf{G}}$ is a well defined function on $\mathbf{G}(\mathbb{A})$ and $0 < \xi_{\mathbf{G}} \leq 1$. Set

$$U = \prod_{v \in R} U_v.$$

Note also that $\xi_{\mathbf{G}}$ is bi-U-invariant.

For $v \in R - \mathcal{T}$, denoting by η_v the product of all positive roots in $\mathcal{S}_v$, set

$$\eta_v(kadk') := \eta_v(a)$$

where $kadk' \in U_v A_v^+ \Omega_v U_v$ for all v with $K_v \neq \mathbb{C}$. As in the case of the definition of ξ_v, if $K_v = \mathbb{C}$, we set $\eta_v = \prod_{\alpha \in \mathcal{S}_v} \alpha^{n_\alpha}$ with the same n_α defined as before. If $v \in \mathcal{T}$, we set $\eta_v = 1$.

Lemma 2.3. *For any $\epsilon > 0$, there is a constant $C_\epsilon > 0$ such that for any $g = (g_v)_v \in \mathbf{G}(\mathbb{A})$,*

$$\prod_{v \in R} |\eta_v(g_v)|^{-1/2} \le \xi_{\mathbf{G}}(g) \le C_\epsilon \cdot \prod_{v \in R} |\eta_v(g_v)|^{-1/2+\epsilon}. \tag{2.4}$$

In particular,

$$\xi_{\mathbf{G}}(g) \to 0 \quad \text{as } g \to \infty \text{ in } \mathbf{G}(\mathbb{A}).$$

Proof. For $v \in R - \mathcal{T}$, it follows from the explicit formula for Ξ_v (cf. [Oh1, 3.7.1]) that for any $\epsilon > 0$, there is a constant $C_{v,\epsilon} > 0$ such that for any $g_v \in \mathbf{G}(K_v)$,

$$|\eta_v(g_v)|^{-1/2} \le \xi_v(g_v) \le C_{v,\epsilon} \cdot |\eta_v(g_v)|^{-1/2+\epsilon}.$$

Moreover $C_{v,\epsilon} = 1$ for almost all v. This implies (2.4).

To see the second claim, first note that for any $g \in \mathbf{G}(\mathbb{A})$,

$$\xi_{\mathbf{G}}(g) \le \xi_v(g_v) \le C_\epsilon \cdot |\eta_v(g_v)|^{-1/2+\epsilon}. \tag{2.5}$$

Now suppose on the contrary that there exists a sequence $g_i \to \infty$ such that

$$\xi_{\mathbf{G}}(g_i) \nrightarrow 0.$$

Then by passing to a subsequence we may assume either that there is a place $v \in R - \mathcal{T}$ such that $g_{i,v} \to \infty$ in $\mathbf{G}(K_v)$ or that there is v_i with $q_{v_i} \to \infty$ such that $g_{i,v_i} \notin U_{v_i}$ and $\Omega_{v_i} = \{e\}$ for all i. If $g_{i,v} \to \infty$ as $i \to \infty$, then $|\eta_v(g_{i,v})| \to \infty$ as $i \to \infty$ and hence $\xi_{\mathbf{G}}(g_i) \to 0$ by (2.4). Therefore the first case cannot happen.

In the second case, note that since $g_{i,v_i} \notin U_{v_i}$ and $\Omega_{v_i} = \{e\}$, we have $|\eta_v(g_{i,v_i})| \ge q_{v_i}$ for each i. Hence by (2.5),

$$\xi_{\mathbf{G}}(g_i) \le C_\epsilon \cdot q_{v_i}^{-1/2+\epsilon}.$$

This gives a contradiction since $q_{v_i} \to \infty$. □

Lemma 2.6. *Let $\iota : \mathbf{G} \to \mathrm{GL_N}$ be a faithful absolutely irreducible representation defined over K and H_ι be a height function on $\mathbf{G}(\mathbb{A})$ associated to ι. Then there exist $m \in \mathbb{N}$ and $C > 0$ such that*

$$\xi_{\mathbf{G}}(g) \le C \cdot H_\iota^{-1/m}(g) \quad \text{for any } g \in \mathbf{G}(\mathbb{A}).$$

Proof. Let χ denote the highest weight of ι. Let $l \in \mathbb{N}$ be such that $\chi|_{A_v^+} \le l \cdot \log_{q_v} \eta_v$ for each $v \in R$. Here $q_v = e$ if $v \in R_\infty$. Without loss of generality, we may assume

$$H_v(\iota(a)) = q_v^{\chi(a)} \qquad \text{for each } a \in A_v^+ \text{ and } v \in R_f.$$

Since $|\eta_v(a_v)| = q_v^{\log_{q_v} |\eta_v(a_v)|}$ for $a_v \in A_v^+$, by Lemma 2.3, there exist some $c_1, c_2 > 0$ such that for any $g = (g_v) \in \mathbf{G}(\mathbb{A})$,

$$\xi_{\mathbf{G}}^{4l}(g) \le c_1 \cdot \prod_v |\eta_v(g_v)|^{-l} \le c_2 \cdot \prod_v H_v^{-1}(\iota(g_v)) = c_2 \cdot H_\iota^{-1}(g).$$

□

Even though we do not need the following fact in this paper, it is of independent interest:

Proposition 2.7. *There exists* $0 < p = p(\mathbf{G}) < \infty$ *such that* $\xi_{\mathbf{G}} \in L^p(\mathbf{G}(\mathbb{A}))$.

Proof. Choose any absolutely irreducible representation of $\mathbf{G}$ defined over K, for instance, the adjoint representation, and let H_ι be a height function on $\mathbf{G}(\mathbb{A})$ associated to ι (see (1.12)). By Theorem 3.4, the height zeta function

$$\mathcal{Z}(s) := \int_{\mathbf{G}(\mathbb{A})} H_\iota(g)^{-s} d\tau(g)$$

converges for $\mathrm{Re}(s) > a_\iota$ where a_ι is defined as in (1.3). Since $\xi_{\mathbf{G}}^m(g) \le C \cdot H_\iota^{-1}(g)$ for some $m \in \mathbb{N}$ by Lemma 2.6, $\xi_{\mathbf{G}}$ is L^p-integrable for any $p > ma_\iota$. □

2.2. **Uniform bound for matrix coefficients of $\mathbf{G}(\mathbb{A})$.** Let $W_f \subset \mathbf{G}(\mathbb{A}_f)$ be a compact open subgroup. Write $W_v = W_f \cap \mathbf{G}(K_v)$ for each $v \in R$. Then $W_v = U_v$ for almost all $v \in R_f$. For each $v \in R_f$, by [Be], there exists $d_{W_v} < \infty$ such that for any irreducible unitary representation ρ of $\mathbf{G}(K_v)$, the dimension of W_v-invariant vectors of ρ is at most d_{W_v}. Moreover d_{W_v} can be taken to 1 whenever W_v is a hyper-special compact subgroup. Hence the following number is well-defined:

$$d_{W_f} := \prod_{v \in R_f} d_{W_v} < \infty.$$

Set $U_f := \prod_{v \in R_f} U_v$ and $U_\infty := \prod_{v \in R_\infty} U_v$.

Theorem 2.8. *Let* $\mathbf{G}$ *be a connected absolutely almost simple* K*-group with* K*-rank at least* 2*. Let* W_f *be a compact open subgroup of* $\mathbf{G}(\mathbb{A}_f)$*. Let* π *be any unitary representation of* $\mathbf{G}(\mathbb{A})$ *without* $\mathbf{G}(K_v)^+$*-invariant vector for every* $v \in R$*. Then for any* U_∞*-finite and* W_f*-invariant unit vectors* x *and* y*,*

(2.9) $$|\langle \pi(g)x, y\rangle| \le d_0 \cdot c_{W_f} \cdot (\dim\langle U_\infty x\rangle \cdot \dim\langle U_\infty y\rangle)^{(r+1)/2} \cdot \xi_{\mathbf{G}}(g) \quad \textit{for all } g \in \mathbf{G}(\mathbb{A})$$

where $c_{W_f} := d_{W_f} \cdot \prod_v [U_v : U_v \cap W_v] \cdot (\max_{d \in \Omega_v} [U_v : dU_v d^{-1}])$ *and* $d_0, r \ge 1$ *depend only on* $\mathbf{G}$*. Moreover if* $\mathbf{G}(K_v) \not\cong \mathrm{Sp}_{2n}(\mathbb{C})$ *locally for any* $v \in R_\infty$*,* $d_0 = 1$ *and* $r = 1$*.*

The proof of above theorem is based on theorems in [Oh1]. More precisely, recall:

Theorem 2.10 (Theorem 1.1-2, [Oh1])**.** *Suppose that the* K_v*-rank of* $\mathbf{G}$ *is at least* 2*. Let* π_v *be a unitary representation of* $\mathbf{G}(K_v)$ *without* $\mathbf{G}(K_v)^+$*-invariant vectors. Then for any* U_v*-finite unit vectors* x *and* y*,*

$$|\langle \pi_v(g)x, y\rangle| \le d_v \cdot c_v \cdot (\dim\langle U_v x\rangle \cdot \dim\langle U_v y\rangle)^{r_v/2} \cdot \xi_v(g) \quad \textit{for any } g \in \mathbf{G}(K_v)$$

where $c_v = \max_{d \in \Omega_v} [U_v : dU_v d^{-1}]$ *and* $d_v, r_v \ge 1$ *depend only on* $\mathbf{G}(K_v)$*. Moreover whenever* $\mathbf{G}(K_v) \not\cong \mathrm{Sp}_{2n}(\mathbb{C})$ *locally,* $d_v = 1$ *and* $r_v = 1$*.*

In the case when $\mathbf{G}(K_v) \cong \mathrm{Sp}_{2n}(\mathbb{C})$ locally, the above theorem was stated only for U_v-invariant vectors in [Oh1]. However using the remark following Prop. 2.7 in [Oh1], the proof can be modified for the above claim.

Proof of Theorem 2.8 For $g = (g_v)_v \in \mathbf{G}(\mathbb{A})$, choose a finite subset S_g of places containing

$$\{v \in R_f : g_v \notin U_v\} \cup R_\infty.$$

Note that for $v \in R - S_g$, we have $g_v \in U_v$ and hence $\xi_v(g_v) = 1$. Therefore for $g = (g_v)_v \in \mathbf{G}(\mathbb{A})$,

$$\xi(g) = \prod_{v \in S_g} \xi_v(g_v).$$

Let $G_g = \prod_{v \in S_g} \mathbf{G}(K_v)$ and $W_g = \prod_{v \in S_g \cap R_f} W_v$. As a G_g representation, π has a Hilbert integral decomposition:

$$\pi = \int_{z \in Z_g} \oplus^{m_z} \rho_z \, d\nu(z)$$

where Z_g is the unitary dual of G_g and ρ_z is irreducible, m_z is a multiplicity for each $z \in Z_g$ and ν is a measure on Z_g (see [Di] or [Section 2.3, Zi]). We may assume that for all z, ρ_z has no $\mathbf{G}(K_v)^+$-invariant vector (see [Prop. 2.3.2, Zi]).

If we write $\mathcal{L}_z = \oplus^{m_z} \rho_z$, $x = \int x_z d\nu(z)$ and $y = \int y_z d\nu(z)$ with

$$x_z = \sum^{m_z} x_{zi} \quad \text{and} \quad y_z = \sum^{m_z} y_{zi} \in \mathcal{L}_z,$$

we have

$$\langle x, y \rangle = \int_{Z_g} \sum_i \langle x_{zi}, y_{zi} \rangle \, d\nu(z).$$

It follows from the definition of a Hilbert direct integral that

$$\dim\langle U_\infty x_{zi} \rangle \le \dim\langle U_\infty x_z \rangle \le \dim\langle U_\infty x \rangle,$$

x_{zi} is W_g-invariant for almost all z and all i, and similarly for y. Without loss of generality, we assume the above holds for all z. We claim that

(2.11)
$$|\langle \rho_z(g) x_{zi}, y_{zi} \rangle| \le c_{W_f} \cdot d_0 \cdot \xi_{\mathbf{G}}(g) \cdot (\dim\langle U_\infty x \rangle \cdot \dim\langle U_\infty y \rangle)^{(r+1)/2} \cdot \|x_{zi}\| \cdot \|y_{zi}\|$$

where $r = \max_v r_v$ and $c_0 = d_{W_f} \prod_v (c_v \cdot [U_v : U_v \cap W_v]) < \infty, d_0 = \prod_v d_v < \infty$ with c_v, d_v, r_v as in Theorem 2.10. Since G_g is a type (I) group [Be], we may write $\rho_z = \otimes_{v \in S_g} \rho_{z(v)}$ where $\rho_{z(v)}$ is an irreducible representation of $\mathbf{G}(K_v)$ without $\mathbf{G}(K_v)^+$-invariant vectors. Since the finite linear combinations of pure tensor vectors are dense, it suffices to prove (2.11) assuming x_{zi} and y_{zi} are finite sums of pure tensors. Hence we can write

$$x_{zi} = \sum_j \otimes_{v \in S_g} x_{zij(v)} \; ; \; y_{zi} = \sum_k \otimes_{v \in S_g} y_{zik(v)}$$

where for each $v \in S_g$, $x_{zij(v)}$ (resp. $y_{zik(v)}$) are mutually orthogonal and the number of summands for x_{zi} (resp. y_{zi}) is at most $\dim\langle U_\infty x\rangle \cdot d_{W_f}$ (resp. $\dim\langle U_\infty y\rangle \cdot d_{W_f}$). Hence by Cauchy-Schwartz inequality, for $x_{zij} = \prod_{v\in S_g} x_{zij(v)}$ and $y_{zij} = \prod_{v\in S_g} y_{zij(v)}$

$$\sum_j \|x_{zij}\| \le (\dim\langle U_\infty x\rangle \cdot d_{W_f})^{1/2}\|x_{zi}\|; \text{ and } \sum_k \|y_{zik}\| \le (\dim\langle U_\infty y\rangle \cdot d_{W_f})^{1/2}\|y_{zi}\|.$$

Since for $v \in R_f$

$$\dim\langle U_v x\rangle \le [U_v : W_v \cap U_v] \quad \text{and} \quad \dim\langle U_v y\rangle \le [U_v : W_v \cap U_v],$$

by Theorem 2.10, we have for $c_0 = \prod_v c_v$,

(2.12)

$$|\langle \rho_z(g)x_{zi}, y_{zi}\rangle| \le \sum_{j,k} \prod_{v\in S_g} |\langle \rho_{z(v)}(g_v)x_{zij(v)}, y_{zik(v)}\rangle|$$

$$\le c_0 \cdot d_0 \cdot \prod_{v\in S_g} \xi_v(g_v) \cdot (\dim\langle U_\infty x\rangle \cdot \dim\langle U_\infty y\rangle)^{r/2}(\prod_{v\in R_f}[U_v : W_v \cap U_v]) \cdot \left(\sum_{j,k} \|x_{zij}\| \cdot \|y_{zik}\|\right)$$

$$\le c_0 \cdot d_0 \cdot \xi_{\mathbf{G}}(g) \cdot (\dim\langle U_\infty x\rangle \cdot \dim\langle U_\infty y\rangle)^{(r+1)/2} \cdot \prod_{v\in R_f}[U_v : W_v \cap U_v]) \cdot d_{W_f} (\|x_{zi}\| \cdot \|y_{zi}\|)$$

$$= c_{W_f} \cdot d_0 \cdot \xi_{\mathbf{G}}(g) \cdot (\dim\langle U_\infty x\rangle \cdot \dim\langle U_\infty y\rangle)^{(r+1)/2} \cdot (\|x_{zi}\| \cdot \|y_{zi}\|)$$

proving (2.11). Therefore again by Cauchy-Schwartz inequality,

(2.13)

$$|\langle(\oplus^{m_z}\rho_z)(g)(x_z), y_z\rangle| \le \sum_i |\langle \rho_z(g)x_{zi}, y_{zi}\rangle|$$

$$\le c_{W_f} \cdot d_0 \cdot \xi_{\mathbf{G}}(g) \cdot (\dim\langle U_\infty x\rangle \cdot \dim\langle U_\infty y\rangle)^{(r+1)/2} \cdot \|x_z\| \cdot \|y_z\|.$$

By integrating over Z_g, we obtain (2.9).

Since U-finite vectors form a dense subset by Peter-Weyl theorem, the above implies an adelic version of Howe-Moore theorem [HM] on the vanishing of matrix coefficients:

Corollary 2.14. *Let* $\mathbf{G}$ *and* π *be as in Theorem 2.8. Then for any vectors* x *and* y,

$$\langle \pi(g)x, y\rangle \to 0 \quad \text{as } g \to \infty \text{ in } \mathbf{G}(\mathbb{A}).$$

2.3. **Automorphic bound for $\mathbf{G}(\mathbb{A})$.** If $\mathbf{G}$ has K-rank at most one, the analogue of Theorem 2.8 does not hold in general. However if we look at those infinite dimensional representations occurring in $L^2(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$, we still obtain a similar upper bound.

We now consider the unitary representation of $\mathbf{G}(\mathbb{A})$ on the space $L^2(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$ by right translations. Let $L^2_{00}(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$ denote the orthogonal complement to the direct sum of all one-dimensional representations occurring in $L^2(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$. If $\mathbf{G}$ is simply connected, it follows from the strong approximation property that

$L^2_{00}(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$ is the complement to the space of constant functions (cf. Lemma 2.19).

We first state the following conjecture:

Conjecture 2.15. *Let* $\mathbf{G}$ *be a connected absolutely almost simple* K*-group. Let* W_f *be a compact open subgroup of* $\mathbf{G}(\mathbb{A}_f)$*. Then for any* $U_\infty \times W_f$*-invariant unit vectors* $f, h \in L^2_{00}(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$,

$$|\langle f, g.h\rangle| \le c_{W_f} \cdot \xi_{\mathbf{G}}(g) \quad \text{for all } g \in \mathbf{G}(\mathbb{A})$$

where $c_{W_f} > 0$ *is a constant depending only on* $\mathbf{G}$ *and* W_f*.*

The above holds for groups of K-rank at least 2 by Theorem 2.8. For $\mathbf{G} = \mathrm{PGL}_2$, Conjecture 2.15 is essentially equivalent to the Ramanujan conjecture. We will prove a weaker statement of Conjecture 2.15 where the function $\xi_{\mathbf{G}}$ is replaced by a function $\tilde{\xi}_{\mathbf{G}}$ with slower decay such that $\xi_{\mathbf{G}} \le \tilde{\xi}_{\mathbf{G}} \le \xi_{\mathbf{G}}^{1/2}$.

Definition 2.16. *Set* $R_1 := \{v \in R : \mathrm{rank}_{K_v}(\mathbf{G}) = 1\}$*. Define*

$$\tilde{\xi}_{\mathbf{G}}(g) := \prod_{v\in R_1} \xi_v(g_v)^{1/2} \prod_{v\in R-R_1} \xi_v(g_v)$$

where $g = (g_v)_v \in \mathbf{G}(\mathbb{A})$*.*

Theorem 2.17 (Automorphic bounds)**.** *Let* $\mathbf{G}$ *be a connected absolutely almost simple* K*-group. Let* W_f *be a compact open subgroup of* $\mathbf{G}(\mathbb{A}_f)$*. Then for any* U_∞*-finite and* W_f*-invariant unit vectors* $x, y \in L^2_{00}(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$,

$$|\langle x, g.y\rangle| \le c_{W_f} \cdot (\dim\langle U_\infty x\rangle \cdot \dim\langle U_\infty y\rangle)^{(r+1)/2} \cdot \tilde{\xi}_{\mathbf{G}}(g) \quad \text{for all } g \in \mathbf{G}(\mathbb{A})$$

where $c_{W_f} > 0, r = r(\mathbf{G}) \ge 1$*. Moreover* $r = 1$ *provided for any* $v \in R$*,* $\mathbf{G}(K_v) \not\simeq \mathrm{Sp}_{2n}(\mathbb{C})$ $(n \ge 2)$ *locally.*

Recall that for unitary representations ρ_1 and ρ_2 of $\mathbf{G}(K_v)$, ρ_1 is said to be weakly contained in ρ_2 if every diagonal matrix coefficients of ρ_1 can be approximated uniformly on compact subsets by convex combinations of diagonal matrix coefficients of ρ_2. For each $v \in R$, denote by $\hat{\mathbf{G}}_v$ the unitary dual of $\mathbf{G}(K_v)$ and by $\hat{\mathbf{G}}_v^{\mathrm{Aut}} \subset \hat{\mathbf{G}}_v$ the automorphic dual of $\mathbf{G}(K_v)$, i.e., the subset of unitary dual of $\mathbf{G}(K_v)$ consisting of representations which are weakly contained in the representations appearing as $\mathbf{G}(K_v)$ components of $L^2(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))^{O_f}$ for some compact open subgroup O_f of $\mathbf{G}(\mathbb{A}_f)$.

Theorem 2.18 (Burger-Sarnak [BS], Clozel-Ullmo [CU])**.** *Let* $\mathbf{G}$ *be a connected absolutely almost simple* K*-group. Let* $\mathbf{H} \subset \mathbf{G}$ *be a connected semisimple* K*-subgroup. Then for any* $v \in R$ *and for any* $\rho_v \in \hat{\mathbf{G}}_v^{\mathrm{Aut}}$*, any irreducible representation of* $\mathbf{H}(K_v)$ *weakly contained in* $\rho_v|_{\mathbf{H}(K_v)}$ *is contained in* $\hat{\mathbf{H}}_v^{\mathrm{Aut}}$*.*

Lemma 2.19. *For any* $v \in R$ *such that* $\mathbf{G}(K_v)$ *is non-compact,* $L^2_{00}(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$ *has no non-zero* $\mathbf{G}(K_v)^+$*-invariant function.*

Proof. (cf. proof of Lemma 3.8 in [GaO].) Let $\mathcal{L}_v$ denote the set of $f \in L^2_{00}(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$ fixed by $\mathbf{G}(K_v)^+$. We need to show that $\mathcal{L}_v = \{0\}$. Consider the subgroup $\mathbf{G}^{\{v\}}$ of $\mathbf{G}(\mathbb{A})$ consisting of elements whose v-component is trivial, and consider continuous functions in $\mathbf{G}^{\{v\}}$ with compact support of the form $\prod_{w\in R-\{v\}} f_w$ where $f_w \in C_c(\mathbf{G}(K_w))$ for all $w \in R - \{v\}$ and $f_w|_{U_w} = 1$ for almost all w. By considering the convolutions with these functions, we obtain a dense family of the continuous functions of $\mathcal{L}_v$. Hence it suffices to show that any continuous function $f \in \mathcal{L}_v$ is trivial. Let $f \in \mathcal{L}_v$ be continuous. Let $\tilde{\mathbf{G}}$ be the simply connected cover of $\mathbf{G}$ and denote by $\mathrm{pr} : \tilde{\mathbf{G}} \to \mathbf{G}$ the covering map. Consider the projection map

$$\tilde{\mathbf{G}}(K)\backslash\tilde{\mathbf{G}}(\mathbb{A}) \to \mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}).$$

Let $\tilde{f}$ be the pull back of f. Since the image of $\tilde{\mathbf{G}}(K_v)$ is $\mathbf{G}(K_v)^+$ under the map pr, the function $\tilde{f}$ is left $\tilde{\mathbf{G}}(K)$-invariant and right $\tilde{\mathbf{G}}(K_v)$-invariant. On the other hand, the strong approximation property implies that $\tilde{\mathbf{G}}(K)\tilde{\mathbf{G}}(K_v)$ is dense in $\tilde{\mathbf{G}}(\mathbb{A})$ (cf. [Theorem 7.12, PR]). Therefore $\tilde{f}$ is constant. It follows that f is a sum of characters of $\mathbf{G}(\mathbb{A})$, and hence 0 since $f \in L^2_{00}(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$. □

Proof of Theorem 2.17. The case when K-rank is at least 2 follows from Theorem 2.8 and Lemma 2.19. Suppose first that $\mathbf{G}$ has K-rank one. Then there is a K-embedding of a K-group $\mathbf{H}$ where $\mathbf{H}$ is either SL_2 or PGL_2. Using a Gelbart-Jacquet [GJ] estimate toward the Ramanujan conjecture for SL_2 or PGL_2 and Theorem 2.18, we obtain for $v \in R_1$, any infinite dimensional $\rho_v \in \hat{\mathbf{G}}_v^{\mathrm{Aut}}$, and U_v-finite vectors x_v, y_v, (cf. Theorem 3.4 [COU])

$$|\langle \rho_v(g)(x_v), y_v\rangle| \le c_v \cdot \xi_v(g)^{1/2} \cdot (\dim\langle U_v x_v\rangle \cdot \dim\langle U_v y_v\rangle)^{1/2} \tag{2.20}$$

for any $g \in \mathbf{G}(K_v)$. Combining this with Theorem 2.10, we can derive the desired bound by the same argument as in the proof of Theorem 2.8.

Now suppose $\mathbf{G}$ is K-anisotropic. If $R_1 \neq \emptyset$, it follows from the classification theorem by Tits [Ti1] that $\mathbf{G}$ is of Dynkin type $\mathcal{A}$. Applying [Theorem 1.1, Cl], we deduce that there exists a K-embedding of K-subgroup $\mathbf{H}$ of type $\mathcal{A}$ such that $\mathbf{H}$ has K_v-rank one whenever $v \in R_1$. Let $v \in R_1$. Then up to isogeny, we have either that $\mathbf{H} = \mathrm{PGL}_1(D)$ for a quaternion algebra D over K and $\mathbf{H} = \mathrm{PGL}_2$ over K_v, or $\mathbf{H} = \mathrm{PGU}(D, *)$ for a division algebra D of prime degree d over a quadratic extension F of K with a second kind involution $*$, and $\mathbf{H} = \mathrm{PGU}(n-1, 1)$ over K_v (with $n \ge 3$). In the former case we use the Jacquet-Langlands correspondence [JL] to transfer the Gelbart-Jacquet automorphic bound of PGL_2 to $\mathbf{H}(K_v)$ via Theorem 2.18. In the second case which is hardest, it is explained in [Cl1] that by the base changes obtained by Rogawski [Ro] and Clozel [Cl2], we can use the bound of $\mathrm{PGL}_n(F_w)$ to get a bound for $\mathbf{H}(K_v)$ where w is a place of F lying above v and F_w is a quadratic extension of K_v. This gives us (2.20) for $v \in R_1$ again in view of Theorem 2.18 and Theorem 2.10. Combining with Theorem 2.10 for those places $v \in R - (R_1 \cup \mathcal{T})$ as in the proof of Theorem 2.8, we obtain the desired bound. This finishes the proof. □

Let $X_1, \cdots, X_m$ be an orthonormal basis of the Lie algebra $\mathrm{Lie}(U_\infty)$ with respect to an Ad-invariant scalar product. Then the elliptic operator

$$\mathcal{D} := 1 - \sum_{i=1}^{m} X_i^2 \tag{2.21}$$

lies in the center of the universal enveloping algebra of $\mathrm{Lie}(U_\infty)$. We say a function f on $\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A})$ is smooth if f is W_f-invariant for some compact open subgroup of $\mathbf{G}(\mathbb{A}_f)$ and smooth for the action of $\mathbf{G}_\infty$.

In Theorem 2.17, we can relax U_∞-finite conditions to smooth conditions provided we replace the L^2-norms by L^2-Sobolev norms:

Theorem 2.22. *Let $\mathbf{G}$ be a connected absolutely almost simple K-group. Let W_f be a compact open subgroup of $\mathbf{G}(\mathbb{A}_f)$. Then for any W_f-invariant smooth functions $f, h \in L^2_{00}(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$ with compact support,*

$$|\langle f, g.h\rangle| \le c_{W_f} \cdot \tilde{\xi}_{\mathbf{G}}(g) \cdot \|\mathcal{D}^l(f)\| \cdot \|\mathcal{D}^l(h)\| \quad \text{for all } g \in \mathbf{G}(\mathbb{A})$$

where $c_{W_f} > 0$ and l is any sufficiently large integer.

Proof. Deducing this from Theorem 2.17 is quite standard in view of the results of Harish-Chandra explained in [Ch 4, Wa]. We give a sketch of the proof. Denote by π the representation $L^2_{00}(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$. Then $\pi = \oplus_{\nu\in\hat{U}_\infty}\pi_\nu$ where π_ν is the ν-isotypic component of π and $\mathcal{D}$ acts as a scalar, say, c_ν on each π_ν. We write $f = \sum_{\nu\in\hat{U}_\infty} f_\nu$ and $h = \sum_{\nu\in\hat{U}_\infty} h_\nu$. One has $\|f_\nu\| = c_\nu^{-l}\|\mathcal{D}^l f_\nu\|$ and similarly for h. Then

$$|\langle f, g.h\rangle| \le \sum_{(\nu_1,\nu_2)\in\hat{U}_\infty\times\hat{U}_\infty} |\langle f_{\nu_1}, g.h_{\nu_2}\rangle|$$

Using Theorem 2.17, we then obtain

$$\begin{aligned} &|\langle f, g.h\rangle| \\ &\le c_{W_f} \cdot \tilde{\xi}_{\mathbf{G}}(g) \left(\sum_{\nu\in\hat{U}_\infty} \|f_\nu\| \dim\langle U_\infty f_\nu\rangle^{(r+1)/2}\right)\left(\sum_{\nu\in\hat{U}_\infty} \|h_\nu\| \dim\langle U_\infty h_\nu\rangle^{(r+1)/2}\right) \\ &\le c_{W_f} \cdot \tilde{\xi}_{\mathbf{G}}(g) \cdot \|\mathcal{D}^l(f)\| \cdot \|\mathcal{D}^l(h)\| \cdot \sum_{\nu\in\hat{U}_\infty} {c_\nu}^{-2l} \dim(\nu)^{r+1} \end{aligned}$$

Now if $l \in \mathbb{N}$ is sufficiently large, then $\sum_\nu c_\nu^{-2l}\dim(\nu)^{r+1} < \infty$ [Wa]. This proves the claim. □

If $\mathbf{G}$ is a connected semisimple K-group, we say that a sequence $\{g_i \in \mathbf{G}(\mathbb{A})\}$ tends to infinity strongly if for any non-trivial connected simple normal K-subgroup $\mathbf{H}$ of $\mathbf{G}$, $\pi(g_i)$ tends to ∞ as $i \to \infty$, where $\pi : \mathbf{G}(\mathbb{A}) \to \mathbf{G}(\mathbb{A})/\mathbf{H}(\mathbb{A})$ denotes the canonical projection.

Theorem 2.23 (Mixing for $L^2(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$). *Let $\mathbf{G}$ be a product of connected absolutely almost simple K-groups. Then for any $f, h \in L^2_{00}(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$,*

$$\langle f, g.h\rangle \to 0$$

as $g \in \mathbf{G}(\mathbb{A})$ tends to infinity strongly.

Proof. Write $\mathbf{G} = \mathbf{G}_1 \times \cdots \times \mathbf{G}_m$ where each $\mathbf{G}_i$ is a connected absolutely almost simple K-group. By Theorem 2.17 and Peter-Weyl theorem, for each $1 \le i \le m$, and for any $f_i, h_i \in L^2_{00}(\mathbf{G}_i(K)\backslash\mathbf{G}_i(\mathbb{A}))$,

$$\langle f_i, g_i.h_i\rangle \to 0 \tag{2.24}$$

as $g_i \to \infty$ in $\mathbf{G}_i(\mathbb{A})$.

Consider $\otimes_{i=1}^m L^2(\mathbf{G}_i(K)\backslash\mathbf{G}_i(\mathbb{A}))$ as a subset of $L^2(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$. Then the finite sums of the functions of the form $h = \otimes_{i=1}^m h_i \in L^2(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$, $h_j \in L^2(\mathbf{G}_j(K)\backslash\mathbf{G}_j(\mathbb{A}))$, such that for at least one j, $h_j \in L^2_{00}(\mathbf{G}_j(K)\backslash\mathbf{G}_j(\mathbb{A}))$ form a dense subset of the space $L^2_{00}(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$. Hence it suffices to prove the claim for $f = \otimes_{i=1}^m f_i$ and $h = \otimes_{i=1}^m h_i$ of such type. Suppose $h_j \in L^2_{00}(\mathbf{G}_j(K)\backslash\mathbf{G}_j(\mathbb{A}))$ for some $1 \le j \le m$. If $g = (g_1, \cdots, g_m)$ with $g_i \in \mathbf{G}_i(\mathbb{A})$, then

$$|\langle f, g.h\rangle| = \prod_{i=1}^m |\langle f_i, g_i.h_i\rangle| \le |\langle f_j, g_j.h_j\rangle| \cdot (\prod_{i\ne j} \|f_i\| \cdot \|h_i\|).$$

If f_j' denotes the projection of f_j to $L^2_{00}(\mathbf{G}_j(K)\backslash\mathbf{G}_j(\mathbb{A}))$, then

$$\langle f_j, g_j.h_j\rangle = \langle f_j', g_j.h_j\rangle.$$

Since $g \to \infty$ strongly and hence $g_j \to \infty$, we obtain $\langle f_j', g_j.h_j\rangle \to 0$ by (2.24). This proves the claim. □

3. Volume asymptotics

Let $\mathbf{G}$ be a connected adjoint semisimple algebraic group over K. As in the previous section, let $L^2_{00}(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$ be the orthogonal complement to the direct sum of all one dimensional representations in $L^2(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$. Denote by Λ an orthogonal basis for $L^2_{00}(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))^\perp$ consisting of continuous unitary characters of $\mathbf{G}(\mathbb{A})$. Then

$$L^2(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A})) = L^2_{00}(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A})) \oplus \sum_{\chi\in\Lambda} \mathbb{C}\chi.$$

For a compact open subgroup W_f of $\mathbf{G}(\mathbb{A}_f)$, set

$$\Lambda^{W_f} := \{\chi \in \Lambda : u.\chi = \chi \text{ for any } u \in W_f\};$$
$$\ker(\Lambda^{W_f}) := \cap\{\ker(\chi) : \chi \in \Lambda^{W_f}\}.$$

Note that $\ker(\Lambda^{W_f})$ is a normal subgroup of $\mathbf{G}(\mathbb{A})$. Moreover:

Lemma 3.1. *We have*

(1) $\mathbf{G}(K)\mathbf{G}_\infty^\circ W_f \subset \ker(\Lambda^{W_f})$.

(2) $\#\Lambda^{W_f} = [\mathbf{G}(\mathbb{A}) : \ker(\Lambda^{W_f})] < \infty$.
(3) *For any* $g \in \mathbf{G}(\mathbb{A})$,

$$\sum_{\chi\in\Lambda^{W_f}} \chi(g) = \begin{cases} \#\Lambda^{W_f} & \text{if } g \in \ker(\Lambda^{W_f}) \\ 0 & \text{otherwise} \end{cases}.$$

Proof. Note that an element of Λ is precisely a continuous homomorphism $\chi : \mathbf{G}(\mathbb{A}) \to S^1$ which contains $\mathbf{G}(K)$ in the kernel, where S^1 is the unit circle. Since $\mathbf{G}_\infty^\circ$ is a connected semisimple group, $\mathbf{G}_\infty^\circ \subset \ker(\chi)$ for any $\chi \in \Lambda$. Hence

$$\mathbf{G}(K)\mathbf{G}_\infty^\circ W_f \subset \ker(\Lambda^{W_f}).$$

Since $\mathbf{G}_\infty^\circ$ has a finite index in $\mathbf{G}_\infty$, it follows from [Theorem 5.1, PR] that there exist finitely many $u_1, \cdots, u_h \in \mathbf{G}(\mathbb{A})$ such that

$$\mathbf{G}(\mathbb{A}) = \cup_{i=1}^h \mathbf{G}(K)u_i\mathbf{G}_\infty^\circ W_f.$$

It follows $[\mathbf{G}(\mathbb{A}) : \ker(\Lambda^{W_f})] < \infty$. Clearly the quotient $\mathbf{G}(\mathbb{A})/\ker(\Lambda^{W_f})$ is a finite abelian group whose dual is isomorphic to Λ^{W_f}. Hence (2) and (3) easily follow from the duality of finite abelian groups. □

Let $\iota : \mathbf{G} \to \mathrm{GL}_N$ be an absolutely irreducible faithful representation defined over K. We give a definition of a height function on $\mathbf{G}(\mathbb{A})$ associated to ι which is slightly more general than those considered in the introduction (see (1.12)). It is this class of the functions for which we prove our main theorems.

Definition 3.2. *A height function* H_ι *on* $\mathbf{G}(\mathbb{A})$ *is defined by the product* $\prod_{v\in R} H_{\iota,v}$ *where* $H_{\iota,v}$ *is a function on* $\mathbf{G}(K_v)$ *for* $v \in R$ *satisfying the following:*

(1) *there exists a finite subset* $S \subset R$ *such that*

$$H_{\iota,v}(g) = \max_{ij} |\iota(g)_{ij}|_v \quad \text{for all } v \in R - S;$$

(2) *for* $v \in S$, *there exists* $C > 0$ *such that*

$$C^{-1} \cdot \max_{ij} |\iota(g)_{ij}|_v \le H_{\iota,v}(g) \le C \cdot \max_{ij} |\iota(g)_{ij}|_v;$$

(3) *for any* $v \in S \cap R_\infty$, *there exists* $b > 0$ *such that for any small* $\epsilon > 0$,

$$(1 - b\cdot\epsilon)H_{\iota,v}(x) \le H_{\iota,v}(gxh) \le (1 + b\cdot\epsilon)H_{\iota,v}(x)$$

for any $x \in \mathbf{G}(K_v)$ *and any* g, h *in the* ϵ*-neighborhood of* e *in* $\mathbf{G}(K_v)$ *with respect to a Riemannian metric;*

(4) *for any* $v \in S \cap R_f$, $H_{\iota,v}$ *is bi-invariant under a compact open subgroup of* $\mathbf{G}(K_v)$.

Fix a height function H_ι on $\mathbf{G}(\mathbb{A})$. Note that the height function H_ι is bi-invariant under a compact open subgroup, say W_f, of $\mathbf{G}(\mathbb{A}_f)$.

For $T > 0$, set

$$B_T := \{g \in \mathbf{G}(\mathbb{A}) : H_\iota(g) < T\}.$$

For a finite subset S of R, let $\mathbf{G}^S$ denote the subgroup of $\mathbf{G}(\mathbb{A})$ consisting of (g_v), with $g_v = e$ for all $v \in S$, and set $\mathbf{G}_S := \prod_{v \in S} \mathbf{G}(K_v)$. Note that $\mathbf{G}^\emptyset = \mathbf{G}(\mathbb{A})$ and $\mathbf{G}(\mathbb{A}) = \mathbf{G}_S\mathbf{G}^S$. Let τ_S and τ^S denote Haar measures on $\mathbf{G}_S$ and $\mathbf{G}^S$ respectively.

The following lemma easily follows from the decomposition of a Haar measures dg_v in terms of Cartan decomposition.

Lemma 3.3. *There exists $\epsilon > 0$ such that for every finite set $S \subset R$,*

$$\tau_S(B_T \cap \mathbf{G}_S) = O(T^{a_\iota - \epsilon}).$$

where the implied constant depends on S. In particular, for any $v \in R$,

$$\int_{\mathbf{G}(K_v)} H_{\iota,v}(g_v)^{-a_\iota} dg_v < \infty.$$

Given an automorphic character χ, we consider the following integral

$$\mathcal{Z}^S(s,\chi) := \int_{\mathbf{G}^S} H_\iota(g)^{-s}\chi(g)\, d\tau^S(g).$$

The following follows from Theorem 7.1 in [STT2] and the properties of the Hecke L-functions:

Theorem 3.4. *Let S be a finite subset of R and a_ι, b_ι as in (1.3). Then $\mathcal{Z}^S(s,\chi)$ converges absolutely when* $\mathrm{Re}(s) > a_\iota$*, and there exists $\epsilon > 0$ such that $\mathcal{Z}^S(s,\chi)$ has a meromorphic continuations to* $\mathrm{Re}(s) > a_\iota - \epsilon$ *with unique poles at $s = a_\iota$ of order at most b_ι. The order of the pole is exactly b_ι for $\chi = 1$. Moreover, for some constants $\kappa \in \mathbb{R}$ and $k > 0$,*

$$\left| \frac{(s-a_\iota)^{b_\iota}\mathcal{Z}^S(s,\chi)}{s^{b_\iota}} \right| \le k \cdot |1 + \mathrm{Im}(s)|^\kappa$$

for $\mathrm{Re}(s) > a_\iota - \epsilon$*.*

For $g \in \mathbf{G}(\mathbb{A})$, define

$$\delta_S(g) := \sum_{\chi \in \Lambda^{W_f}} c_{S,\chi}\chi(g) \quad \text{with} \quad c_{S,\chi} = \lim_{s\to a_\iota} (s-a_\iota)^{b_\iota}\mathcal{Z}^S(s,\chi). \tag{3.5}$$

It will be a consequence of Theorem 3.9 that for all g, $\delta_S(g) > 0$.

We use the following version of Ikehara Tauberian theorem to deduce the volume asymptotics from Theorem 3.4.

Theorem 3.6. *Fix $a > 0$ and $\delta > 0$. Let $\alpha(t)$ be a non-negative non-decreasing function on (δ, ∞) such that*

$$f(s) := \int_\delta^\infty t^{-s}\, d\alpha$$

converges for $\mathrm{Re}(s) > a$*. Suppose that*

- *$f(s)$ has a meromorphic continuation to the half plane* $\mathrm{Re}(s) > a - \epsilon > 0$ *and has a unique pole at $s = a$ with order b;*

- *For some* $\kappa \in \mathbb{R}$ *and* $k > 0$,
$$\left|\frac{f(s)(s-a)^b}{s^b}\right| \le k \cdot |1 + \mathrm{Im}(s)|^\kappa$$
for $\mathrm{Re}(s) > a - \epsilon$.

Then for some $\delta > 0$,
$$\int_\delta^T d\alpha = \alpha(T) - \alpha(\delta) = \frac{c}{a(b-1)!} \cdot T^a P(\log T) + O(T^{a-\delta}) \quad \text{as } T \to \infty$$
where $c = \lim_{s\to a}(s-a)^b f(s)$ *and* $P(x)$ *is a monic polynomial of degree* $b-1$.

Proof. This can be proven by repeating the same argument as in the appendix of [CT1] simply replacing the sum $\sum_n n^{-s}\alpha_n$ by the integral $\int_\delta^\infty t^{-s}\,d\alpha(t)$. □

Lemma 3.7.
- *We have*

$$\delta_0 := \inf_{g\in \mathbf{G}(\mathbb{A})} H_\iota(g) > 0. \tag{3.8}$$

- *For each* $T > 0$, B_T *is a relatively compact subset of* $\mathbf{G}(\mathbb{A})$. *In particular, the height function* $H_\iota : \mathbf{G}(\mathbb{A}) \to [\delta_0, \infty)$ *is proper.*

Proof. By Definition 3.2, there exists a finite subset R_0, such that for all $v \in R - R_0$, $H_v(\iota(g)) \ge 1$ for any $g \in \mathbf{G}(K_v)$. Let $0 < \delta \le 1$ be such that $H_v(\iota(g)) \ge \delta$ for $v \in R_0$ and $\delta_0 = \delta^{\#R_0}$. Then $H_\iota(g) \ge \delta_0$ for all $g \in \mathbf{G}(\mathbb{A})$. Note that
$$B_T \subset \mathbf{G}(\mathbb{A}) \cap \prod_v \{g_v \in \mathbf{G}(K_v) : H_v(\iota(g_v)) \le \delta_0^{-1}T\}.$$
Since for almost all $v \in R_f$, $H_v(\iota(g_v)) \ge q_v$ whenever $g_v \notin \mathbf{G}(\mathcal{O}_v)$, it follows that for some finite subset $R_1 \subset R$, we have
$$B_T \subset \{(g_v)_v \in \mathbf{G}(\mathbb{A}) : H_v(\iota(g_v)) \le \delta_0^{-1}T \text{ for } v \in R_1,\ \ g_v \in \mathbf{G}(\mathcal{O}_v) \text{ otherwise}\}$$
and hence B_T is a relatively compact subset of $\mathbf{G}(\mathbb{A})$. □

Theorem 3.9. *Let* $a_\iota \in \mathbb{Q}^+$ *and* $b_\iota \in \mathbb{N}$ *be as in (1.3). Then for any finite subset* $S \subset R$ *and* $g \in \mathbf{G}^S$,
$$\tau^S(B_T \cap g\ker(\Lambda^{W_f}) \cap \mathbf{G}^S) = \frac{\delta_S(g^{-1})}{\#\Lambda^{W_f} \cdot a_\iota(b_\iota - 1)!} \cdot T^{a_\iota} P(\log T) + O(T^{a_\iota - \delta}) \tag{3.10}$$
where the leading term is positive, $P(x)$ *is a monic polynomial of degree* $b_\iota - 1$, *and* $\delta > 0$.

Proof. By the above lemma, B_T is a relatively compact subset of $\mathbf{G}(\mathbb{A})$ and hence $\tau^S(B_T \cap \mathbf{G}^S) < \infty$ for each $T \ge 1$ and for any finite S. Let δ_0 be as in Lemma 3.7,
$$\alpha(t) = \tau^S(B_t \cap g\ker(\Lambda^{W_f}) \cap \mathbf{G}^S) \quad \text{for } t \in [\delta_0, \infty),$$

and

$$f(s) = \int_{\delta_0}^{\infty} t^{-s}\, d\alpha.$$

Then by Lemma 3.1(3),

$$\begin{aligned} f(s) &= \int_{g\ker(\Lambda^{W_f})\cap \mathbf{G}^S} H_\iota(h)^{-s}\, d\tau^S(h) \\ &= (\#\Lambda^{W_f})^{-1} \sum_{\chi\in\Lambda^{W_f}} \int_{\mathbf{G}^S} H_\iota(h)^{-s}\chi(g^{-1}h)\, d\tau^S(h) \\ &= (\#\Lambda^{W_f})^{-1} \sum_{\chi\in\Lambda^{W_f}} \mathcal{Z}^S(s,\chi)\chi(g^{-1}). \end{aligned}$$

Hence, (3.10) follows from Theorems 3.4 and 3.6.

Now we show that the leading term in (3.10) is nonzero. For any $g \in \mathbf{G}(\mathbb{A})$, there exists $c(g) > 0$ such that $gB_T \subset B_{c(g)T}$. Since $[\mathbf{G}^S : (\ker(\Lambda^{W_f}) \cap \mathbf{G}^S)] < \infty$, there exists $c > 0$ such that

$$\begin{aligned} \tau^S(B_T \cap \mathbf{G}^S) &= \sum_{h\in \mathbf{G}^S/(g\ker(\Lambda^{W_f})g^{-1}\cap \mathbf{G}^S)} \tau^S(h^{-1}B_T \cap g\ker(\Lambda^{W_f}) \cap \mathbf{G}^S) \\ &\le [\mathbf{G}^S : (\ker(\Lambda^{W_f}) \cap \mathbf{G}^S)] \cdot \tau^S(B_{cT} \cap g\ker(\Lambda^{W_f}) \cap \mathbf{G}^S). \end{aligned}$$

Since $\mathcal{Z}^S(s,1)$ has a pole of order exactly b_ι at $s = a_\iota$,

$$\lim_{T\to\infty} \frac{\tau^S(B_T \cap \mathbf{G}^S)}{T^{a_\iota}(\log T)^{b_\iota-1}} > 0.$$

Hence,

$$\lim_{T\to\infty} \frac{\tau^S(B_T \cap g\ker(\Lambda^{W_f}) \cap \mathbf{G}^S)}{T^{a_\iota}(\log T)^{b_\iota-1}} > 0,$$

and $\delta_S(g) > 0$. □

Proposition 3.11. *For $g \in \mathbf{G}(\mathbb{A})$ and a co-finite subgroup V_f of W_f,*

$$\tau(B_T \cap g\ker(\Lambda^{W_f})) \sim [\ker(\Lambda^{W_f}) : \ker(\Lambda^{V_f})] \cdot \tau(B_T \cap g\ker(\Lambda^{V_f})).$$

Proof. Note that if $\chi \in \Lambda^{V_f} - \Lambda^{W_f}$ then $\chi(w) \neq 1$ for some $w \in W_f$. Since H_ι is W_f-invariant,

$$\mathcal{Z}^\emptyset(s,\chi) = \int_{\mathbf{G}(\mathbb{A})} H_\iota^{-s}(wg)\chi(wg)\, d\tau(g) = \chi(w)\mathcal{Z}^\emptyset(s,\chi),$$

and hence $c_{\emptyset,\chi} = 0$. Therefore,

$$\sum_{\chi\in\Lambda^{W_f}} c_{\emptyset,\chi}\chi(g) = \sum_{\chi\in\Lambda^{V_f}} c_{\emptyset,\chi}\chi(g),$$

and the claim follows from Theorem 3.9 and Lemma 3.1(2). □

4. Equidistribution for saturated cases

Let $\mathbf{G}$ be a product of connected adjoint absolutely simple groups defined over K and ι be a faithful absolutely irreducible representation of $\mathbf{G}$. Recall the compact spaces X_ι and $X(\mathbb{A})$ defined in Theorems 1.7 and 1.10 respectively. Since the arguments for both spaces are essentially identical, we consider the space X_ι. Without loss of generality, we may consider $\mathbf{G}(K_v)$ as a subset of $X_{\iota,v}$ and $\mathbf{G}(\mathbb{A})$ as a subset of X_ι. Fix a height function $H_\iota = \prod_{v\in R} H_{\iota,v}$ on the associated adele group $\mathbf{G}(\mathbb{A})$ relative to ι as in Definition 3.2.

Setting $H_{\iota,S} = \prod_{v\in S} H_{\iota,v}$, we define

$$m_{\iota,S} := \int_{\mathbf{G}_S} \delta_S(g) H_{\iota,S}(g)^{-a_\iota}\, d\tau_S(g)$$

where δ_S is given in (3.5). By Lemma 3.3, $m_{\iota,S} < \infty$. We also define a probability measure on $\mathbf{G}_S$:

$$\mu_{\iota,S} := m_{\iota,S}^{-1} \cdot \delta_S(g) \cdot H_{\iota,S}(g)^{-a_\iota}\, d\tau_S(g).$$

It gives a probability measure on $X_{\iota,S} = \prod_{v\in S} X_{\iota,v} \supset \bar{\iota}(\mathbf{G}_S)$. Note that $\mu_{\iota,S}$ is given by (1.6) with

$$c_u = \delta_S(u) \cdot m_{\iota,S}^{-1} \cdot \left(\int_{\mathbf{G}_S} H_{\iota,S}(g)^{-a_\iota} d\tau_S(g) \right). \tag{4.1}$$

Lemma 4.2. *For finite sets $S \subset T$ in R, the projection $\mathbf{G}_T \to \mathbf{G}_S$ maps the measure $\mu_{\iota,T}$ to the measure $\mu_{\iota,S}$.*

Proof. It follows from (3.5) that

$$c_{S,\chi} = c_{T,\chi} \left(\int_{\mathbf{G}_{T-S}} H_{\iota,T-S}(h)^{-a_\iota} \chi(h) d\tau_{T-S}(h) \right).$$

Then

$$\begin{aligned}
&\int_{\mathbf{G}_{T-S}} \delta_T(gh) H_{\iota,T}(gh)^{-a_\iota}\, d\tau_{T-S}(h) \\
&= \int_{\mathbf{G}_{T-S}} \sum_{\chi\in\Lambda^{W_f}} c_{T,\chi} \chi(gh) H_{\iota,T}(gh)^{-a_\iota} d\tau_{T-S}(h) \\
&= \sum_{\chi\in\Lambda^{W_f}} c_{T,\chi} \left(\int_{\mathbf{G}_{T-S}} H_{\iota,T-S}(h)^{-a_\iota} \chi(h) d\tau_{T-S}(h) \right) \chi(g) H_{\iota,S}(g)^{-a_\iota} \\
&= \left(\sum_{\chi\in\Lambda^{W_f}} c_{S,\chi}\chi(g) \right) H_{\iota,S}(g)^{-a_\iota} = \delta_S(g) H_{\iota,S}(g)^{-a_\iota}.
\end{aligned}$$

This implies the claim. ☐

Lemma 4.2 implies that there exists a unique probability measure μ_ι on X_ι such that the image of μ_ι under the projection map $X_\iota \to X_{\iota,S}$ is equal to $\mu_{\iota,S}$.

This section is devoted to a proof of the following:

Theorem 4.3. *Suppose that ι is saturated. Then for any $f \in C(X_\iota)$,*

$$\lim_{T\to\infty} \frac{1}{\tau(B_T \cap \mathbf{G}_{H_\iota})} \sum_{g\in \mathbf{G}(K):H_\iota(g)<T} f(g) = \int_{X_\iota} f\, d\mu_\iota,$$

where τ is the Haar measure on $\mathbf{G}_{H_\iota}$ normalized so that $\tau(\mathbf{G}(K)\backslash \mathbf{G}_{H_\iota}) = 1$.

Note that Theorem 4.3 implies Theorem 1.15 (by taking $f = 1$) and Theorems 1.5 and 1.7. The proof of Theorem 1.10 is similar. Combining Theorem 4.3 with Theorem 3.9 and Lemma 4.11, we deduce Theorems 1.2 and 1.9 for the saturated case. The rate of convergence as well as nonsaturated case are discussed in the next section.

For a compact open subgroup W_f of $\mathbf{G}(\mathbb{A}_f)$ under which H_ι is bi-invariant, we consider the subgroup $\ker(\Lambda^{W_f})$ of $\mathbf{G}(\mathbb{A})$ where Λ^{W_f} is the set of all automorphic characters in $L^2(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}))$ which are W_f-invariant, and let τ_{W_f} denote the normalized Haar measure on $\ker(\Lambda^{W_f})$ so that

$$\tau_{W_f}(\mathbf{G}(K)\backslash \ker(\Lambda^{W_f})) = 1.$$

In the following, we fix a compact open subgroup W_f of $\mathbf{G}(\mathbb{A}_f)$ under which H_ι is bi-invariant and set

$$\mathbf{G}_{H_\iota} = \ker(\Lambda^{W_f}) \quad \text{and} \quad \tau = \tau_{W_f}.$$

Lemma 4.4. *For any $f \in C(X_\iota)$,*

$$\lim_{T\to\infty} \frac{1}{\tau(B_T \cap \mathbf{G}_{H_\iota})} \int_{B_T\cap \mathbf{G}_{H_\iota}} f\, d\tau = \int_{X_\iota} f\, d\mu_\iota.$$

Proof. To derive a formula for the Haar measure τ on $\mathbf{G}_{H_\iota}$, we observe that by the weak approximation, $\mathbf{G}_{H_\iota}$ projects onto $\mathbf{G}_S$. Hence,

$$\mathbf{G}_{H_\iota} = \bigcup_{g\in \mathbf{G}_S/(\mathbf{G}_{H_\iota}\cap \mathbf{G}_S)} g(\mathbf{G}_{H_\iota}\cap \mathbf{G}_S)s_g(\mathbf{G}_{H_\iota}\cap \mathbf{G}^S)$$

where

$$g \mapsto s_g : \mathbf{G}_S \to \mathbf{G}^S$$

is map which factors through the finite group $\mathbf{G}_S/(\mathbf{G}_S \cap \mathbf{G}_{H_\iota})$ and $\chi(g) = \chi(s_g^{-1})$ for every $\chi \in \Lambda^{W_f}$. Then one can choose a Haar measure τ^S on $\mathbf{G}^S$ such that

$$\int_{\mathbf{G}_{H_\iota}} f\, d\tau = \int_{g\in \mathbf{G}_S}\int_{h\in \mathbf{G}^S\cap \mathbf{G}_{H_\iota}} f(gs_gh)\; d\tau^S(h)\, d\tau_S(g). \tag{4.5}$$

for any $f \in C_c(\mathbf{G}_{H_\iota})$.

First, we prove the lemma for a function $f \in C(X_\iota)$ that factors through $X_{\iota,S}$ for some finite $S \subset R$, i.e.,

$$f((x_v)_{v\in R}) := f_S((x_v)_{v\in S}) \quad \text{for some } f_S \in C(X_{\iota,S}).$$

It follows from (4.5) that

$$\int_{B_T\cap \mathbf{G}_{H_\iota}} f\, d\tau = \int_{\mathbf{G}_S} f_S(g)\tau^S(B_{T\cdot H_{\iota,S}(g)^{-1}} \cap s_g\mathbf{G}_{H_\iota} \cap \mathbf{G}^S)\, d\tau_S(h). \tag{4.6}$$

By Theorem 3.9, there exists a constant $c > 0$ such that for any $g \in \mathbf{G}_S$ with $H_{\iota,S}(g) \le T/2$,

$$\tau^S(B_{T\cdot H_{\iota,S}(g)^{-1}} \cap s_g\mathbf{G}_{H_\iota} \cap \mathbf{G}^S) \le cH_{\iota,S}(g)^{-a_\iota}T^{a_\iota}(\log TH_{\iota,S}(g)^{-1})^{b_\iota -1}.$$

With δ_0 as in Lemma 3.7, there exists $d > 0$ such that for any $g \in \mathbf{G}_S$ satisfying $T/2 \le H_{\iota,S}(g) \le T\delta_0^{-1}$,

$$\tau^S(B_{T\cdot H_{\iota,S}(g)^{-1}} \cap s_g\mathbf{G}_{H_\iota} \cap \mathbf{G}^S) \le \tau^S(B_2 \cap s_g\mathbf{G}_{H_\iota} \cap \mathbf{G}^S) \le d\cdot H_{\iota,S}(g)^{-a_\iota}T^{a_\iota}.$$

Also, for $H_{\iota,S}(g) > T\delta_0^{-1}$, the above volume is zero.

Setting

$$y_T(g) := \frac{\tau^S(B_{T\cdot H_{\iota,S}(g)^{-1}} \cap s_g\mathbf{G}_{H_\iota} \cap \mathbf{G}^S)}{\tau^S(B_T \cap \mathbf{G}_{H_\iota} \cap \mathbf{G}^S)},$$

we deduce that for some constant $C > 0$,

$$y_T(g) \le C\cdot H_{\iota,S}(g)^{-a_\iota} \quad \text{for any } g \in \mathbf{G}_S.$$

In particular, y_T is in $L^1(\mathbf{G}_S)$ by Lemma 3.3. Hence, by Theorem 3.9,

$$y_T(g) \to \delta_S(s_g^{-1})H_{\iota,S}(g)^{-a_\iota}\delta_S(e)^{-1} \quad \text{as } T\to\infty.$$

Since $\delta_S(s_g^{-1}) = \delta(g)$, we apply the dominated convergence theorem to (4.6) and deduce that

$$\int_{B_T\cap \mathbf{G}_{H_\iota}} f\, d\tau \sim \mu_{\iota,S}(f_S)\cdot m_{\iota,S}\cdot \delta_S(e)^{-1}\cdot \tau^S(B_T\cap \mathbf{G}_{H_\iota}\cap \mathbf{G}^S) \quad \text{as } T\to\infty. \tag{4.7}$$

Note that $\mu_{\iota,S}(f_S) = \mu_\iota(f)$. Taking $f = 1$, we also get

$$\tau(B_T\cap \mathbf{G}_{H_\iota}) \sim m_{\iota,S}\cdot \delta_S(e)^{-1}\cdot \tau^S(B_T\cap \mathbf{G}_{H_\iota}\cap \mathbf{G}^S) \quad \text{as } T\to\infty. \tag{4.8}$$

Therefore, for any f that factors through $X_{\iota,S}$,

$$\int_{B_T\cap \mathbf{G}_{H_\iota}} f\, d\tau \sim \mu_\iota(f)\cdot \tau(B_T\cap \mathbf{G}_{H_\iota}) \quad \text{as } T\to\infty. \tag{4.9}$$

Let $f \in C(X_\iota)$. Fix any $\epsilon > 0$. We can find a finite subset $S \subset R$ and continuous functions f^+, f^- that factor through $X_{\iota,S}$ such that

$$f^- \le f \le f^+ \quad \text{and} \quad \|f^+ - f^-\|_\infty < \epsilon.$$

By (4.9),

$$\int_{B_T \cap \mathbf{G}_{H_\iota}} f^{\pm}\, d\tau \sim \mu_\iota(f^{\pm}) \cdot \tau(B_T \cap \mathbf{G}_{H_\iota}) \quad \text{as } T \to \infty.$$

Since $\mu(f^+ - f^-) \le \epsilon$ and $\epsilon > 0$ is arbitrary, it is easy to deduce that

$$\int_{B_T \cap \mathbf{G}_{H_\iota}} f\, d\tau \sim \mu_\iota(f) \cdot \tau(B_T \cap \mathbf{G}_{H_\iota}) \quad \text{as } T \to \infty.$$

This finishes the proof. □

Corollary 4.10. *Let V_f be a compact open subgroup of $\mathbf{G}(\mathbb{A}_f)$ under which H_ι is bi-invariant. Then*

(1) $\tau(B_T \cap \mathbf{G}_{H_\iota}) \sim_{T\to\infty} \tau_{V_f}(B_T \cap \ker(\Lambda^{V_f}))$;
(2) *for any* $f \in C(X_\iota)$,

$$\int_{B_T \cap \mathbf{G}_{H_\iota}} f\, d\tau \sim_{T\to\infty} \int_{B_T \cap \ker(\Lambda^{V_f})} f\, d\tau_{V_f}.$$

Proof. It suffices to prove the claim for the case when $V_f \subset W_f$. Since the restriction of τ to $\ker(\Lambda^{V_f})$ is equal to $[\ker(\Lambda^{W_f}) : \ker(\Lambda^{V_f})] \cdot \tau_{V_f}$, the first claim follows from Proposition 3.11. The second claim follows from the claim (1) and Lemma 4.4. □

Lemma 4.11. *In the notation of Theorem 1.9, let $\mathcal{L}$ and $\mathcal{L}'$ be metrizations of the line bundle L and (B_T, W_f), and (B'_T, W'_f) be defined as above with respect to $\mathcal{L}$ and $\mathcal{L}'$ respectively. Then*

$$\lim_{T\to\infty} \frac{\tau(B_T \cap \ker(\Lambda^{W_f}))}{\tau(B'_T \cap \ker(\Lambda^{W'_f}))} = \frac{[\mathbf{G}(\mathbb{A}) : \ker(\Lambda^{W_f})] \cdot \tau_{\mathcal{L}}(\mathbf{G}(\mathbb{A}))}{[\mathbf{G}(\mathbb{A}) : \ker(\Lambda^{W'_f})] \cdot \tau_{\mathcal{L}'}(\mathbf{G}(\mathbb{A}))}$$

where τ is a Haar measure on $\mathbf{G}(\mathbb{A})$.

Proof. Let $V_f = W_f \cap W'_f$. By Proposition 3.11, it suffices to show that

$$\lim_{T\to\infty} \frac{\tau(B_T \cap \ker(\Lambda^{V_f}))}{\tau(B'_T \cap \ker(\Lambda^{V_f}))} = \frac{\tau_{\mathcal{L}}(\mathbf{G}(\mathbb{A}))}{\tau_{\mathcal{L}'}(\mathbf{G}(\mathbb{A}))}. \tag{4.12}$$

Let S be a finite set such that $H_{\mathcal{L},v} = H_{\mathcal{L}',v}$ for every $v \in R - S$. If we set $H_{\mathcal{L},S} = \prod_{v\in S} H_{\mathcal{L},v}$, then it follows from (1.8) and Theorem 3.4 that

$$\frac{\tau_{\mathcal{L}}(\mathbf{G}(\mathbb{A}))}{\tau_{\mathcal{L}'}(\mathbf{G}(\mathbb{A}))} = \frac{\int_{\mathbf{G}_S} H_{\mathcal{L},S}(g)^{-a_L} \delta_S(g)\, d\tau_S}{\int_{\mathbf{G}_S} H_{\mathcal{L}',S}(g)^{-a_L} \delta_S(g)\, d\tau_S}. \tag{4.13}$$

Theorem 3.9 with $S = \emptyset$ and (4.13) imply that the both sides of (4.12) stay the same when $H_{\mathcal{L},S}$ and $H_{\mathcal{L}',S}$ are replaced by constant multiples. Hence, we can assume that

$$H_{\mathcal{L},S}(e) = H_{\mathcal{L}',S}(e) = 1. \tag{4.14}$$

As in the proof of Lemma 4.4, we obtain

$$\begin{aligned}\tau(B_T \cap \ker(\Lambda^{V_f})) &= \int_{g\in \mathbf{G}_S} \tau^S(B_{TH_{\mathcal{L},S}(g)^{-1}} \cap s_g \ker(\Lambda^{V_f}) \cap \mathbf{G}^S)\; d\tau_S(g) \\ &\sim \left(\int_{g\in\mathbf{G}_S} H_{\mathcal{L},S}(g)^{-a_L}\delta_S(g) d\tau_S(g)\right)\cdot \tau^S(B_T\cap \ker(\Lambda^{V_f})\cap \mathbf{G}^S).\end{aligned}$$

Similarly,

$$\tau(B'_T \cap \ker(\Lambda^{V_f})) \sim \left(\int_{g\in\mathbf{G}_S} H_{\mathcal{L}',S}(g)^{-a_L}\delta_S(g) d\tau_S(g)\right)\cdot \tau^S(B'_T\cap \ker(\Lambda^{V_f})\cap \mathbf{G}^S).$$

Since by (4.14),

$$B_T\cap \mathbf{G}^S = B'_T \cap \mathbf{G}^S,$$

this finishes the proof. □

For a fixed $f\in C(X_\iota)$, we define a function F_T on $\mathbf{G}_{H_\iota}\times \mathbf{G}_{H_\iota}$ by

$$F_T(g,h) = \sum_{\gamma\in\mathbf{G}(K)} f(g^{-1}\gamma h)\cdot \chi_{B_T}(g^{-1}\gamma h)$$

Clearly F_T is well defined as a function on $Y\times Y$ where $Y=\mathbf{G}(K)\backslash \mathbf{G}_{H_\iota}$.

Note that

$$F_T(e,e) = \sum_{\gamma\in\mathbf{G}(K): H_\iota(\gamma)\le T} f(\gamma).$$

Proposition 4.15 (Weak-convergence)**.** *Suppose that ι is saturated. Let $f\in C(X_\iota)$. For $i=1,2$, let $\alpha_i\in C(Y)$ be a W_f-invariant function and $\int_Y \alpha_i\, d\tau = 1$. If $\alpha(x,y) := \alpha_1(x)\alpha_2(y)$, then*

$$\lim_{T\to\infty}\frac{1}{\tau(B_T\cap\mathbf{G}_{H_\iota})}\int_{Y\times Y} F_T\cdot\alpha\; d(\tau\times\tau) = \lim_{T\to\infty}\frac{1}{\tau(B_T\cap\mathbf{G}_{H_\iota})}\int_{B_T\cap\mathbf{G}_{H_\iota}} f\, d\tau.$$

Proof. Observe that

(4.16)

$$\begin{aligned}\langle F_T,\alpha\rangle_{Y\times Y} &= \int_{x\in Y}\int_{y\in Y}\left(\sum_{\gamma\in\mathbf{G}(K)} f(x^{-1}\gamma y)\chi_{B_T}(x^{-1}\gamma y)\right)\alpha_1(x)\alpha_2(y)\, d\tau(y)d\tau(x) \\ &= \int_{x\in Y}\int_{h\in\mathbf{G}_{H_\iota}} f(x^{-1}h)\chi_{B_T}(x^{-1}h)\alpha_1(x)\alpha_2(h)\, d\tau(h)d\tau(x) \\ &= \int_{g\in\mathbf{G}_{H_\iota}} f(g)\chi_{B_T}(g)\left(\int_{x\in Y}\alpha_1(x)\alpha_2(xg)\, d\tau(x)\right) d\tau(g) \\ &= \int_{g\in B_T\cap\mathbf{G}_{H_\iota}} f(g)\langle\alpha_1, g\cdot\alpha_2\rangle\, d\tau(g)\end{aligned}$$

Write $\mathbf{G} = \mathbf{G}_1 \cdots \mathbf{G}_m$ as a product of connected absolutely simple K-groups. Since the height function H_ι is proper, we have that $g \to \infty$ strongly in $\mathbf{G}(\mathbb{A})$ if and only if $H_\iota(g_i) \to \infty$ for each $i = 1, \dots, m$ where $g = g_1 \cdots g_m$, $g_i \in \mathbf{G}_i(\mathbb{A})$.

For $C > 0$, define

$$B^C := \{g_1 \cdots g_m \in \mathbf{G}(\mathbb{A}) : H_\iota(g_i) > C \text{ for each } i = 1, \cdots, m\}.$$

Note that $\alpha_1 - 1, \alpha_2 - 1 \in L^2_{00}(Y)$. Hence, by Theorem 2.23, for any given $\epsilon > 0$, there exists $C > 0$ such that

$$|\langle \alpha_1, g \cdot \alpha_2 \rangle - 1| = |\langle \alpha_1 - 1, g \cdot (\alpha_2 - 1) \rangle| < \epsilon \quad \text{for all } g \in B^C. \tag{4.17}$$

Hence

$$\left| \int_{g \in B_T \cap \mathbf{G}_{H_\iota}} f(g) \langle \alpha_1, g.\alpha_2 \rangle \, d\tau(g) - \int_{g \in B_T \cap \mathbf{G}_{H_\iota}} f(g) \, d\tau(g) \right|$$
$$< \max f \cdot (\|\alpha_1\| \cdot \|\alpha_2\| + 1) \cdot \tau((B_T - B^C) \cap \mathbf{G}_{H_\iota}) + \max f \cdot \tau(B_T \cap B^C \cap \mathbf{G}_{H_\iota}) \cdot \epsilon.$$

Using that ι is saturated, we deduce that

$$\lim_{T \to \infty} \frac{\tau((B_T - B^C) \cap \mathbf{G}_{H_\iota})}{\tau(B_T \cap \mathbf{G}_{H_\iota})} = 0.$$

Since $\epsilon > 0$ is arbitrary, by (4.16), this proves the claim. □

By Lemma 4.4, the following theorem implies Theorem 4.3:

Theorem 4.18. *Suppose that ι is saturated. For any $f \in C(X_\iota)$, we have*

$$\lim_{T \to \infty} \frac{1}{\tau(B_T \cap \mathbf{G}_{H_\iota})} \sum_{g \in \mathbf{G}(K) : H_\iota(g) < T} f(g) = \lim_{T \to \infty} \frac{1}{\tau(B_T \cap \mathbf{G}_{H_\iota})} \int_{B_T \cap \mathbf{G}_{H_\iota}} f \, d\tau.$$

Proof. Recall that $\mathbf{G}_{H_\iota} = \ker(\Lambda^{W_f})$ and $\tau = \tau_{W_f}$. It suffices to prove our theorem for non-negative functions $f \in C(X_\iota)$. Fix $\epsilon > 0$. Let W_∞ be a symmetric neighborhood of e in $\mathbf{G}^\circ_\infty$ such that

$$W_\infty B_T W_\infty \subset B_{(1+\epsilon)T} \quad \text{and} \quad B_{(1-\epsilon)T} \subset \cap_{g,h \in W_\infty} h B_T g \quad \text{for all } T > 1.$$

By uniform continuity, there exists a cofinite subgroup V_f of W_f such that

$$f(g^{-1}xh) - \epsilon \le f(x) \le f(g^{-1}xh) + \epsilon \quad \text{for all } x \in X \text{ and } g, h \in V_f. \tag{4.19}$$

Replacing W_∞ by a smaller one if necessary, we may assume that (4.19) holds for any $g, h \in W$ where $W := W_\infty \times V_f$. It follows that for any $T > \delta_0$ and for any $g, h \in W$,

$$F^-_{(1-\epsilon)T}(g, h) \le F_T(e, e) \le F^+_{(1+\epsilon)T}(g, h) \tag{4.20}$$

where

$$F^\pm_T(g, h) = \sum_{\gamma \in \mathbf{G}(K)} (f(g^{-1}\gamma h) \pm \epsilon) \cdot \chi_{B_T}(g^{-1}\gamma h).$$

Set $Y = \mathbf{G}(K)\backslash \ker(\Lambda^{V_f})$. Now let $\psi \in C_c(Y)$ be a non-negative V_f-invariant function such that $\mathrm{supp}(\psi) \subset \mathbf{G}(K)\backslash\mathbf{G}(K)W$ and $\int_Y \psi\, d\tau_{V_f} = 1$. By integrating (4.20) over $Y \times Y$ against the function $\alpha(x,y) = \psi(x)\cdot\psi(y)$, we obtain

$$\langle F^-_{(1-\epsilon)T}, \alpha\rangle \le F_T(e,e) \le \langle F^+_{(1+\epsilon)T}, \alpha\rangle.$$

Note that Theorem 3.9 implies the following: there exist constants $a_\epsilon \ge 1$ and $b_\epsilon \le 1$ tending to 1 as $\epsilon \to 0$ such that for all sufficiently small $\epsilon > 0$,

$$b_\epsilon \le \liminf_T \frac{\tau_{V_f}(B_{(1-\epsilon)T} \cap \ker(\Lambda^{V_f}))}{\tau_{V_f}(B_T \cap \ker(\Lambda^{V_f}))} \le \limsup_T \frac{\tau_{V_f}(B_{(1+\epsilon)T} \cap \ker(\Lambda^{V_f}))}{\tau_{V_f}(B_T \cap \ker(\Lambda^{V_f}))} \le a_\epsilon. \tag{4.21}$$

Hence by applying Proposition 4.15,

$$\begin{aligned}
&\limsup_T \frac{F_T(e,e)}{\tau_{V_f}(B_T \cap \ker(\Lambda^{V_f}))} \le \limsup_T \frac{\langle F^+_{(1+\epsilon)T}, \alpha\rangle}{\tau_{V_f}(B_T \cap \ker(\Lambda^{V_f}))}\\
&\le \limsup_T \frac{\langle F^+_{(1+\epsilon)T}, \alpha\rangle}{\tau_{V_f}(B_{(1+\epsilon)T} \cap \ker(\Lambda^{V_f}))} \cdot \limsup_T \frac{\tau_{V_f}(B_{(1+\epsilon)T} \cap \ker(\Lambda^{V_f}))}{\tau_{V_f}(B_T \cap \ker(\Lambda^{V_f}))}\\
&\le a_\epsilon \cdot \limsup_T \frac{\int_{B_T\cap\ker(\Lambda^{V_f})} (f+\epsilon)d\tau}{\tau_{V_f}(B_T \cap \ker(\Lambda^{V_f}))} \le a_\epsilon \cdot \left(\limsup_T \frac{\int_{B_T\cap\ker(\Lambda^{V_f})} f d\tau}{\tau_{V_f}(B_T \cap \ker(\Lambda^{V_f}))} + \epsilon \right)\\
&\le a_\epsilon \cdot \left(\limsup_T \frac{\int_{B_T\cap\ker(\Lambda^{W_f})} f d\tau}{\tau(B_T \cap \ker(\Lambda^{W_f}))} + \epsilon \right)
\end{aligned}$$

by Corollary 4.10, and similarly,

$$b_\epsilon \cdot \left(\liminf_T \frac{\int_{B_T\cap\ker(\Lambda^{W_f})} f d\tau}{\tau(B_T \cap \ker(\Lambda^{W_f}))} - \epsilon \right) \le \liminf_T \frac{F_T(e,e)}{\tau_{V_f}(B_T \cap \ker(\Lambda^{V_f}))}.$$

Taking $\epsilon \to 0$, this implies by Corollary 4.10 that

$$\lim_T \frac{F_T(e,e)}{\tau(B_T \cap \mathbf{G}_{H_\iota})} = \lim_T \frac{\int_{B_T\cap\mathbf{G}_{H_\iota}} f\, d\tau}{\tau(B_T \cap \mathbf{G}_{H_\iota})}.$$

□

To derive the asymptotic formula for the number of K-rational points, it suffices to take $f = 1$ in Theorem 4.18. In this cases the above computation simplifies significantly, and it applies to general families of balls B_T, which we presently introduce.

For an increasing sequence $\{B_T\}$ of relatively compact subsets of $\mathbf{G}(\mathbb{A})$ and a compact open subgroup $W_f \subset \mathbf{G}(\mathbb{A}_f)$, we call $\{B_T\}$ W_f*-well rounded* if the following holds:

(1) $W_f B_T W_f = B_T$ for any $T > 1$;

(2) for any small $\epsilon > 0$, there exists a neighborhood $W_\epsilon \subset \mathbf{G}_\infty^\circ$ of e such that

$$W_\epsilon B_T W_\epsilon \subset B_{(1+\epsilon)T} \quad \text{and} \quad B_{(1-\epsilon)T} \subset \cap_{g,h\in W_\epsilon} g B_T h$$

for all $T > 1$;

(3) $\tau_{W_f}(B_T \cap \ker(\Lambda^{W_f})) \to \infty$ as $T \to \infty$ and there exist constants $a_\epsilon \geq 1$ and $b_\epsilon \leq 1$ tending to 1 as $\epsilon \to 0$ such that for all sufficiently small $\epsilon > 0$,

$$b_\epsilon \leq \liminf_T \frac{\tau_{W_f}(B_{(1-\epsilon)T} \cap \ker(\Lambda^{W_f}))}{\tau_{W_f}(B_T \cap \ker(\Lambda^{W_f}))} \leq \limsup_T \frac{\tau_{W_f}(B_{(1+\epsilon)T} \cap \ker(\Lambda^{W_f}))}{\tau_{W_f}(B_T \cap \ker(\Lambda^{W_f}))} \leq a_\epsilon.$$

The proof of Theorem 4.18 gives

Proposition 4.22. *Suppose that* $\mathbf{G}$ *is absolutely almost simple. Let* W_f *be a compact open subgroup of* $\mathbf{G}(\mathbb{A}_f)$*. Then for any* W_f*-well rounded sequence* $\{B_T\}$ *of relatively compact subsets of* $\mathbf{G}(\mathbb{A})$*,*

$$\#\mathbf{G}(K) \cap B_T \sim_{T\to\infty} \tau_{W_f}(B_T \cap \ker(\Lambda^{W_f})).$$

Proposition 4.22 can be used, in particular, to compute the asymptotics of integral points. Let S be a finite subset of R containing R_∞ and

$$B_T = \{g = (g_v) \in \mathbf{G}(\mathbb{A}) :\ H_\iota(g) < T,\ g_v \in \mathbf{G}(\mathcal{O}_v) \text{ for } v \notin S\}.$$

Then

$$\mathbf{G}(K) \cap B_T = \{g \in \mathbf{G}(\mathcal{O}_S) :\ H_\iota(g) < T\}$$

where $\mathcal{O}_S$ denotes the ring of S-integers. Clearly, the sequence $\{B_T\}$ satisfies properties (1) and (2) for a suitable open subgroup W_f. Verification of (3) reduces to the computation of the asymptotics of the volume $\tau_{W_f}(B_T \cap \ker(\Lambda^{W_f}))$ as $T \to \infty$. We refer to [Ba] where some results on asymptotics of the number of S-integral points were obtained.

We also state a version of Proposition 4.22 with error term, which follows from the proof of Proposition 5.5 below. A proper function $H : \mathbf{G}(\mathbb{A}) \to \mathbb{R}^+$ is called *W_f-well rounded* if the following properties hold:

(1) H is bi-W_f-invariant;

(2) there exists $b > 0$ such that for any small $\epsilon > 0$,

$$(1 - b\epsilon)H(x) \leq H(gxh) \leq (1 + b\epsilon)H(x) \quad \text{for any } g, h \in W_\epsilon \text{ and } x \in \mathbf{G}(\mathbb{A}),$$

where W_ϵ denotes the Riemannian ball at e of radius ϵ in $\mathbf{G}_\infty$;

(3) there exist $a = a(H) > 0$ and $b = b(H) \geq 1$ such that the associated zeta function

$$\mathcal{Z}(s) := \sum_{\chi\in\Lambda^{W_f}} \int_{\mathbf{G}(\mathbb{A})} H(g)^{-s}\chi(g)\, d\tau(g)$$

has a meromorphic continuation to $\mathrm{Re}(s) > a - \epsilon$ with the unique pole at $s = a$ of order b and of positive residue, and for some constants $\kappa \in \mathbb{R}$ and

$k > 0$,

$$\left|\frac{(s-a)^b \mathcal{Z}(s)}{s^b}\right| \le k \cdot |1 + \mathrm{Im}(s)|^\kappa$$

for $\mathrm{Re}(s) > a - \epsilon$.

Proposition 4.23. *Suppose that* $\mathbf{G}$ *is absolutely almost simple. Let* W_f *be a compact open subgroup of* $\mathbf{G}(\mathbb{A}_f)$*. Then for any* W_f*-well rounded function* $H : \mathbf{G}(\mathbb{A}) \to \mathbb{R}^+$,

$$\#\{g \in \mathbf{G}(K) : H(g) < T\} \sim_{T\to\infty} T^{a(H)} P(\log T) + T^{a(H)-\delta}$$

where $P(x)$ *is a polynomial of degree* $b(H) - 1$ *and* $\delta > 0$.

5. Arithmetic fibrations

In this section we prove Theorems 1.2 and 1.9 for a general case, that is, without the saturation assumption on ι. Let $\mathbf{G}$, ι and H_ι be as in the beginning of Section 4. Fix a compact open subgroup W_f of $\mathbf{G}(\mathbb{A}_f)$ under which H_ι is bi-invariant.

Let $\mathbf{M}$ be the smallest connected normal K-subgroup of $\mathbf{G}$ whose root system contains the set

$$\{\alpha \in \Delta : \frac{u_\alpha + 1}{m_\alpha} = a_\iota\}.$$

Note that ι is saturated if and only if $\mathbf{M} = \mathbf{G}$. There exists a connected normal K-subgroup $\mathbf{N}$ of $\mathbf{G}$ so that $\mathbf{G} = \mathbf{MN}$ and $\mathbf{M} \cap \mathbf{N} = \{e\}$. Let $\pi : \mathbf{G} \to \mathbf{N}$ be the canonical projection. Note that any element of $\mathbf{G}(\mathbb{A})$ can be uniquely written as $g_1 g_2$ with $g_1 \in \mathbf{M}(\mathbb{A})$ and $g_2 \in \mathbf{N}(\mathbb{A})$.

As is well known, the restriction $\iota|_{\mathbf{M}}$ of ι is a direct sum of finitely many copies of an irreducible representation, say, ι', of $\mathbf{M}$ with the highest weight given by the restriction of that of ι. The definition of $\mathbf{M}$ implies that ι' is saturated, $a_{\iota'} = a_\iota$ and $b_{\iota'} = b_\iota$. For $x \in \mathbf{N}(K)$, the function $g \mapsto H_\iota(gx)$ defines a height function on $\mathbf{M}(\mathbb{A})$ as in the definition 3.2. Hence Theorem 1.2 for the saturated cases implies that for each $x \in \mathbf{N}(K)$,

$$N_{\pi^{-1}(x)}(H_\iota, T) = \#\{g \in \mathbf{M}(K) : H_\iota(gx) < T\} \sim c_x \cdot T^{a_\iota} (\log T)^{b_\iota - 1} \tag{5.1}$$

for some $c_x > 0$.

Noting that $N(H_\iota, T) = \sum_{x \in \mathbf{N}(K)} N_{\pi^{-1}(x)}(H_\iota, T)$, we prove the following:

Theorem 5.2. *Under the assumptions of Theorem 1.2, we have for some* $\delta > 0$,

$$N(H_\iota, T) = c(H_\iota) \cdot T^{a_\iota} (\log T)^{b_\iota - 1} (1 + O((\log T)^{-\delta})) \tag{5.3}$$

where $c(H_\iota) := \sum_{x \in \mathbf{N}(K)} c_x < \infty$.

The rest of this section is devoted to the proof of the above theorem.

Lemma 5.4. *Let* $\mathbf{G}_1$ *and* $\mathbf{G}_2$ *be normal algebraic* K*-subgroups of* $\mathbf{G}$ *with* $\mathbf{G} = \mathbf{G}_1\mathbf{G}_2$ *and* $\mathbf{G}_1 \cap \mathbf{G}_2 = \{e\}$*. There exists* $\kappa > 1$ *such that for any* $g_1 \in \mathbf{G}_1(\mathbb{A})$ *and* $g_2 \in \mathbf{G}_2(\mathbb{A})$,

$$\kappa^{-1} \cdot H_\iota(g_1) H_\iota(g_2) \le H_\iota(g_1 g_2) \le \kappa \cdot H_\iota(g_1) H_\iota(g_2).$$

Proof. Let χ denote the highest weight of ι. Then there exists a finite subset $S \subset R$ such that for any $v \in R - S$,

$$\mathbf{G}(K_v) = U_v A_v^+ U_v \quad \text{and} \quad H_v(\iota(g)) = \chi(a) \quad \text{for } g = u_1 a u_2 \in \mathbf{G}(K_v)$$

where U_v and A_v^+ are defined as in Section 2. In particular, it follows that for each $v \in R - S$, and for any $g_1 \in \mathbf{G}_1(K_v)$ and $g_2 \in \mathbf{G}_2(K_v)$,

$$H_v(\iota(g_1 g_2)) = H_v(\iota(g_1)) H_v(\iota(g_2)).$$

On the other hand, for $v \in S$, $H_{\iota,v}$ is equivalent to χ in the sense that there exists $\kappa_v > 1$ such that

$$\kappa_v^{-1} \cdot \chi(a) \le H_{\iota,v}(g) \le \kappa_v \cdot \chi(a) \quad \text{for } g = u_1 a d u_2 \in U_v A_v^+ \Omega_v U_v = \mathbf{G}(K_v).$$

This implies the lemma. □

A key ingredient in deducing Theorem 5.2 is the following stronger version of (5.1):

Proposition 5.5. *There exist $\beta, \delta, d > 0$ such that for each $x \in \mathbf{N}(K)$ and for any $T \ge \beta \cdot H_\iota(x)$,*

$$|N_{\pi^{-1}(x)}(H_\iota, T) - c_x \cdot T^{a_\iota} (\log T)^{b_\iota - 1}| \le d \cdot d_x \cdot T^{a_\iota} (\log T)^{b_\iota - 1 - \delta}$$

where $d_x = H_\iota(x)^{-a_\iota} (\log H_\iota(x))^{b_\iota - 1}$. Moreover $c_x = O(H_\iota(x)^{-a_\iota})$.

Proof. Let $\delta_0 > 0$ be such that $H_\iota(g) \ge \delta_0$ for $g \in \mathbf{G}(\mathbb{A})$ (Lemma 3.7). For each $x \in \mathbf{N}(\mathbb{A})$, we define a function H_ι^x on $\mathbf{M}(\mathbb{A})$ by

$$H_\iota^x(g) := H_\iota(gx), \qquad g \in \mathbf{M}(\mathbb{A}).$$

It is easy to see that H_ι^x is a height function as in the definition 3.2 with respect to the representation ι'. Set

$$B_T^x = \{g \in \mathbf{M}(\mathbb{A}) : \, H_\iota^x(g) < T\}.$$

Since x commutes with $\mathbf{M}(\mathbb{A})$, the group $\mathbf{M}_{H_\iota^x}$ is independent of x, and we denote it by $\mathbf{M}_{H_\iota}$. Let $Y = \mathbf{M}(K) \backslash \mathbf{M}_{H_\iota}$ and τ be the invariant probability measure on Y. For each $x \in \mathbf{N}(K)$, set

$$F_T^x(g, h) := \sum_{\gamma \in \mathbf{M}(K)} \chi_{B_T^x}(g^{-1} \gamma h), \quad g, h \in \mathbf{M}(\mathbb{A}).$$

We may consider F_T^x as a function on $Y \times Y$. Write $\mathbf{M} = \mathbf{M}_1 \cdots \mathbf{M}_r$ as a product of connected absolutely simple K-groups. For a collection of smooth $(W_f \cap \mathbf{M}_i(\mathbb{A}))$-invariant functions $\psi_i \in C_c(\mathbf{M}_i(K) \backslash \mathbf{M}_i(\mathbb{A}) \cap \mathbf{M}_{H_\iota})$, $1 \le i \le r$, define $\psi \in C_c(Y)$ and $\alpha \in C_c(Y \times Y)$ by

$$\psi(z_1, \cdots, z_r) := \prod_{i=1}^r \psi_i(z_i) \quad \text{and} \quad \alpha(y_1, y_2) := \psi(y_1) \psi(y_2).$$

Assume that $\int \psi_i \, d\tau_i = 1$ for each i where τ_i is the invariant probability measure on $\mathbf{M}_i(K)\backslash \mathbf{M}_i(\mathbb{A}) \cap \mathbf{M}_{H_\iota}$. We claim that for a sufficiently large $l \in \mathbb{N}$, independent of x, we have for any $x \in \mathbf{N}(K)$,

$$\langle F_T^x, \alpha \rangle_{Y\times Y} = c_x \cdot T^{a_\iota} (\log T)^{b_\iota - 1} + O(d_x \cdot C'_\psi \cdot T^{a_\iota} (\log T)^{b_\iota - 1 - \delta}).$$

where $C'_\psi = \max(1, \max_i \|\mathcal{D}^l \psi_i\|^{2r})$ for some large l and $\mathcal{D}$ is the elliptic operator defined in (2.21).

As in the proof of Proposition 4.15, we derive that

$$\langle F_T^x, \alpha \rangle = \int_{g \in B_T^x \cap \mathbf{M}_{H_\iota}} \langle \psi, g.\psi \rangle \; d\tau(g)$$

Note that

$$\begin{aligned} |\, \langle \psi, g.\psi \rangle - 1| &= |\prod_{i=1}^{r} \langle \psi_i, g_i.\psi_i \rangle - 1| \\ &= |\sum_{i=1}^{r} (\prod_{j=1}^{i-1} \langle \psi_j, g_j.\psi_j \rangle)(\langle \psi_i, g_i.\psi_i \rangle - 1)| \\ &\le r \cdot C_\psi \cdot \max_i |\langle \psi_i, g_i.\psi_i \rangle - 1| \\ &= r \cdot C_\psi \cdot \max_i |\langle \psi_i - 1, g_i.(\psi_i - 1) \rangle| \end{aligned}$$

where $C_\psi = \max(1, \max_i \|\psi_i\|^{2r-2})$. Since $\psi_i - 1 \in L^2_{00}(\mathbf{M}_i(K)\backslash \mathbf{M}_i(\mathbb{A}) \cap \mathbf{M}_{H_\iota})$ for each i, we deduce from Theorem 2.22 that

(5.6)

$$|\langle F_T^x, \alpha \rangle - \tau(B_T^x \cap \mathbf{M}_{H_\iota})| \le 2r \cdot (\prod_i c_{W_f \cap \mathbf{M}_i(\mathbb{A})}) \cdot C'_\psi \cdot \int_{g = g_1 \cdots g_r \in B_T^x \cap \mathbf{M}_{H_\iota}} \max_i \tilde{\xi}_{\mathbf{M}_i}(g_i) \, d\tau(g)$$

where $C'_\psi = \max(1, \max_i \|\mathcal{D}^l \psi_i\|^{2r})$ for some large l.

Since $\tilde{\xi}_{\mathbf{M}_i} \le {\xi_{\mathbf{M}_i}}^{1/2}$, it follows from Lemma 2.6 that there exist $m \in \mathbb{N}$ and $C_1 > 0$ such that for any $1 \le i \le r$,

$$\tilde{\xi}_{\mathbf{M}_i}(g_i) < C_1 \cdot H_\iota(g_i)^{-1/m} \quad \text{for any } g_i \in \mathbf{M}_i(\mathbb{A}).$$

Define a function on $\mathrm{M}(\mathbb{A})$ by

$$\tilde{H}(g_1 \cdots g_r) := \min_i H_\iota(g_i), \quad g_i \in \mathbf{M}_i(\mathbb{A}).$$

Let κ be as in Lemma 5.4 for $\mathbf{G}_1 = \mathbf{M}$ and $\mathbf{G}_2 = \mathbf{N}$ so that $B_T^x \subset B_{\kappa T \cdot H_\iota(x)^{-1}}$. It then follows from (5.6) that

$$|\langle F_T^x, \alpha \rangle - \tau(B_T^x \cap \mathbf{M}_{H_\iota})| < C_2 \cdot C'_\psi \cdot \int_{B_{\kappa T \cdot H_\iota(x)^{-1}} \cap \mathbf{M}_{H_\iota}} \tilde{H}(g)^{-1/m} \, d\tau(g) \tag{5.7}$$

for a constant $C_2 > 0$ independent of x.

Since ι' is saturated, for every proper normal K-subgroup $\mathbf{L}$ of $\mathbf{M}$,

$$\tau_{\mathbf{L}}(B_T \cap \mathbf{L}_{H_\iota}) \ll (\log T)^{-1}\tau(B_T \cap \mathbf{M}_{H_\iota})$$

where $\tau_{\mathbf{L}}$ is a Haar measure on $\mathbf{L}(\mathbb{A})$.

For each $C > 1$, set

$$B^C = \{g \in \mathbf{M}(\mathbb{A}) : \tilde{H}(g) > C\}.$$

Note that

$$(B_T - B^C) \cap \mathbf{M}_{H_\iota} \subset \cup_{i=1}^r \Omega_i$$

where $\Omega_i = \{g = g_1 \cdots g_r \in \mathbf{M}_{H_\iota} : H_\iota(g_i) \le C, \quad H_\iota(g) < T\}$. Now denoting by $\mathbf{L}^{(i)}$ the subgroup of $\mathbf{M}$ generated by $\mathbf{M}_1, \cdots, \mathbf{M}_{i-1}, \mathbf{M}_{i+1}, \cdots, \mathbf{M}_r$, let $\kappa_i > 1$ be a constant as in Lemma 5.4 for $\mathbf{G}_1 = \mathbf{M}_i$ and $\mathbf{G}_2 = \mathbf{L}^{(i)}$. Then for any $C \gg 1$,

$$\begin{aligned}\tau(\Omega_i) &\le \int_{H_\iota(g_i)<C} \tau_{\mathbf{L}_i(\mathbb{A})}(B_{\kappa_i \delta_0^{-1} T} \cap \mathbf{L}^{(i)}_{H_\iota})\, d\tau_{\mathbf{M}_i}(g_i)\\ &\ll C^{a_\iota}(\log C)^{b_\iota - 1}(\log T)^{-1}\tau(B_{\kappa_0 T} \cap \mathbf{M}_{H_\iota})\end{aligned}$$

where $\kappa_0 = \max_i(\kappa_i \delta_0^{-1})$.

Hence for any $C \gg 1$ and $T \gg C$,

$$\tau((B_T - B^C) \cap \mathbf{M}_{H_\iota}) \ll C^{a_\iota}(\log C)^{b_\iota - 1}(\log T)^{-1}\tau(B_{\kappa_0 T} \cap \mathbf{M}_{H_\iota})$$

Therefore

(5.8)

$$\begin{aligned}\int_{B_T \cap \mathbf{M}_{H_\iota}} \tilde{H}^{-1/m}\, d\tau &= \int_{B_T \cap B^C \cap \mathbf{M}_{H_\iota}} \tilde{H}^{-1/m}\, d\tau + \int_{(B_T - B^C)\cap \mathbf{M}_{H_\iota}} \tilde{H}^{-1/m}\, d\tau\\ &\ll (C^{-1/m} + \delta_0^{-1/m} \cdot C^{a_\iota}(\log C)^{b_\iota - 1}(\log T)^{-1}) \cdot \tau(B_{\kappa_0 T} \cap \mathbf{M}_{H_\iota})\\ &\ll (\log T)^{-\delta} \cdot \tau(B_{\kappa_0 T} \cap \mathbf{M}_{H_\iota}) \qquad \text{for } C = (\log T)^{1/(2a_\iota)}\end{aligned}$$

for some $\delta > 0$. We now deduce from (5.7) and (5.8) that

$$\langle F_T^x, \alpha \rangle = \tau(B_T^x \cap \mathbf{M}_{H_\iota}) + O\left(C'_\psi \cdot (\log T)^{-\delta} \cdot \tau(B_{\kappa_0 \kappa T \cdot H_\iota(x)^{-1}} \cap \mathbf{M}_{H_\iota})\right) \tag{5.9}$$

for some $\delta > 0$.

Let $S \subset R$ be as in the proof of Lemma 5.4. As in (4.6), we get

$$\tau(B_T^x \cap \mathbf{M}_{H_\iota}) = \int_{g \in B_{\delta_0^{-1}\kappa T \cdot H_\iota^{-1}(x)} \cap \mathbf{M}_S} \tau^S(B_{\kappa T \cdot H_\iota^{-1}(gx)} \cap s_g \mathbf{M}_{H_\iota} \cap \mathbf{M}^S)\, d\tau_S(g). \tag{5.10}$$

By Theorem 3.9,

$$\tau^S(B_T \cap s_g \mathbf{M}_{H_\iota} \cap \mathbf{M}^S) = c_0 \delta_S(s_g^{-1}) T^{a_\iota}(\log T)^{b_\iota - 1} + O(T^{a_\iota}(\log T)^{b_\iota - 2})$$

for some $c_0 > 0$ and $T \gg 1$. Note that $\delta_S(s_g^{-1}) = \delta_S(g) = \delta_S(gx)$ and it is bounded. We deduce that when $H_\iota(gx) \ll T/\delta_0$,

$$\begin{aligned}&\tau^S(B_{\kappa T H_\iota^{-1}(gx)} \cap \mathbf{M}_{H_\iota} \cap \mathbf{M}^S)\\ =&c \cdot \delta_S(g)(T \cdot H_\iota^{-1}(gx))^{a_\iota}(\log T)^{b_\iota - 1} + O((T \cdot H_\iota^{-1}(gx))^{a_\iota}(\log H_\iota(gx))^{b_\iota-1}(\log T)^{b_\iota-2}).\end{aligned}$$

for $c = c(S, W_f, \kappa) > 0$. To estimate the integral over the domain with $H_\iota(gx) \gg T/\delta_0$, it suffices to note that by Lemma 3.3,

$$\tau_S(B_{T\cdot H_\iota^{-1}(x)} \cap \mathbf{M}_S) \ll (TH_\iota^{-1}(x))^{a_\iota - \epsilon}.$$

Since by Lemmas 3.3 and 5.4,

$$\int_{g\in\mathbf{M}_S} \delta_S(g) H_\iota(gx)^{-a_\iota}(\log H_\iota(gx))^{b_\iota-1}\, d\tau_S(g) \ll (\log H_\iota(x))^{b_\iota-1} H_\iota(x)^{-a_\iota},$$

it follows from the above estimates that for $T \gg H_\iota(x)$,

$$\tau(B_T^x \cap \mathbf{M}_{H_\iota}) = c_x T^{a_\iota}(\log T)^{b_\iota-1} + O(d_x T^{a_\iota}(\log T)^{b_\iota-2}),$$

where

$$c_x = c \cdot \int_{g\in\mathbf{M}_S} \delta(gx) H_\iota(gx)^{-a_\iota}\, d\tau_S(g) \ll H_\iota(x)^{-a_\iota}. \tag{5.11}$$

Hence combining (5.9) and (5.10), we have for $T \gg H_\iota(x)$,

$$\langle F_T^x, \alpha\rangle = c_x \cdot T^{a_\iota}(\log T)^{b_\iota-1} + O(d_x \cdot C'_\psi \cdot T^{a_\iota}(\log T)^{b_\iota-1-\delta}).$$

Denote by τ_∞ and τ_f Haar measures on $\mathbf{G}_\infty$ and $\mathbf{G}(\mathbb{A}_f)$ respectively so that $\tau = \tau_\infty \times \tau_f$. Let ϕ_ϵ be a smooth symmetric nonnegative function on $\mathbf{M}_\infty$, which is a product $\prod_{i=1}^r \phi_{i,\epsilon}$ of smooth functions on the simple factors of $\mathbf{M}_\infty$, $\int_{\mathbf{M}_\infty} \phi_\epsilon\, d\tau_\infty = 1$ and $\mathrm{supp}(\phi_\epsilon)$ is contained in the Riemannian ball at e in $\mathbf{M}_\infty$ of radius ϵ, and for some $\rho > 0$, $\max_i \|\mathcal{D}^l\phi_{i,\epsilon}\|^{2r} \ll \epsilon^{-\rho}$ (see, for example, Lemma 4.4 in [GaO]). By the definition of H_ι in 3.2, there exists $b > 0$ such that

$$\mathrm{supp}(\phi_\epsilon) \cdot B_T^x \cdot \mathrm{supp}(\phi_\epsilon) \subset B_{(1+b\epsilon)T}^x$$

for every $T > 1$ and $x \in \mathbf{N}(K)$.

Define

$$\psi_\epsilon(g) = \frac{1}{\tau_f(W_f)} \sum_{\gamma\in\mathbf{M}(K)} \phi_\epsilon(\gamma g_\infty)\cdot \chi_{W_f}(\gamma g_f), \quad g = g_\infty g_f \in \mathbf{M}_\infty \mathbf{M}(\mathbb{A}_f).$$

Define $\alpha_\epsilon(y_1, y_2) = \psi_\epsilon(y_1)\psi_\epsilon(y_2)$ for $(y_1, y_2) \in Y \times Y$. Then

$$\begin{aligned}&N_{\pi^{-1}(x)}(H_\iota, T) \le \left\langle F_{(1+b\epsilon)T}^x, \alpha_\epsilon\right\rangle\\ =&c_x T^{a_\iota}(\log T)^{b_\iota-1} + O(c_x\cdot\epsilon\cdot T^{a_\iota}(\log T)^{b_\iota-1} + d_x\cdot \epsilon^{-\rho}T^{a_\iota}(\log T)^{b_\iota-1-\delta}).\end{aligned}$$

Setting $\epsilon = (\log T)^{-\delta/(\rho+1)}$, we derive the upper estimate for $N_{\pi^{-1}(x)}(H_\iota, T)$. The lower estimate is proved similarly. □

Proof of Theorem 5.2 According to the choice of $\mathbf{N}$, for any simple root $\alpha \in \Delta$ whose restriction to $\mathbf{N}$ is a root, we have

$$\frac{u_\alpha + 1}{m_\alpha} < a_\iota.$$

Hence it follows from Theorem 3.9 that $N_{\mathbf{N}}(H_\iota, T) = O(T^{a_\iota - \epsilon})$ for some $\epsilon > 0$. Since $c_x \ll H_\iota(x)^{-a_\iota}$ by (5.11) and $d_x = H_\iota(x)^{-a_\iota}(\log H_\iota(x))^{b_\iota - 1}$, it follows that

$$C(H_\iota) := \sum_{x \in \mathbf{N}(K)} c_x < \infty \quad \text{and} \quad \sum_{x \in \mathbf{N}(K)} d_x < \infty.$$

Let $\delta_0 > 0$ be as in (3.8). Applying Lemma 5.4 for $\mathbf{M}$ and $\mathbf{N}$ with κ therein, we have

$$\begin{aligned}
&\sum_{x \in \mathbf{N}(K) : H_\iota(x) > \beta^{-1} T} N_{\pi^{-1}(x)}(H_\iota, T) \\
&= \#\{xy \in \mathbf{N}(K)\mathbf{M}(K) : H_\iota(x) > \beta^{-1}T, \;\; H_\iota(xy) < T\} \\
&\le N_{\mathbf{M}}(H_\iota, \kappa\beta^{-1}) \cdot N_{\mathbf{N}}(H_\iota, \kappa T \delta_0^{-1}) \\
&= O(T^{a_\iota - \epsilon}).
\end{aligned}$$

Now applying Proposition 5.5 with β, δ therein, since $\sum_{x \in \mathbf{N}(K)} d_x < \infty$,

$$\begin{aligned}
&\sum_{x \in \mathbf{N}(K) : H_\iota(x) \le \beta^{-1} T} N_{\pi^{-1}(x)}(H_\iota, T) \\
&= \left(\sum_{x \in \mathbf{N}(K) : H_\iota(x) \le \beta^{-1} T} c_x \right) T^{a_\iota} (\log T)^{b_\iota - 1} + O(T^{a_\iota} (\log T)^{b_\iota - 1 - \delta}).
\end{aligned}$$

Therefore as $T \to \infty$,

$$\begin{aligned}
N(H_\iota, T) &= \sum_{x \in \mathbf{N}(K) : H_\iota(x) \le \beta^{-1} T} N_{\pi^{-1}(x)}(H_\iota, T) + O(T^{a_\iota - \epsilon}) \\
&= \left(\sum_{x \in \mathbf{N}(K) : H_\iota(x) \le \beta^{-1} T} c_x \right) T^{a_\iota} (\log T)^{b_\iota - 1} (1 + O((\log T)^{-\delta})).
\end{aligned}$$

Since $\sum_{x \in \mathbf{N}(K) : H_\iota(x) \le \beta^{-1} T} c_x = C(H_\iota) + O(T^{-\epsilon})$, we have

$$N(H_\iota, T) = C(H_\iota) \cdot T^{a_\iota} (\log T)^{b_\iota - 1} (1 + O((\log T)^{-\delta}))$$

finishing the proof.

6. Examples

Let $\mathbf{G}$ be a connected semisimple adjoint algebraic group defined over a number field K. For simplicity, we assume that $\mathbf{G}$ is split over K. Let $\iota : \mathbf{G} \to \mathrm{GL}_N$ be an absolutely irreducible representation defined over K with the highest weight λ_ι. We define a_ι and b_ι as in (1.3) and set

$$\Delta_\iota = \{\alpha \in \Delta : \frac{u_\alpha + 1}{m_\alpha} = a_\iota\}.$$

For $\alpha \in \Delta$, we denote by $\check{\alpha}$ the corresponding coroot. Given a height function H_ι as in Definition 3.2, we denote by W_f the compact open subgroup of $\mathbf{G}(\mathbb{A}_f)$ that leaves H_ι bi-invariant and by Λ^{W_f} the finite set of W_f-invariant automorphic characters of $\mathbf{G}(\mathbb{A})$. It follows from Theorem 7.1 in [STT2] that if for a finite subset $S \subset R$ and an automorphic character χ,

$$c_{S,\chi} = \lim_{s \to a_\iota^+} (s - a_\iota)^{b_\iota} \int_{\mathbf{G}^S} H_\iota(g)^{-s} \chi(g)\, dg \neq 0,$$

then

$$\chi(\check{\alpha}) = 1 \quad \text{for all } \alpha \in \Delta_\iota, \tag{6.1}$$

and conversely if (6.1) holds, then $c_{S,\chi} \neq 0$ for all sufficiently large $S \subset R$.

We consider several examples:

(i) Suppose that λ_ι is a multiple of $2\rho + \sum_{\alpha\in\Delta} \alpha$. In particular, this holds for λ_ι corresponding to the anticanonical class and for all rank 1 groups. In this case, $\Delta_\iota = \Delta$. If a character $\chi \in \Lambda^{W_f}$ satisfies (6.1) then it follows from the Cartan decomposition (2.1) that $\chi(\mathbf{G}^S) = 1$ for sufficiently large S and by the weak approximation, $\chi = 1$. This shows that $c_{S,\chi} = 0$ for every finite $S \subset R$ and every $\chi \in \Lambda^{W_f} - \{1\}$. Hence, Theorem 3.9 implies that for a Haar measure τ on $\mathbf{G}(\mathbb{A})$,

$$\tau(B_T) \sim [\mathbf{G}(\mathbb{A}) : \mathbf{G}_{H_\iota}] \cdot \tau(B_T \cap \mathbf{G}_{H_\iota}).$$

In this case, Theorem 1.15 can be stated as

$$\#\mathbf{G}(K) \cap B_T \sim \tau(B_T) \tag{6.2}$$

where τ is normalized so that $\tau(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A})) = 1$. Also, (1.6) together with (4.1) imply that the asymptotic distribution of rational points is given by

$$\mu_\iota = \prod_{v\in R} \frac{H_{\iota,v}(g_v)^{-a_\iota}\, dg_v}{\int_{\mathbf{G}(K_v)} H_{\iota,v}(g_v)^{-a_\iota}\, dg_v}. \tag{6.3}$$

(ii) Suppose that $K = \mathbb{Q}$ and $W_f = \prod_{v\in R_f} \mathbf{G}(\mathbb{Z}_p)$ (with respect to the canonical model over $\mathbb{Z}$). Then according to Remark in Section 2 in [GaO],

$$\mathbf{G}(\mathbb{A}) = \mathbf{G}(\mathbb{Q})\mathbf{G}(\mathbb{R})^\circ W_f.$$

Hence, $\Lambda^{W_f} = \{1\}$, (6.2) holds, and the measure μ_ι is given by (6.3).

(iii) According to [PR, §8.2], there exists a lattice $L \subset K^N$ such that $\mathbf{G}$ has class number 1 with respect to L, i.e.,

$$\mathbf{G}(\mathbb{A}) = \mathbf{G}(K)\mathbf{G}_\infty \left(\prod_{v\in R_f} \mathrm{Stab}_{\mathbf{G}(K_v)}(L\otimes \mathcal{O}_v) \right). \tag{6.4}$$

We take the height function $H = \prod_{v\in R} H_v$ where H_v is the maximum norm with respect to L for $v \in R_f$ and H_v is a norm invariant under a maximal compact subgroup U_v of $\mathbf{G}(K_v)$ for $v \in R_\infty$. Then

$$W_f = \prod_{v\in R_f} \mathrm{Stab}_{\mathbf{G}(K_v)}(L\otimes \mathcal{O}_v). \tag{6.5}$$

This implies that for any $\chi \in \Lambda^{W_f}$ which is not U_v-invariant for some $v \in R_\infty$ and for any finite $S \subset R_f$, we have $c_{S,\chi} = 0$. On the other hand, using that $\mathbf{G}(K_v) = U_v\mathbf{G}(K_v)^\circ$ for $v \in R_\infty$ (see Ch. III, [PR]), we deduce from (6.4) that if $\chi \in \Lambda^{W_f}$ is U_v-invariant for all $v \in R_\infty$, then $\chi = 1$. Hence, $c_{S,\chi} = 0$ for all finite $S \subset R_f$ and $\chi \in \Lambda^{W_f} - \{1\}$. This implies that (6.2) holds and

$$\mu_{\iota,R_f} = \prod_{v\in R_f} \frac{H_{\iota,v}(g_v)^{-a_\iota}\, dg_v}{\int_{\mathbf{G}(K_v)} H_{\iota,v}(g_v)^{-a_\iota}\, dg_v}.$$

(iv) (cf. Example 8.8, [STT2]) Let $\mathbf{G} = \mathrm{PGL}_4$ and ι be the adjoint representation. By [PR, §8.2], there exists a lattice $L \subset K^{15}$ such that $\mathbf{G}$ has class number 2 with respect to L. We take the height function $H = \prod_{v\in R} H_v$ where H_v is the maximum norm with respect to L for $v \in R_f$. The group W_f is given by (6.5). By [PR, §8.2], $\mathbf{G}(K)\mathbf{G}_\infty W_f$ is a normal subgroup of index 2 in $\mathbf{G}(\mathbb{A})$. If we additionally assume that the number field K is totally complex, then $\mathbf{G}_\infty$ is connected and, hence, $\Lambda^{W_f} = \{1, \chi\}$ for some automorphic character χ of order 2. Every automorphic character of $\mathbf{G}(\mathbb{A})$ is of the form $\eta \circ \det$ where η is a Hecke character such that $\eta^4 = 1$. Since the map $\det : \mathrm{PGL}_4(K_v) \to K_v^\times/(K_v^\times)^4$ is surjective for every $v \in R$, it follows that $\chi = \eta\circ\det$ with $\eta^2 = 1$. In this case, the roots and coroots are given by

$$\alpha_i(\mathrm{diag}(a_1, \dots, a_4)) = a_i a_{i+1}^{-1}, \quad \check{\alpha}_i(t) = \mathrm{diag}(\underbrace{t, \dots, t}_{i}, 1, \dots, 1)$$

for $i = 1, 2, 3$, and

$$\lambda_\iota = \alpha_1 + \alpha_2 + \alpha_3, \quad 2\rho = 3\alpha_1 + 4\alpha_2 + 3\alpha_3.$$

Hence, $a_\iota = 5$, $b_\iota = 1$, $\Delta_\iota = \{\alpha_2\}$. Then (6.1) is equivalent to $\eta^2 = 1$, and we deduce that $c_{S,\chi} \neq 0$ for sufficiently large finite $S \subset R$. Since the function $\delta_S = c_{S,1} + c_{S,\chi}\chi$ restricted to $\mathbf{G}_S$ is not constant for sufficiently large $S \subset R$, we conclude that the weights c_u defined in (4.1) are not constant and

$$\mu_{\iota,S} \neq \prod_{v\in S} \frac{H_{\iota,v}(g_v)^{-a_\iota}\, dg_v}{\int_{\mathbf{G}(K_v)} H_{\iota,v}(g_v)^{-a_\iota}\, dg_v}$$

in Theorem 1.5.

We also note that in this case, Theorem 3.9 implies that for a Haar measure τ on $\mathbf{G}(\mathbb{A})$ and an automorphic character χ such that $c_{\emptyset,\chi} \neq 0$, we have

$$\lim_{T\to\infty} \frac{\tau(B_T \cap \ker(\chi))}{\tau(B_T)} = c_{\emptyset,1}^{-1} \cdot \frac{1}{2}(c_{\emptyset,1} + c_{\emptyset,\chi}) \neq \frac{1}{2}.$$

In particular, it might happen that in Theorem 1.15, $\tau(B_T)$ is not asymptotic to $[\mathbf{G}(\mathbb{A}) : \mathbf{G}_{H_\iota}] \cdot \tau(B_T \cap \mathbf{G}_{H_\iota})$ as $T \to \infty$.

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Mathematics 253-37, Caltech, Pasadena, CA 91125, USA// IRMAR-Campus de Beaulieu, 35042 RENNES Cedex, France //Mathematics 253-37, Caltech, Pasadena, CA 91125, USA

E-mail address: gorodnik@its.caltech.edu

E-mail address: francois.maucourant@univ-rennes1.fr

E-mail address: heeoh@its.caltech.edu