PROXIMALITY AND EQUIDISTRIBUTION ON THE FURSTENBERG BOUNDARY

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Abstract. Let $G$ be a connected semisimple Lie group with finite center and without compact factors, $P$ a minimal parabolic subgroup of $G$, and $\Gamma$ a lattice in $G$. We prove that every $\Gamma$-orbits in the Furstenberg boundary $G/P$ is equidistributed for the averages over Riemannian balls. The proof is based on the proximality of the action of $\Gamma$ on $G/P$.

1. Introduction

Let $G$ be a connected semisimple Lie group with finite center and without compact factor, and $\Gamma$ a lattice in $G$, that is, a discrete subgroup of $G$ such that $\Gamma\backslash G$ has finite volume. In this article we investigate the distribution of orbits of $\Gamma$ acting on the Furstenberg boundary of $G$. Recall that the Furstenberg boundary can be identified with the factor space $G/P$, where $P$ is a minimal parabolic subgroup of $G$. It is known that every orbit of $\Gamma$ in $G/P$ is dense (see [Mo]). We show that orbits of $\Gamma$ are equidistributed with respect to the averages over Riemannian balls.

Since we study the action of a nonamenable group on a space without a finite invariant measure, our result lies outside the scope of the classical ergodic theory. The published results about distribution of dense orbits of nonamenable groups are limited to a few special examples. Arnold and Krylov showed in [AK] that dense orbits of groups generated by two rotations acting on the 2-dimensional sphere are equidistributed. A similar problem was considered by Kazhdan in [Ka] where he studied the action of a group generated by two affine isometries on the plane $\mathbb{R}^2$. Distribution of dense orbits of a lattice in $\text{SL}(2, \mathbb{R})$ acting on $\mathbb{R}^2$ was investigated by Ledrappier [L] and Nogueira [N].

Let $X$ be the symmetric space of $G$ equipped with a right invariant Riemannian metric $d$. Note that $X$ can be identified with $L\backslash G$ for a maximal compact subgroup $L$ of $G$.

Fix $x, \hat{x} \in X$ and denote by $K$ and $\hat{K}$ the stabilizers of $x$ and $\hat{x}$ respectively. Let $\nu$ and $\hat{\nu}$ be the probability Haar measures on $K$ and $\hat{K}$ and $m_\hat{x}$ the harmonic measures at $\hat{x}$ on $G/P$, that is, the unique $\hat{K}$-invariant probability measure on $G/P$. For $S \subset G$
and $T > 0$, define

$$S_T(\tilde{x}) = \{ s \in S : d(x, \tilde{x}s) < T \},$$
$$S_T = S_T(\tilde{x}).$$

Our main result is the following theorem.

**Theorem 1.** For every $f \in C(G/P)$, $\tilde{x} \in X$, and $y \in G/P$,

$$\lim_{T \to \infty} \frac{1}{|\Gamma_T(\tilde{x})|} \sum_{\gamma \in \Gamma_T(\tilde{x})} f(\gamma y) = \int_{G/P} f dm_\tilde{x},$$

Moreover, the convergence is uniform for $y \in G/P$.

We remark that it was shown in [EM] (see also [DRS]) that

$$|\Gamma_T(\tilde{x})| \sim_{T \to \infty} \frac{\text{Vol}(G_T(\tilde{x}))}{\text{Vol}(\Gamma \backslash G)},$$

and the exact asymptotics of the volume $\text{Vol}(G_T(\tilde{x})) = \text{Vol}(G_T)$ as $T \to \infty$ was computed in [Kn].

The first result in the direction of Theorem 1 was established in [Ma], where the case of the real hyperbolic spaces was considered. A similar problem for $G = \text{SL}(n, \mathbb{R})$ was considered in [G], where the averages were defined with respect to a matrix norm. A different proof of Theorem 1 is given in [GO]. The approach presented here allows us to show that the convergence is uniform whereas in [GO], the uniformness doesn’t seem obvious from the method of the proof. While the proof in [GO] uses equidistribution of solvable flows on $\Gamma \backslash G$, our proof is based on the strong proximality of the action of $G$ on $G/P$ (see Theorem 2 below). This result is of independent interest, and it might be useful for other applications.

Recall that an action of a group $H$ on a compact metric space $(Y, d)$ is called proximal if for every $u, v \in Y$ there exists a sequence \{ $h_n$ \} $\subset H$ such that $d(h_nu, h_nv) \to 0$ as $n \to \infty$. The fact that the action of $G$ on $G/P$ is proximal plays important role in the study of random walks on $G$ (see, for example, [F]). It turns out that a typical sequence in $G$ acts on $G/P$ in proximal fashion.

**Theorem 2 (Strong proximality).** Let $O$ be a neighborhood of the diagonal in $G/P \times G/P$ and $u, v \in G/P$. Then

$$\lim_{T \to \infty} \frac{\text{Vol} \left( \{ g \in G_T(\tilde{x}) : (gu, gv) \notin O \} \right) }{\text{Vol}(G_T(\tilde{x}))} = 0$$

and

$$\lim_{T \to \infty} \frac{| \{ \gamma \in \Gamma_T(\tilde{x}) : (\gamma u, \gamma v) \notin O \} |}{| \Gamma_T(\tilde{x}) |} = 0$$

uniformly on $u, v$.

In the case of the real hyperbolic space, Theorem 2 was proved in [Ma] using geometric methods.
2. Proof of Theorem 2

2.1. Cartan decomposition. Let $G = K \exp(\mathfrak{p})$ be the Cartan decomposition of $G$ and $A \subset \exp(\mathfrak{p})$ a split Cartan subgroup of $G$, that is, a maximal connected abelian subgroup in $\exp(\mathfrak{p})$. We fix a system of positive roots $\Sigma^+$ on $\mathfrak{a} = \text{Lie}(A)$, and let

$$A^+ = \{a \in A : \alpha(\log a) \geq 0 \text{ for all } \alpha \in \Sigma^+ \}$$

denote the closed positive Weyl chamber in $A$. Then $G = KA^+K$, and a Haar measure on $G$ can be given by

$$\int_G \psi(g)dg = \int_K \int_{A^+} \int_K \psi(k_1ak_2)\xi(\log a)d\nu(k_1)d\nu(k_2), \quad \psi \in C_c(G),$$

where $da$ denotes the Lebesgue measure on $A$,

$$\xi(s) = \prod_{\alpha \in \Sigma^+} \sinh(\alpha(s))^{ma}, \quad s \in \mathfrak{a},$$

and $m_\alpha$ denotes the dimension of the root space for the root $\alpha \in \Sigma^+$.

Let $\tilde{g} \in G$ be such that $x\tilde{g} = \tilde{x}$. Then $G = \tilde{g}^{-1}KA^+K$, $G_T(\tilde{x}) = \tilde{g}^{-1}KA^+_TK$, and

$$\int_G \psi(g)dg = \int_K \int_{A^+} \int_K \psi(\tilde{g}^{-1}k_1ak_2)\xi(\log a)d\nu(k_1)d\nu(k_2), \quad \psi \in C_c(G).$$

In particular, it follows that

$$\text{Vol}(G_T(\tilde{x})) = \text{Vol}(G_T) = \int_{A^+_+} \xi(\log a)da.$$

2.2. Reduction to maximal parabolics. Fix a system of simple roots

$$\Pi = \{\alpha_1, \ldots, \alpha_r \} \subset \Sigma^+.$$

Here $r = \dim A$ is the $\mathbb{R}$-rank of $G$. It is well-known that the closed subgroups of $G$ that contain $P$ are in one-to-one correspondence with the subsets of $\Pi$ (see [W, Sec. 1.2]). In particular, $P_i = P_{\{\alpha_i\}}$, $i = 1, \ldots, r$, are the maximal parabolic subgroups of $G$ and

$$P = \bigcap_{i=1}^r P_i.$$

We consider the projection maps

$$\pi_i : G/P \times G/P \to G/P_i \times G/P_i, \quad i = 1, \ldots, r.$$

Let $\Delta$ and $\Delta_i$ denote the diagonals in $G/P \times G/P$ and $G/P_i \times G/P_i$ respectively. Then

$$\Delta = \bigcap_{i=1}^r \pi_i^{-1}(\Delta_i).$$
Since
\[ \prod_{i=1}^{r} \pi_i : G/P \times G/P \to \prod_{i=1}^{r} G/P_i \times G/P_i \]
is a continuous injective map from a compact space to a Hausdorff space, it is a homeomorphism onto its image. It follows that for any neighborhood \( \mathcal{O} \) of \( \Delta \) in \( G/P \times G/P \), there exist neighborhoods \( \mathcal{O}_i \) of \( \Delta_i \) in \( G/P_i \times G/P_i \) such that
\[ \mathcal{O} \supset \bigcap_{i=1}^{r} \pi_i^{-1}(\mathcal{O}_i). \]

Then for every \((u, v) \in G/P \times G/P\),
\[ \{g \in G : g \cdot (u, v) \notin \mathcal{O}\} \subset \bigcup_{i=1}^{r} \{g \in G : g \cdot \pi_i(u, v) \notin \mathcal{O}_i\}. \]

This inclusion shows that it suffices to prove Theorem 2 under the assumption that \( P \) is a maximal parabolic subgroup of \( G \). We keep this assumption until the end of this section.

2.3. Dynamics on projective space. By a result from [T], there is an irreducible representation \( G \to \text{GL}(V) \) such that the highest weight space is one-dimensional, and the stabilizer of this space is \( P \). We consider the induced action of \( G \) on the projective space \( \mathbb{P}(V) \), and let \( w^+ \in \mathbb{P}(V) \) be the direction of the highest weight space. The map \( g \mapsto gw^+ \) defines an embedding of \( G/P \) in \( \mathbb{P}(V) \). Note that if \( \lambda \) is the highest weight, the other weights of the representation are of the form \( \lambda - \sum_{\alpha \in \Sigma^+} n_\alpha \alpha \) for integers \( n_\alpha \geq 0 \). We denote by \( V^< \) the sum of all root spaces with weights other than \( \lambda \). We fix a \( K \)-invariant scalar product on \( V \), which gives rise to a metric \( d \) on \( \mathbb{P}(V) \), which is \( K \)-invariant. Put \( d(w_1, w_2) = d(\hat{g}w_1, \hat{g}w_2) \). Let \( V^<_\epsilon \) be the open \( \epsilon \)-neighborhood of \( V^< \) in \( \mathbb{P}(V) \) with respect to the metric \( d \).

For \( w \in \mathbb{P}(V) \) and \( \tau > 0 \), define
\[ K_\tau(w) = \{k \in K : kw \notin V^<_\tau\}. \]

**Lemma 3.** For every \( w \in G \cdot w^+ \),
\[ \lim_{\tau \to 0^+} \nu(K - K_\tau(w)) = 0. \]

**Proof.** It follows from the Iwasawa decomposition that \( G \cdot w^+ = K \cdot w^+ \). Thus, without loss of generality, we may assume that \( w = w^+ \). By the continuity of the measure, it suffices to prove that
\[ \nu(\{k \in K : kw^+ \in V^<\}) = 0. \]

Suppose that this is false. For a subspace \( W \) of \( V \), define
\[ K_W = \{k \in K : kw^+ \in W\}. \]
Let $W$ be a minimal subspace of $V^<$ such that $\nu(K_W) > 0$. We claim that $\text{Stab}_K(W) = K$. If $\text{Stab}_K(W)$ has infinite index in $K$, then there exist $k_i \in K$, $i \geq 1$, such that $k_i W \neq k_j W$ for $i \neq j$. Since all sets $k_i K_W \subset K$, $i \geq 1$, have the same positive measure, it follows that for some $i \neq j$, $k_i K_W \cap k_j K_W$ has positive measure. Then $k_j^{-1} k_i K_W \cap K_W$ has positive measure too, and for $k \in k_j^{-1} k_i K_W \cap K_W$, $kw^+ \in k_j^{-1} k_i W \cap W$.

Since $k_j^{-1} k_i W \cap W$ is a proper subspace of $W$, this contradicts the choice of $W$. Thus, $\text{Stab}_K(W)$ is a closed subgroup of finite index in $K$. Since $K$ is connected, it follows that $K = \text{Stab}_K(W)$. Then $w^+ \in K_W^{-1} W \subset V^<$. This contradiction proves the lemma.

We consider the sets

$$\begin{align*}
A_T^\eta &= \{ a \in A_T : \alpha(\log a) \geq \eta \text{ for all } \alpha \in \Sigma^+ \}, \\
G_{T, \varepsilon}(u, v) &= \{ g \in G_T(\tilde{x}) : \tilde{d}(gu, gv) > \varepsilon \}, \\
\Omega_{T, \tau}^\eta(u, v) &= \tilde{g}^{-1} K A_T^\eta(K_\tau(u) \cap K_\tau(v))
\end{align*}$$

defined for $T, \eta, \tau, \varepsilon > 0$ and $u, v \in \mathbb{P}(V)$.

**Lemma 4.** For every $\varepsilon > 0$ and $\tau > 0$, there exists $\eta > 0$ such that for every $T > 0$ and $u, v \in G \cdot w^+$,

$$\Omega_{T, \tau}^\eta(u, v) \cap G_{T, \varepsilon}(u, v) = \emptyset.$$  

**Proof.** Note that an element $a \in A_T^\eta$ acts by diagonal transformations on $V$ with respect to some fixed basis, and the eigenvalue associated to the vector $w^+$ is at least $e^\eta$ times greater than the other eigenvalues. Therefore, for all $w \notin V_T^<$ and sufficiently large $\eta$ (depending only on $\tau$ and $\varepsilon$), we have $d(aw, w^+) < \varepsilon/2$ when $a \in A_T^\eta$. Thus, for $\tilde{g}^{-1} k_1 a k_2 \in \Omega_{T, \tau}^\eta(u, v) = \tilde{g}^{-1} K A_T^\eta(K_\tau(u) \cap K_\tau(v))$, we have

$$\tilde{d}(\tilde{g}^{-1} k_1 a k_2 u, \tilde{g}^{-1} k_1 a k_2 v) = d(ak_2 u, ak_2 v) \leq d(ak_2 u, w^+) + d(ak_2 v, w^+) < \varepsilon,$$

This proves the lemma.

### 2.4. Completion of the proof.

By (3),

$$\text{Vol}(\Omega_{T, \tau}^\eta(u, v)) = \left( \int_{A_T^\eta} \xi(\log a)da \right) \cdot \nu(K_\tau(u) \cap K_\tau(v)).$$

Let $\varepsilon, \delta \in (0, 1)$. Using Lemma 3, we choose $\tau > 0$ such that $\nu(K_\tau(u) \cap K_\tau(v)) > 1 - \delta$. 
Let $\eta > 0$ be as Lemma 4. By Lemma 9(a), for sufficiently large $T$,

$$\int_{A^\eta_T} \xi(a)da \geq (1 - \delta) \int_{A^+T} \xi(\log a)da.$$  

Thus, it follows from (4) and (7) that

$$\text{Vol}(\Omega^n_{T,T}(u, v)) \geq (1 - \delta)^2 \text{Vol}(G_T(\tilde{x})).$$

for sufficiently large $T > 0$. Therefore, by (6),

$$\text{Vol}(G_{T,T}(u, v)) \leq (1 - (1 - \delta)^2) \text{Vol}(G_T(\tilde{x}))$$

for all $\delta \in (0, 1)$ and sufficiently large $T > 0$. Since the sets

$$\{(g_1P, g_2P) : \tilde{d}(g_1w^+, g_2w^+) < \varepsilon\}, \quad \varepsilon > 0,$$

form a base of the neighborhoods of the diagonal in $G/P \times G/P$, this proves the first part of Theorem 2.

To prove the second part of Theorem 2, we choose a neighborhood $\mathcal{P}$ of $e$ in $G$ and a neighborhood $\mathcal{Q}$ of the diagonal in $G/P \times G/P$ such that

(8) \[ \mathcal{P}^{-1} \mathcal{P} \cap \Gamma = \{e\}, \]

(9) \[ \mathcal{P}^{-1} \cdot \mathcal{Q} \subset \mathcal{O}, \]

(10) \[ \mathcal{P} \cdot G_T(\tilde{x}) \subset G_{T+c}(\tilde{x}). \]

for fixed $c > 0$ and all $T > 0$. Here we use that $\Gamma$ is discrete, the space $G/P$ is compact, and the metric on the symmetric space is uniformly continuous. By (9), for every $\gamma \in \Gamma$ such that $\gamma \cdot (u, v) \notin \mathcal{O}$, we have $\mathcal{P} \gamma \cdot (u, v) \cap \mathcal{Q} = \emptyset$. Thus, using (10), we deduce that

$$\mathcal{P} \cdot \{\gamma \in \Gamma_T(\tilde{x}) : \gamma \cdot (u, v) \notin \mathcal{O}\} \subset \{g \in G_{T+c}(\tilde{x}) : g \cdot (u, v) \notin \mathcal{Q}\}.$$

Then by (8), $\mathcal{P} \gamma_1 \cap \mathcal{P} \gamma_2 = \emptyset$ for $\gamma_1, \gamma_2 \in \Gamma$, $\gamma_1 \neq \gamma_2$, and

$$|\{\gamma \in \Gamma_T(\tilde{x}) : \gamma \cdot (u, v) \notin \mathcal{O}\}| \leq \frac{1}{\text{Vol}(\mathcal{P})} \text{Vol}(\{g \in G_{T+c}(\tilde{x}) : g \cdot (u, v) \notin \mathcal{Q}\})$$

$$= o(\text{Vol}(G_{T+c}(\tilde{x})))$$

as $T \to \infty$. Now the second statement of Theorem 2 follows from Lemma 9(d) and (1).

3. EQUIDISTRIBUTION ON $\Gamma \backslash G$

Recall that $K$ is a maximal compact subgroups of $G$, and $\nu$ is the probability Haar measure on $K$. Denote by $\varrho$ a right Haar measure on the minimal parabolic subgroup $P$. For a suitable normalization of $\varrho$, the Haar measure on $G$ is given by

(11) \[ \int_G \psi(g)dg = \int_K \int_P \psi(kp)dg(p)d\nu(k), \quad \psi \in C_c(G). \]
We also define a measure \( \mu \) on \( G \) by
\[
\int_{G} \psi(g) d\mu(g) = \int_{K} \int_{P} \psi(kp^{-1}) d\varphi(p) d\nu(k), \quad \psi \in C_c(G).
\]
Note that \( \mu \) is left \( K \)-invariant.

The first step in the proof of Theorem 1 is the following result.

**Proposition 5.** For every \( \Psi \in C_c(\Gamma \backslash G) \) and \( z \in \Gamma \backslash G \),
\[
\lim_{T \to \infty} \frac{1}{\mu(G_T)} \int_{G_T} \Psi(zg) d\mu(g) = \frac{1}{\text{Vol}(\Gamma \backslash G)} \int_{\Gamma \backslash G} \Psi \, dg
\]
where \( G_T = \{ g \in G : d(x, zg) < T \} \).

Proposition 5 is a consequence of the equidistribution of translates of \( K \) in \( \Gamma \backslash G \) proved by Eskin and McMullen in [EM] (see also [S] for a more general result). They showed that for every strongly divergent sequence \( \{g_n\} \subset G \),
\[
\lim_{n \to \infty} \int_{P} \Psi(zkg_n) d\nu(k) = \frac{1}{\text{Vol}(\Gamma \backslash G)} \int_{\Gamma \backslash G} \Psi \, dg.
\]

Recall that a sequence \( \{g_n\} \subset G \) is **strongly divergent** if the projection of \( \{g_n\} \) on every noncompact simple factor of \( G \) is divergent. Note that (13) was proved in [EM] under the condition that the lattice \( \Gamma \) is irreducible. Since the proof of (13) is based on mixing properties of the action of \( G \) on \( \Gamma \backslash G \), it is applicable to the case of a reducible lattice \( \Gamma \) provided that the sequence \( \{g_n\} \) is strongly divergent.

Denote by \( \pi_i : G \to G_i, \ i = 1, \ldots, s \), the projections of \( G \) onto its simple factors. Let \( C_{i,j} \subset G_i, j \geq 1 \), be an increasing sequence of compact subsets such that \( G_i = \bigcup_{j \geq 1} C_{i,j} \). Define
\[
G_{T,n} = G_T - \bigcup_{1 \leq i \leq s} \pi_i^{-1}(C_{i,n}).
\]

**Lemma 6.** For every \( n \geq 1 \), \( \mu(G_{T,n}) \sim \mu(G_T) \) as \( T \to \infty \).

**Proof.** It suffices to show that for every \( i = 1, \ldots, s \) and \( n \geq 1 \),
\[
\mu(G_T \cap \pi_i^{-1}(C_{i,n})) = o(\mu(G_T)) \quad \text{as} \quad T \to \infty.
\]

Fix \( i = 1, \ldots, s \) and \( n \geq 1 \). Note that \( G = DH \), where \( D \) and \( H = \ker(\pi_i) \) are normal connected semisimple Lie subgroups with finite centers, and \( D \) and \( H \) commute. We have \( \pi_i^{-1}(C_{i,n}) = D_{i,n}H \) for some compact set \( D_{i,n} \subset D \). There is a constant \( \delta > 0 \) such that
\[
D_{i,n}H_{T-\delta} \subset (D_{i,n}H)_T \subset D_{i,n}H_{T+\delta} \quad \text{for all} \ T > 0.
\]

We define measures \( \mu_D \) and \( \mu_H \) for the groups \( D \) and \( H \) respectively as in (12). With appropriate normalization, \( \mu = \mu_D \otimes \mu_H \). Thus, it follows from (15) that
\[
\mu(G_T \cap \pi_i^{-1}(C_{i,n})) = \mu((D_{i,n}H)_T) \ll \mu_H(H_{T+\delta}).
\]
Since $G_T = KP_T$ and $P_T^{-1} = P_T$, using (11) and (12), we conclude that

\begin{equation}
(17) \quad \mu(G_T) = \varphi(P_T^{-1}) = \varphi(P_T) = \text{Vol}(G_T).
\end{equation}

Similarly, $H = LQ_T$ where $L$ is a maximal compact subgroup of $H$ contained in $K$, and $Q$ is a minimal parabolic subgroup of $H$. As in (17), we deduce that $\mu_H(H_T) = \text{Vol}_H(H_T).$ By (16), it is sufficient to show that

\begin{equation}
(18) \quad \text{Vol}_H(H_{T+\delta}) = o(\text{Vol}(G_T)) \quad \text{as} \quad T \to \infty.
\end{equation}

Note that with appropriate normalization the Haar measure on $G$ is the product of Haar measures on $D$ and $H$. Without loss of generality, $\text{Vol}_D(D_{i,n}) > 0$. Then by (15),

$$\text{Vol}_H(H_{T+\delta}) \ll \text{Vol}(D_{i,n}H_{T+\delta}) \leq \text{Vol}((D_{i,n}H)_{T+2\delta}).$$

Let $G_T^n$ be defined as in (24). Since the set $D_{i,n}$ is compact, there exists $\eta > 0$ such that

$$(D_{i,n}H)_{T+2\delta} \subset G_{T+2\delta} - G_{T+\delta}^\eta.$$

Thus, (18) follows from Lemma 9(b). \hfill \Box

**Proof of Proposition 5.** The map $K \times A^+ \times K \rightarrow G$ is a diffeomorphism on an open set of full measure. Since the measure $\mu$ is left $K$-invariant and smooth, for some $\sigma \in C(A^+ \times K)$,

$$\int_G \psi(g)d\mu(g) = \int_K \int_{A^+} \int_K \psi(k_1ak_2)\sigma(a,k_2)d\nu(k_1)d\nu(k_2), \quad \psi \in C_c(G).$$

Let $G_{T,n}$ be defined as in (14), and it is $K$-bi-invariant (equivalently, all $C_{i,j}$ are $\pi_i(K)$-bi-invariant). Then

$$G_{T,n} = KA_{T,n}^+K \quad \text{and} \quad \mu(G_{T,n}) = \int_K \int_{A_{T,n}^+} \sigma(a,k_2)d\nu(k_2),$$

where $A_{T,n}^+ = G_{T,n} \cap A^+$.

Let $\varepsilon > 0$. By (13),

$$\left| \int_K \Psi(zk_1ak_2)d\nu(k_1) - \frac{1}{\text{Vol}(\Gamma \backslash G)} \int_{\Gamma \backslash G} \Psi \ d \gamma \right| < \varepsilon$$
for $a \in A^+_T$ and $k_2 \in K$ when $n > n_0(\varepsilon)$. Thus, for $n > n_0(\varepsilon)$,

\begin{equation}
\begin{aligned}
\int_{G_T} \left| \Psi(zg)d\mu(g) - \frac{\mu(G_{T,n})}{\text{Vol}(\Gamma \backslash G)} \int_{\Gamma \backslash G} \Psi \, dg \right| \\
= \int_K \int_{A^+_T} \int_K \Psi(zk_1ak_2)d\nu(k_1)\sigma(a,k_2)d\sigma(k_2) \\
- \frac{\mu(G_{T,n})}{\text{Vol}(\Gamma \backslash G)} \int_{\Gamma \backslash G} \Psi \, dg \leq \int_K \int_{A^+_T} \left| \int_K \Psi(zk_1ak_2)d\nu(k_1) \right| \\
- \frac{1}{\text{Vol}(\Gamma \backslash G)} \int_{\Gamma \backslash G} \Psi \, dg \sigma(a,k_2)d\sigma(k_2) < \varepsilon \mu(G_{T,n}).
\end{aligned}
\end{equation}

By Lemma 6, for every $n \geq 1$,

\[ \int_{G_T} \Psi(zg)d\mu(g) = \int_{G_{T,n}} \Psi(zg)d\mu(g) + o(\mu(G_{T,n})) \]

as $T \to \infty$. Thus, it follows from (19) that

\[ \limsup_{T \to \infty} \left| \frac{1}{\mu(G_T)} \int_{G_T} \Psi(zg)d\mu(g) - \frac{1}{\text{Vol}(\Gamma \backslash G)} \int_{\Gamma \backslash G} \Psi \, dg \right| < \varepsilon \]

for every $\varepsilon > 0$. This proves the proposition.

\[ \qed \]

4. Equidistribution on average

In this section we prove that Theorem 1 holds “on average”. In the case of hyperbolic spaces, the following proposition is a consequence of the work of Roblin [R].

**Proposition 7.** For every $f \in C(G/P)$ and $y \in G/P$,

\[ \lim_{T \to \infty} \frac{1}{\Gamma_T(\tilde{x})} \sum_{\gamma \in \Gamma_T(\tilde{x})} \int_K f(\gamma ky)d\nu(k) = \int_{G/P} f \, dm_{\tilde{x}} \]

where $\Gamma_T(\tilde{x}) = \{ \gamma \in \Gamma : d(x,\tilde{x}\gamma) < T \}$.

**Proof.** There exists $\tilde{p} \in P$ such that $\tilde{x} = x\tilde{p}$. Then $\tilde{K} = \tilde{p}^{-1}K\tilde{p}$, and it follows from (11) that

\begin{equation}
\int_G \psi(g)dg = \int_K \int_P \psi(k\tilde{p}^{-1}p)d\varphi(p)d\nu(k), \quad \psi \in C_c(G).
\end{equation}

Without loss of generality, $f \geq 0$, and since $G = KP$, we may assume that $y = eP$. Let $\varepsilon > 0$, $O_\varepsilon = \{ z \in X : d(x,z) < \varepsilon \}$, and $\phi_\varepsilon \in C_c(X)$ such that

\[ \phi_\varepsilon \geq 0, \quad \text{supp}(\phi_\varepsilon) \subset O_\varepsilon, \quad \int_P \phi_\varepsilon(xp^{-1})d\varphi(p) = 1. \]
Since $X = \tilde{x} P$ and $g$ is right invariant, it follows that
\begin{equation}
\int_{P} \phi_{\varepsilon}(zp^{-1})d\varrho(p) = 1 \quad \text{for every } z \in X.
\end{equation}

Let
\[ \psi_{\varepsilon}(g) = f(gP)\phi_{\varepsilon}(\tilde{x}g), \quad g \in G. \]

Clearly, $\psi_{\varepsilon} \in C_{c}(G)$ and
\[ \Psi_{\varepsilon}(\Gamma g) \overset{\text{def}}{=} \sum_{g \in \Gamma} \psi_{\varepsilon}(\gamma g) \in C_{c}(\Gamma \setminus G). \]

By Proposition 5,
\begin{equation}
\lim_{T \to \infty} \frac{1}{\mu(G_{T})} \sum_{\gamma \in \Gamma} \int_{G_{T}} \psi_{\varepsilon}(\gamma g)d\mu(g) = \frac{1}{\text{Vol}(\Gamma \setminus G)} \int_{\Gamma \setminus G} \Psi_{\varepsilon}(\Gamma g)dg
\end{equation}
and by (20),
\[ \text{Vol}(\Gamma \setminus G) \int_{\Gamma \setminus G} \Psi_{\varepsilon}(\Gamma g)dg = \int_{G} \psi_{\varepsilon}(g)dg = \int_{K} f(kP)d\tilde{\nu}(k) \cdot \int_{P} \phi_{\varepsilon}(\tilde{x}p^{-1})d\varrho(p) \]
\[ = \int_{G/P} fdm_{\tilde{x}} \cdot \int_{P} \phi_{\varepsilon}(xp)d\varrho(p). \]

Denote by $\delta$ the modular function of $P$. By (21),
\[ \left| \int_{P} \phi_{\varepsilon}(xp)d\varrho(p) - 1 \right| = \left| \int_{P} \phi_{\varepsilon}(xp^{-1})(\delta(p) - 1)d\varrho(p) \right| \]
\[ \leq \max\{ |\delta(p) - 1| : xp^{-1} \in \mathcal{O}_{\varepsilon} \}. \]

The sets $\{ p \in P : xp^{-1} \in \mathcal{O}_{\varepsilon} \}, \varepsilon > 0$, form a base of neighborhoods of $P \cap K$ in $P$. Since $\delta|_{P \cap K} = 1$ and $P \cap K$ is compact,
\[ \max\{ |\delta(p) - 1| : xp^{-1} \in \mathcal{O}_{\varepsilon} \} \to 0 \quad \text{as } \varepsilon \to 0^{+}. \]

Thus, it follows from (22) that
\begin{equation}
\lim_{\varepsilon \to 0^{+}} \lim_{T \to \infty} \frac{1}{\mu(G_{T})} \sum_{\gamma \in \Gamma} \int_{G_{T}} \psi_{\varepsilon}(\gamma g)d\mu(g) = \int_{G/P} fdm_{\tilde{x}}.
\end{equation}

Since $G_{T} = KP_{T}$,
\[ \sum_{\gamma \in \Gamma} \int_{G_{T}} \psi_{\varepsilon}(\gamma g)d\mu(g) \]
\[ = \sum_{\gamma \in \Gamma} \int_{K \times P_{T}^{-1}} \psi_{\varepsilon}(\gamma kp^{-1})d\nu(k)d\varrho(p) \]
\[ = \sum_{\gamma \in \Gamma} \int_{K} f(\gamma kP) \left( \int_{P_{T}^{-1}} \phi_{\varepsilon}(\tilde{x}\gamma kp^{-1})d\varrho(p) \right) d\nu(k). \]
For $\gamma \in \Gamma - \Gamma_{T+\varepsilon}(\tilde{x})$, $k \in K$, and $p \in P^{-1}$,
\[d(x, \tilde{x}\gamma kp^{-1}) = d(xpk^{-1}, \tilde{x}\gamma) \geq d(x, \tilde{x}\gamma) - d(x, xpk^{-1}) \geq \varepsilon.
\]
This implies that $\int_{P^{-1}} \phi_\varepsilon(\tilde{x}\gamma kp^{-1})d\varrho(p) = 0$ for $\gamma \in \Gamma - \Gamma_{T+\varepsilon}(\tilde{x})$. Thus,
\[
\sum_{\gamma \in \Gamma} \int_{G_T} \psi_\varepsilon(\gamma g)d\mu(g)
\]
\[= \sum_{\gamma \in \Gamma_{T+\varepsilon}(\tilde{x})} \int_{K} f(\gamma kP) \left( \int_{P^{-1}} \phi_\varepsilon(\tilde{x}\gamma kp^{-1})d\varrho(p) \right) d\nu(k)
\]
\[\leq \sum_{\gamma \in \Gamma_{T+\varepsilon}(\tilde{x})} \int_{K} f(\gamma kP) \left( \int_{P} \phi_\varepsilon(\tilde{x}\gamma kp^{-1})d\varrho(p) \right) d\nu(k)
\]
\[= \sum_{\gamma \in \Gamma_{T+\varepsilon}(\tilde{x})} \int_{K} f(\gamma kP)d\nu(k).
\]
Combining (23), (17), (1) and Lemma 9(c), we deduce that
\[
\liminf_{T \to \infty} \frac{1}{|\Gamma_T(\tilde{x})|} \sum_{\gamma \in \Gamma_T(\tilde{x})} \int_{K} f(\gamma kP)d\nu(k) \geq \int_{G/P} fdm_{\tilde{x}}.
\]
On the other hand, for $\gamma \in \Gamma_{T-\varepsilon}(\tilde{x})$, $k \in K$, and $p \in P$ such that $d(x, \tilde{x}\gamma kp^{-1}) < \varepsilon$,
\[d(x, xpk^{-1}) \leq d(x, \tilde{x}\gamma kp^{-1}) + d(xpk^{-1}, \tilde{x}\gamma kp^{-1}) < T.
\]
This shows that for $\gamma \in \Gamma_{T-\varepsilon}(\tilde{x})$,
\[
\int_{P^{-1}} \phi_\varepsilon(\tilde{x}\gamma kp^{-1})d\varrho(p) = \int_{P} \phi_\varepsilon(\tilde{x}\gamma kp^{-1})d\varrho(p) \stackrel{(21)}{=} 1.
\]
Hence,
\[
\sum_{\gamma \in \Gamma} \int_{G_T} \psi_\varepsilon(\gamma g)d\mu(g)
\]
\[\geq \sum_{\gamma \in \Gamma_{T-\varepsilon}(\tilde{x})} \int_{K} f(\gamma kP) \left( \int_{P^{-1}} \phi_\varepsilon(\tilde{x}\gamma kp^{-1})d\varrho(p) \right) d\nu(k)
\]
\[= \sum_{\gamma \in \Gamma_{T-\varepsilon}(\tilde{x})} \int_{K} f(\gamma kP)d\nu(k).
\]
By (23), (17), (1), and Lemma 9(c),
\[
\limsup_{T \to \infty} \frac{1}{|\Gamma_T(\tilde{x})|} \sum_{\gamma \in \Gamma_T(\tilde{x})} \int_{K} f(\gamma kP)d\nu(k) \leq \int_{G/P} fdm_{\tilde{x}}.
\]
This proves the proposition.
5. Proof of Theorem 1

Now the proof can be completed using the argument from [Ma]. Let \(\varepsilon > 0\). Since the space \(G/P \times G/P\) is compact, there exists a neighborhood \(\mathcal{O}\) of the diagonal in \(G/P \times G/P\) such that for every \((z_1, z_2) \in \mathcal{O}\), we have \(|f(z_1) - f(z_2)| < \varepsilon\). Then for every \(k \in K\),

\[
\left| \sum_{\gamma \in \Gamma_T(\hat{x})} f(\gamma y) - \sum_{\gamma \in \Gamma_T(\hat{x})} f(\gamma ky) \right| \\
\leq \sum_{\gamma \in \Gamma_T(\hat{x}) : (\gamma y, \gamma ky) \in \mathcal{O}} |f(\gamma y) - f(\gamma ky)| + \sum_{\gamma \in \Gamma_T(\hat{x}) : (\gamma y, \gamma ky) \notin \mathcal{O}} |f(\gamma y) - f(\gamma ky)| \\
\leq \varepsilon |\Gamma_T(\hat{x})| + 2 \sup |f| \cdot |\{\gamma \in \Gamma_T(\hat{x}) : (\gamma y, \gamma ky) \notin \mathcal{O}\}|.
\]

Thus, it follows from Theorem 2 that

\[
\lim_{T \to \infty} \frac{1}{|\Gamma_T(\hat{x})|} \left| \sum_{\gamma \in \Gamma_T(\hat{x})} f(\gamma y) - \sum_{\gamma \in \Gamma_T(\hat{x})} f(\gamma ky) \right| = 0
\]

for all \(k \in K\). Hence, by the dominated convergence theorem,

\[
\lim_{T \to \infty} \left| \frac{1}{|\Gamma_T(\hat{x})|} \sum_{\gamma \in \Gamma_T(\hat{x})} f(\gamma y) - \frac{1}{|\Gamma_T(\hat{x})|} \int_K f(\gamma ky) d\nu(k) \right| = 0.
\]

Finally, Theorem 1 follows from Proposition 7.

6. Appendix: Volume Estimates

In this section, we give proofs of volume estimates, which are used in Theorems 1 and 2. There are other ways to establish these volume estimates. For example, one can use the exact asymptotics of the volume of Riemannian balls from [Kn] (see also [GO]). We present a straightforward proof that does not use asymptotics.

Let \(a\) be the Lie algebra of the Cartan subgroup \(A\) and \(a^+\) the positive Weyl chamber with respect to the root system \(\Sigma^+\). The Riemannian metric defines a scalar product on \(a\) and, by duality, on the dual space of \(a\). For \(\alpha \in \Sigma^+\), we denote by \(m_\alpha\) the dimension of the corresponding root space and put \(\rho = \frac{1}{2} \sum_{\beta \in \Sigma^+} m_\beta \beta\).

**Lemma 8.** The maximum of \(\rho\) on \(\{a \in a : \|a\| \leq 1\}\) is achieved at a unique point in the interior of \(a^+\).

**Proof.** Since the set \(\{a \in a : \|a\| = 1\}\) is strictly convex, it is clear that the point of maximum is unique. It is sufficient to show that \((\rho, \alpha) > 0\) for every \(\alpha \in \Sigma^+\). Denote by \(\sigma_\alpha\) the reflection with respect to the hyperplane \(\{\alpha = 0\}\). The map \(\sigma_\alpha\) permutes the elements of the set \(\Sigma^+ - \{\alpha, 2\alpha\}\). Since \(m_{\sigma_\alpha(\beta)} = m_\beta\), we have

\[
\sigma_\alpha(\rho) = \rho - 2m_\alpha \alpha - 4m_{2\alpha} \alpha.
\]
Thus, 
\[
(\rho, \alpha) = (\sigma_\alpha(\rho), \sigma_\alpha(\alpha)) = 2m_\alpha(\alpha, \alpha) + 4m_{2\alpha}(\alpha, \alpha) - (\rho, \alpha)
\]
and \((\rho, \alpha) = (m_\alpha + 2m_{2\alpha})(\alpha, \alpha)\) is positive. \(\square\)

For \(T, \eta > 0\), define
\[
A_T^\eta = \{a \in A_T : \alpha(\log a) \geq \eta \text{ for all } \alpha \in \Sigma^+ \}
\]
(24)
\[
G_T^\eta = KA_T^\eta \cap K.
\]

**Lemma 9.** For every \(\eta > 0\),

(a) \[
\int_{A_T^\eta} \xi(\log a) da \xrightarrow{T \to \infty} \int_{A_T^+} \xi(\log a) da,
\]
(b) \[
\operatorname{Vol}(G_T^\eta) \xrightarrow{T \to \infty} \operatorname{Vol}(G_T),
\]
(c) \[
\liminf_{\varepsilon \to 0^+} \left( \limsup_{T \to \infty} \frac{\operatorname{Vol}(G_{T+\varepsilon})}{\operatorname{Vol}(G_T)} \right) = 1,
\]
(d) \[
\operatorname{Vol}(G_{T+\eta}) \ll \operatorname{Vol}(G_T).
\]

**Proof.** We have
\[
(25) \int_{a_T^+} \xi(a) da = 2^{-|\Sigma^+|} \sum_{i \in I} \int_{a_T^{+i}} e^{\lambda_i(a)} da
\]
where \(\lambda_i\)’s the characters of the form \(2\rho - \sum_{\alpha \in \Sigma^+} n_\alpha \alpha\) for some \(n_\alpha \geq 0\). Let
\[
\delta = \max\{2\rho(a) : a \in a_T^{+i} \},
\]
\[
\delta_i = \max\{\lambda_i(a) : a \in a_T^{+i} \}, \quad i \in I,
\]
\[
\delta_\alpha = \max\{2\rho(a) : a \in a_T^{+\alpha}, \alpha(a) = 0 \}, \quad \alpha \in \Sigma^+.
\]
It follows from Lemma 8 that for \(\lambda_i \neq 2\rho\) and \(\alpha \in \Sigma^+, \delta > \max\{\delta_i, \delta_\alpha\}\). Thus,
\[
(26) \int_{a_T^+} e^{\lambda_i(a)} da \leq \operatorname{Vol}(a_T^+) e^{\delta i T} \ll T^r e^{\delta i T}
\]
where \(r = \dim a\). Let \(\varepsilon > 0\) be such that
\[
\delta - \varepsilon > \max\{\delta_i, \delta_\alpha : i \in I, \alpha \in \Sigma^+ \}.
\]
Then
\[
(27) \int_{a_T^+} e^{2\rho(a)} da = T^\delta \int_{a_T^{+i}} e^{2\rho(a)} da
\]
\[
\geq T^r e^{(\delta - \varepsilon)T} \operatorname{Vol}\{a \in a_T^{+i} : 2\rho(a) \geq \delta - \varepsilon\} \gg T^r e^{(\delta - \varepsilon)T}.
\]
Combining (25), (26), and (27), we deduce that

\[
\int_{a^+_T} \xi(a) da \gg T^r e^{(\delta - \varepsilon)T}.
\]

On the other hand, for \( \alpha \in \Sigma^+ \),

\[
\int_{a^+_T \cap \{ \alpha < \eta \}} \xi(a) da \leq \int_{a^+_T \cap \{ \alpha < \eta \}} e^{2\rho(a)} da \leq \int_{a^+_T \cap \{ \alpha = 0 \}} e^{2\rho(a)} da = T^{r-1} \int_{a^+_T \cap \{ \alpha = 0 \}} e^{2T\rho(a)} da \ll T^{r-1} e^{\delta_0 T} = o(e^{(\delta - \varepsilon)T}).
\]

Since

\[
a^+_T - a^+_T \subset \bigcup_{\alpha \in \Sigma^+} a^+_T \cap \{ \alpha < \eta \}.
\]

This proves part (a) of the lemma. Part (b) follows from (2).

To prove part (c), we note that

\[
\text{Vol}(G_{T+\varepsilon}) = \int_{a^+_T + \varepsilon} \xi(a) da = (T + \varepsilon)^r \int_{a^+_T} \xi((T + \varepsilon)a) da
\]

It is easy to check that there exist \( b > 0 \) such that \( \sinh(t + \varepsilon) \leq e^\varepsilon \sinh(t) + b \) for every \( \varepsilon \in (0, 1) \) and \( t \geq 0 \). Thus, for \( a \in a^+_T \) and sufficiently small \( \varepsilon > 0 \),

\[
\xi((T + \varepsilon)a) \leq \prod_{\alpha \in \Sigma^+} (a_\varepsilon \sinh(\alpha(Ta)) + b)^{m_\alpha} \leq d_\varepsilon \xi(Ta) + C \sum_{i \in I} e^{\lambda_i(a)}
\]

where \( d_\varepsilon \to 1 \) as \( \varepsilon \to 0^+ \), \( C > 0 \), and \( \lambda_i \)'s are characters such that \( 2\rho - \lambda_i < 0 \) in the interior of \( a^+_T \). Thus, it follows from (26) that

\[
\int_{a^+_T} \xi((T + \varepsilon)a) da \leq d_\varepsilon \int_{a^+_T} \xi(Ta) da + o(e^{(\delta - \varepsilon)T}).
\]

Using (4) and (28), we deduce that

\[
\limsup_{T \to \infty} \frac{\text{Vol}(G_{T+\varepsilon})}{\text{Vol}(G_T)} \leq d_\varepsilon,
\]

and part (c) of the lemma follows. The last part of lemma can be proved similarly. \( \square \)

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REFERENCES


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