Equidistribution on homogeneous spaces

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(joint work with Barak Weiss)
Problem

Let $\Gamma$ be a discrete group acting on a compact topological space $X$. We study distribution of orbits of $\Gamma$ in $X$:

- Fix a set of “balls” $\Gamma_T \subset \Gamma$, $T > 0$.
- For $f \in C(X)$ and $x_0 \in X$, what is the asymptotic behavior of
  \[ \sum_{\gamma \in \Gamma_T} f(\gamma \cdot x_0) ? \]
Example (Bogolubov-Krylov + Birkhoff)

For a homeomorphism $T$ of $X$, there exists an invariant measure $\mu$ such that for $\mu$-a.e. $x_0 \in X$ and $f \in C(X)$,

$$
\frac{1}{2N + 1} \sum_{i=-N}^{N} f(T^i x_0) \to \int_X f \, d\mu
$$

as $N \to \infty$.

Essentially, the same result has been extended to the class of amenable groups.
What about nonamenable groups?

Known results about equidistribution:

1. (Arnold-Krylov) Two isometries acting on $S^2$.

2. (Guivarc’h) Translations on homogeneous spaces of compact groups.

3. (Kazhdan, Guivarc’h, Vorobets) Groups acting by affine isometries on $\mathbb{R}^2$.

4. (Nevo-Stein, Grigorchuk, Bufetov) Pointwise ergodic theorem for free groups.

5. (Fujiwara-Nevo) Pointwise ergodic theorem for word-hyperbolic groups.

Let $\Gamma$ be a lattice in a (noncompact) semi-simple Lie group $G$ (e.g., $\Gamma = \text{SL}(n, \mathbb{Z})$). We prove equidistribution of all dense $\Gamma$-orbits for algebraic measure-preserving actions.
“Balls” in $\Gamma$

1. Let $d$ be the Cartan-Killing Riemannian metric on the symmetric space $K\backslash G$ and

$$\Gamma_T = \{ \gamma \in \Gamma : d(K\gamma, K) < \log T \}.$$

2. Let $\rho : G \to \text{GL}(V)$ be a proper homomorphism, $\| \cdot \|$ a norm on $\text{End}(V)$, and

$$\Gamma_T = \{ \gamma \in \Gamma : \| \rho(\gamma) \| < T \}.$$

If $G$ is not simple, we also need the following condition:

**Balanced condition.** For a decomposition $G = G_1 \cdot G_2$, where $G_1$ and $G_2$ are closed connected normal subgroups of $G$, and compact $C \subset G_1$,

$$\frac{\#(\Gamma_T \cap C \cdot G_2)}{\#\Gamma_T} \to 0 \quad \text{as} \quad T \to \infty.$$

In case 1, this condition is equivalent to nonexistence of compact factors for $G$; in case 2, this condition may fail even for irreducible $\Gamma$. 
Algebraic measure preserving actions

1. $\Gamma \subset G$ acts on $L/\Lambda$ by left multiplication where
   - $G$ is a closed subgroup of a Lie group $L$.
   - $\Lambda$ is a lattice in $L$.

2. $\Gamma \subset G$ acts on $L/\Lambda$ by automorphisms (e.g., $\text{SL}(n, \mathbb{Z})$ action on $\mathbb{T}^n$) where
   - $L$ is a Lie group.
   - $\Lambda$ is a lattice in $L$.
   - $G$ is a closed subgroup of $\text{Aut}(L)$ such that $\Gamma$ stabilizes $\Lambda$. 
Main result

**Theorem 1** Consider an algebraic measure preserving action of $\Gamma$ on a space $X$ equipped with finite measure $\mu$. Then for every $x_0 \in X$ such that $\Gamma \cdot x_0$ is dense in $X$ and every $f \in C_c(X)$,

$$\frac{1}{\# \Gamma_T} \sum_{\gamma \in \Gamma_T} f(\gamma \cdot x_0) \to \frac{1}{\mu(X)} \int_X f \, d\mu.$$ 

Previously, Hee Oh proved equidistribution of the action $\Gamma$ by left multiplication on $\SL(n, \mathbb{R})/\Lambda$, where $\Gamma$ and $\Lambda$ are lattices in $\SL(n, \mathbb{R})$. 
Sketch of the proof (inducing action)

Let

\[ Y = (G \times X)/\sim, \]

where the equivalence relation is defined by

\[(g, x) \sim (g\gamma^{-1}, \gamma x) \quad \text{for } \gamma \in \Gamma,\]

equipped with the measure

\[ \nu = \text{Vol}_{G/\Gamma} \otimes m. \]

The group \( G \) acts on \( Y \) by

\[ g' \cdot [(g, x)] = [(g'g, x)]. \]

We show that Theorem 1 is equivalent to equidistribution of \( G \)-orbits in \( Y \):

For \( y_0 = [(e, x_0)] \) and every \( \phi \in C_c(Y) \),

\[ (*) \quad \frac{1}{\text{Vol}(G_T)} \int_{G_T} \phi(g \cdot y_0) \, dg \to \frac{1}{\nu(Y)} \int_Y \phi \, d\nu. \]
Sketch of the proof (proof of (∗))

We use Cartan decomposition $G = KA^+K$, where $K$ is a maximal compact subgroup of $G$ and $A^+$ is positive Weyl chamber. Then

$$dg = dk d\rho(a) dl, \quad g = kal.$$ 

Let $\alpha_i \in C(K)$, $i = 1, \ldots, N$, be a partition of unity and $l_i \in \text{supp}(\alpha_i)$.

$$\int_{GT} \phi(g \cdot y_0) dg = \int_K dk \int_K dl \int_{a: ||kal|| < T} \phi(kaly_0) d\rho(a)$$

$$= \sum_{i=1}^N \int_K dk \int_K \alpha_i(l) dl \int_{a: ||kal|| < T} \phi(kaly_0) d\rho(a)$$

$$\approx \sum_{i=1}^N \int_K dk \int_{a: ||kal|| < T} \left( \int_K \phi(kaly_0) \alpha_i(l) dl \right) d\rho(a).$$

Using Ratner theory (N. Shah), one shows that for $\phi \in C_c(Y)$ and $\alpha \in C(K)$,

$$\int_K \phi(aly_0) \alpha(l) dl \rightarrow \left( \int_K \alpha dl \right) \cdot \frac{1}{\nu(Y)} \int_Y \phi d\nu$$

as $a \to \infty$ in strong sense.
Spaces with infinite volume

Example (Ledrappier, Nogueira): Consider action of \( \Gamma = \text{SL}(n, \mathbb{Z}) \) on \( \mathbb{R}^n \). If \( x_0 \in \mathbb{R}^n \) is not a multiple of a rational vector, then \( \Gamma \cdot x_0 \) is dense in \( \mathbb{R}^n \) and for every \( f \in C_c(\mathbb{R}^n - \{0\}) \),

\[
\frac{1}{T(n-1)^2} \sum_{\gamma \in \Gamma: \|\gamma\| < T} f(\gamma \cdot x_0) \rightarrow c_n \int_{\mathbb{R}^n - \{0\}} f(x) \frac{dx}{\|x\|}
\]

for some \( c_n > 0 \).

Note that

1. \( \#\{\gamma \in \Gamma : \|\gamma\| < T\} \sim c \cdot T^{n^2-n} \).

2. The measure \( \frac{dx}{\|x\|} \) is not invariant under \( \Gamma \).
Example
Infinite volume homogeneous spaces

Let $G$ be a connected Lie group, $H$ a closed subgroup of $G$, and $\Gamma$ a lattice in $G$. We investigate distribution of dense orbits for the action of $\Gamma$ on $G/H$ by left multiplication.

Asymptotics for discrete $\Gamma$-orbits in $G/H$ (for reductive $H$) was obtained by Eskin, Mozes, Shah.

**Skew balls:** For $g_1, g_2 \in G$, define

$$H_T[g_1, g_2] = g_1^{-1}G_Tg_2^{-1} \cap H.$$  

**Moderate volume growth condition:** for any bounded $B \subset G$ and $\varepsilon > 0$, there exists a neighborhood of $e$ in $H$ such that for any $g_1, g_2 \in B$, 

$$\text{Vol}(O \cdot H_T[g_1, g_2]) \leq (1 + \varepsilon)\text{Vol}(H_T[g_1, g_2]).$$
Asymptotics

Idea:

\[
\begin{array}{c}
\Gamma \backslash G \\
\phi
\end{array}
\xleftarrow{\pi} \xrightarrow{\tau} G/H
\]

**Theorem 2** For \( g_0H \in G/H \) such that

\[
\overline{\Gamma g_0H} = G,
\]

the following is equivalent:

- For any \( g \in G \) and \( \phi \in C_c(\Gamma \backslash G) \),

\[
\frac{1}{\text{Vol}(H_T[g, g_0])} \int_{H_T[g, g_0]} \phi(\Gamma g_0h^{-1})dh \rightarrow \int_{\Gamma \backslash G} \phi.
\]

- For every \( f \in C_c(G/H) \),

\[
\sum_{\gamma \in \Gamma_T} f(\gamma \cdot g_0H) \sim \int_{G_T} f(g \cdot g_0H) dg.
\]
The first condition (equidistribution of skew-balls in $\Gamma \backslash G$) can be proved for semisimple groups $H$ that satisfies balanced condition and for some unipotent groups $H$.

**Balanced condition.** For a decomposition $H = H_1 \cdot H_2$, where $H_1$ and $H_2$ are closed connected normal subgroups of $H$, and compact $C \subset H_1$,

$$\frac{\text{Vol}(H_T \cap C \cdot H_2)}{\text{Vol}(H_T)} \to 0 \quad \text{as } T \to \infty.$$
Equidistribution

Let $Y \subset G$ be measurable section of the projection map $\tau : G \to G/H$, i.e., the product map $Y \times H \to G$ is a Borel isomorphism. Then

$$dg = d\nu_Y(y) \otimes dh, \quad g = yh,$$

where $dg$ and $dh$ are right Haar measures on $G$ and $H$, and $\nu_Y$ is a measure on $Y$. We assume existence of the limit

$$\lim_{T \to \infty} \frac{\operatorname{Vol}(H_T[g_1, g_2])}{\operatorname{Vol}(H_T[e, e])} \overset{\text{def}}{=} \alpha(g_1, g_2).$$

Let

$$\nu_{g_0}(y) = \alpha(y, g_0^{-1})d\nu_Y(y).$$
Main result

**Theorem 3** Assuming the moderate volume growth condition, (+), and equidistribution for the \( H \)-orbit in \( \Gamma \backslash G \) (as in Theorem 2), we have

\[
\frac{1}{\text{Vol}(H_T)} \sum_{\gamma \in \Gamma_T} f(\gamma \cdot g_0 H) \rightarrow \int_{G/H} f \, d\nu_{g_0}.
\]

**Proof.**

\[
\int_{G_T} f(g \cdot g_0 H) \, dg
\]

\[
= \int_{g: \|gg_0^{-1}\| < T} f(gH) \, dg
\]

\[
= \int_{Y} d\nu_{Y}(y) \int_{h: \|yhg_0^{-1}\| < T} f(yhH) \, dh
\]

\[
= \int_{Y} f(yH) \text{Vol}(H_T[y, g_0^{-1}]) \, d\nu_{Y}(y)
\]

\[
\sim \text{Vol}(H_T) \cdot \int_{G/H} f \, d\nu_{g_0}.
\]
Lattice points in affine symmetric spaces

$G/H$ – affine symmetric space,
$S$ – symmetric spaces of $G$,
$d$ – Cartan-Killing metric on $S$,
$\Gamma$ – lattice in $G$.

For $u, v \in S$,

$\Gamma_t = \{ \gamma \in \Gamma : d(u \cdot \gamma, v) \leq t \}$,

$H_t = \{ h \in H : d(u \cdot h, v) \leq t \}$.

**Corollary 4** For

$x \in G/H$ such that $\Gamma \cdot x = G/H$,

we have

$$\frac{1}{\text{Vol}(H_t)} \sum_{\gamma \in \Gamma_t} \delta_{\gamma x} \to c_x \cdot \mu$$

for $c_x > 0$ and a smooth measure $\mu$ on $G/H$. 
Measure $\mu$

$K \subset H$ – maximal compact subgroup,
$s_0 \in S$ – fixed point of $K$,
$a \subset \text{Lie}(H)$ – Cartan subalgebra,
$a_0 \in a$ – barycenter (direction of maximal volume growth),
\[ \delta = \lim_{t \to \infty} \frac{\log \text{Vol}(G_t)}{t}, \]
$\nu$ – $G$-invariant measure on $G/H$.

For $v \in S$ and $a \in a$, Busemann function:
\[ \beta(v, a) = \lim_{t \to \infty} (d(s_0 \exp(ta), v) - t). \]

\[ d\mu(y) = \left( \int_K \exp(-\delta \beta(vyk, a_0)) dk \right) d\nu(y) \]
Oppenheim conjecture

Let

\[ Q(x) = \sum_{i,j=1}^{d} a_{ij}x_i x_j, \quad x \in \mathbb{R}^d, \]

be a real nondegenerate indefinite quadratic form. Assume that \( d \geq 3 \) and \( Q \) is not a multiple of a rational form.

**Theorem 5 (Margulis)** \( Q(\mathbb{Z}^d) \) is dense in \( \mathbb{R} \).

For example, the set

\[ \{ m^2 + n^2 - \sqrt{2}k^2 : m, n, k \in \mathbb{Z} \} \]

is dense in \( \mathbb{R} \).

**Theorem 6 (Eskin,Margulis,Mozes)** If

\[ \text{sign}(Q) \neq (2,1), (2,2), \]

then for any \( (a, b) \subset \mathbb{R} \),

\[ \# \{ x \in \mathbb{Z}^d : a < Q(x) < b, \| x \| < T \} \sim c_Q(b-a)T^{d-2} \]

for some \( c_Q > 0 \).
Oppenheim conjecture for frames

Denote by $B$ the bilinear form that corresponds to the quadratic form $Q$. Let

$$\mathcal{F}^d = \{ (f_1, \ldots, f_d) : f_i \in \mathbb{R}^d, \text{Vol}(f_1, \ldots, f_d) = 1 \}$$

be the spaces of integral unimodular bases of $\mathbb{R}^d$. For $f \in \mathcal{F}^d$, the matrix

$$B(f) \xrightarrow{\text{def}} (B(f_i, f_j))_{1 \leq i, j \leq d}$$

belongs to the variety

$$\mathcal{V}_Q = \left\{ g \in \text{GL}(d, \mathbb{R}) : \begin{array}{c} \det(g) = \det(Q) \\ \text{sign}(g) = \text{sign}(Q) \end{array} \right\}$$

One can show that $B(\mathcal{F}_Z^d)$ is dense in $\mathcal{V}_Q$, and moreover,

**Theorem 7** Let $\text{sign}(B) = (p, q)$, $p \leq q$. For a "nice" bounded $\Omega \subset \mathcal{V}_Q$,

$$\# \{ f \in \mathcal{F}_Z^d : B(f) \in \Omega, \|f_i\| < T \} \sim m_Q(\Omega) \cdot \left\{ \begin{array}{c} T^{p(q-1)}, \quad p < q, \\ T^p(p-1)(\log T), \quad p = q, \end{array} \right. \quad \text{where } m_Q \text{ is an (explicit) smooth measure on } \mathcal{V}_Q.$$