Homework set 4 (due Wed., April 26)

1. A tangent bundle $TM$ is called trivial if there exists a diffeomorphism $f : TM \to M \times \mathbb{R}^d$, $d = \dim M$, such that the following diagram commute:

$$
\begin{array}{ccc}
TM & \xrightarrow{f} & M \times \mathbb{R}^d \\
\downarrow & & \downarrow \\
M & \downarrow & \end{array}
$$

and $f$ is linear on the fibers.

(a) Prove that a tangent bundle is trivial if and only if there exist vector fields $X_1, \ldots, X_d$ such that $X_1(p), \ldots, X_d(p)$ are linearly independent for every $p \in M$.

(b) Give an example of a tangent bundle which is not trivial.

2. A Lie group $G$ is a group with a structure of a smooth manifold such that the maps $(g, h) \mapsto gh$ and $g \mapsto g^{-1}$ are smooth.

(a) Prove that the tangent bundle $TG$ is trivial.

(b) Prove that the tangent bundle $TS^3$ is trivial. (Hint: use the group SU(2).)

3. Show that the following set is a submanifold of $\mathbb{C}^{n+1}$

$$
\left\{ (z_0, \ldots, z_n) \in \mathbb{C}^{n+1} : \sum_{i=0}^{n} z_i^2 = 1 \right\}
$$

and prove that it is diffeomorphic to the tangent bundle $TS^n$.

4. Give an example of a flow on a projective space $\mathbb{RP}^2$, which has exactly one fixed point and all other orbits are periodic.

5. Let $M = \mathbb{R}^2$ and $X = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}$ be a vector field on $M$. Find the corresponding flow on $M$. 
