Homework set 2 (due Wed., April 12)

1. Prove that the following sets of $n \times n$ real matrices are submanifolds of $\mathbb{R}^{n^2}$ and determine their dimensions:
   (a) $\text{SL}(n, \mathbb{R}) = \{ X : \det X = 1 \}$,
   (b) $\text{O}(n, \mathbb{R}) = \{ X : \; ^tX \cdot X = E \}$.

2. Show that the set of $n \times n$ real matrices of rank $n - 1$ is a submanifold of $\mathbb{R}^{n^2}$. What is its dimension?

3. (a) Let $M$ be a smooth manifold and $N$ its closed submanifold. Prove that for any smooth function $f : N \to \mathbb{R}$ has a smooth extension, that is, there exists a smooth function $F : M \to \mathbb{R}$ such that $F|_N = f$.
   (b) Show that without the assumption that $N$ is closed, the previous claim is false.

4. (a) Let $M$ be a smooth manifold, $C$ a closed subset of $M$, and $U \supset C$ an open subset of $M$. Prove that there exists a smooth function $f : M \to [0, 1]$ such that $f|_C = 0$ and $f|_{M \setminus U} = 1$.
   (b) A function $f : M \to \mathbb{R}$ is called proper if the preimage of every compact subset of $\mathbb{R}$ is compact. Prove that on every smooth manifold $M$, there exists a smooth proper function. This result can be useful for proving the Whitney Embedding Theorem for noncompact manifolds.