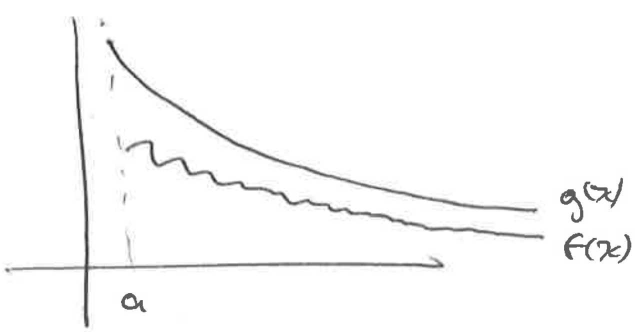
<u>Comparison</u>: sometimes we can determine whether $\int_a^\infty f(x)\,dx$ converges/diverges even if we can't evaluate the integral directly:

<u>Theorem</u>: Sps $f(x), g(x)$ are continuous for $a \le x < \infty$ and $0 \le f(x) \le g(x)$ for $a \le x \lneqq \infty$. Then:

1) if $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges

2) if $\int_a^\infty f(x)\,dx$ diverges, so does $\int_a^\infty g(x)\,dx$



<u>ex</u>: determine convergence / divergence of

1) $\int_1^\infty \frac{\sin^2 x^2}{x^2}\, dx$

2) $\int_1^\infty \frac{1+e^{-x}}{x^2}\, dx$

<u>Sol'n</u>: ① - Since $0 \le \sin^2 x^2 \le 1$ we know

$$\frac{\sin^2 x^2}{x^2} \le \frac{1}{x^2} \quad \text{for } 1 \le x < \infty$$

- Since $\int_1^\infty \frac{1}{x^2}\, dx$ converges

we must have $\int_1^\infty \frac{\sin^2 x^2}{x^2}\, dx$

converges as well by comparison theorem

② Since $1+e^{-x} > 1$ for $1 \le x < \infty$

we know $\frac{1+e^{-x}}{x} > \frac{1}{x}$ " "

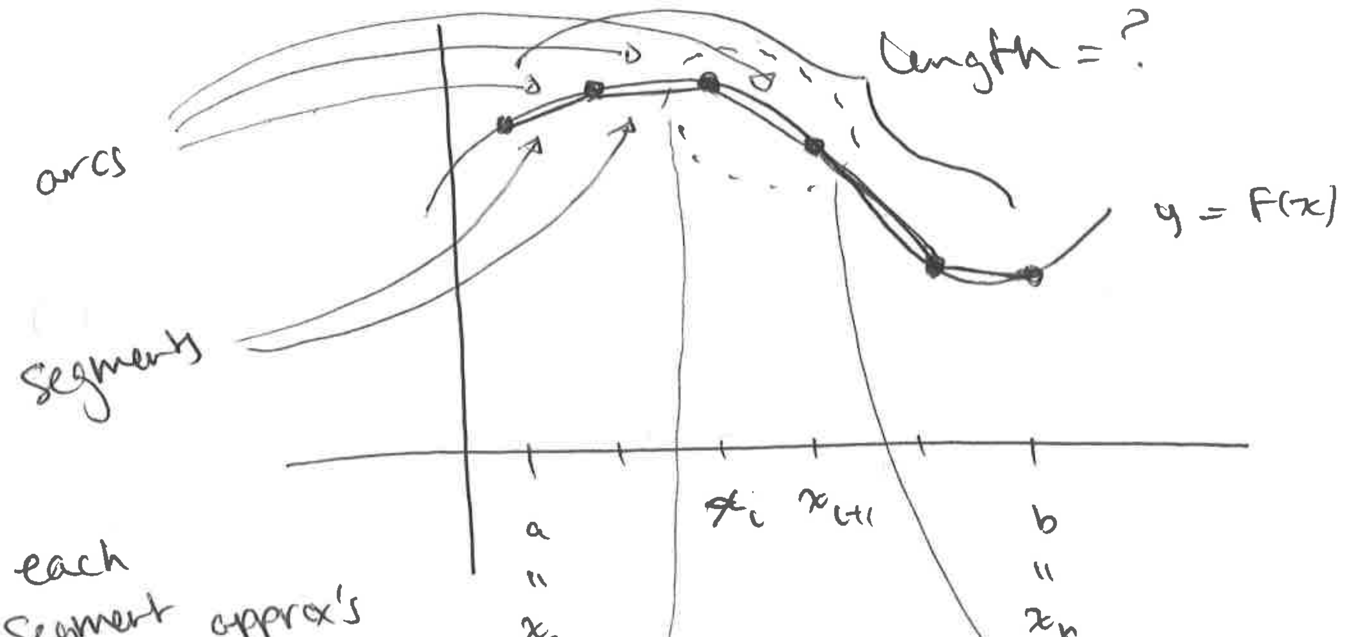
Since $\int_1^\infty \frac{1}{x}\, dx$ diverges, $\int_1^\infty \frac{1+e^{-x}}{x}\, dx$

must diverge as well

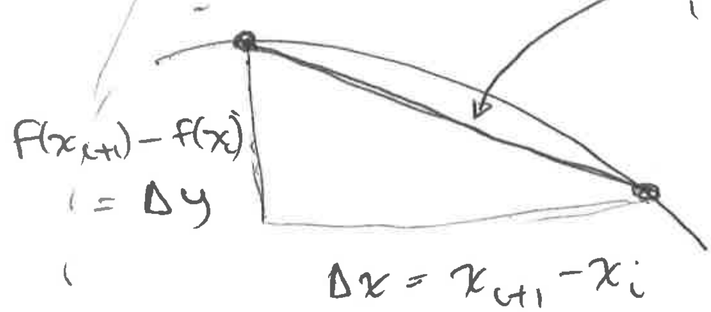
8.1 Arc Length

Q: How can we find the <u>length</u> of a curve $y = f(x)$ over some interval $[a, b]$?

A: approximate (with line segments) and ~~take~~ a limit!



each segment approx's each arc in length - and we can find length of each segment!



$$= \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$= \sqrt{(\Delta x)^2\left(1 + \left(\frac{\Delta y}{\Delta x}\right)^2\right)}$$

$$= \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\ \Delta x$$

$$= \sqrt{1 + \left(\frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}\right)^2}\ \Delta x$$

So the ith segment has length

$$\sqrt{1 + \left(\frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}\right)^2} \; \Delta x$$

if we add these together we approximate actual length of curve:

$$\text{actual length} = L \approx \sum_{i=1}^{n} \sqrt{1 + \left(\frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}\right)^2} \; \Delta x$$

In the limit, this approx'n becomes exact. Also in the limit the quantity $\left(\frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}\right)$ becomes $f'(x_i)$

So:

$$L = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{1 + (\cdots)^2} \; \Delta x$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{1 + (f'(x_i))^2} \; \Delta x$$

but this is just $\int_a^b \sqrt{1 + (f'(x))^2} \, dx$!

3 We've derived the arc length formula:

Thm: assuming $f'(x)$ is continuous on $[a,b]$, length of curve $y = f(x)$ for $a \leq x \leq b$ given by:

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

$$= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

(if instead want to compute length of $x = g(y)$ over $a \leq y \leq b$ we can derive:

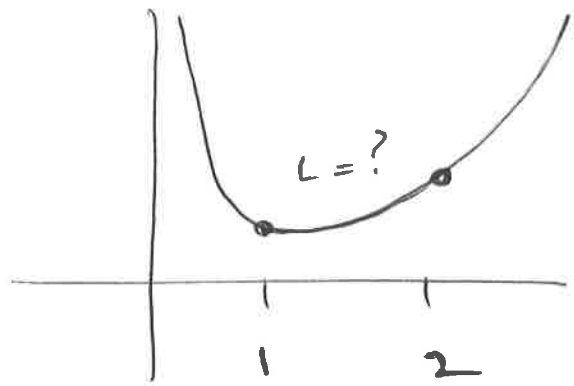
$$L = \int_a^b \sqrt{1 + (g'(y))^2}\, dy$$

$$= \int_a^b \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy\ )$$

④

ex: Find length of $y = \frac{1}{3}x^3 + \frac{1}{4x} = f(x)$

over $1 \le x \le 2$.



$$L = \int_1^2 \sqrt{1+(f')^2}\, dx$$

$$f'(x) = x^2 - \frac{1}{4}x^{-2}$$

$$= \int_1^2 \sqrt{1+\left(x^2-\frac{1}{4}x^{-2}\right)^2}\, dx$$

$$= \int_1^2 \sqrt{1+x^4-\frac{1}{2}+\frac{1}{16}x^{-4}}\, dx$$

$$= \int_1^2 \sqrt{x^4+\frac{1}{2}+\frac{1}{16}x^{-4}}\, dx$$

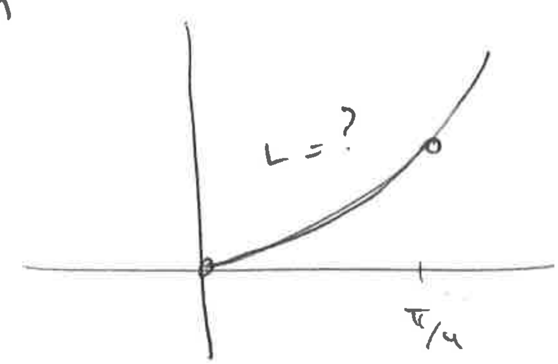
$$= \int_1^2 \sqrt{\left(x^2+\frac{1}{4}x^{-2}\right)^2}\, dx$$

$$= \int_1^2 x^2+\frac{1}{4}x^{-2}\, dx$$

$$= \frac{1}{3}x^3 - \frac{1}{4}x^{-1}\Big|_1^2 = \frac{1}{3}x^3 - \frac{1}{4x}\Big|_1^2$$

$$= \left(\frac{8}{3}-\frac{1}{8}\right) - \left(\frac{1}{3}-\frac{1}{4}\right) = 59/24$$

ex: Find length of $y = \ln(\sec x) = f(x)$ from $x = 0$ to $x = \pi/4$



$$L = \int_0^{\pi/4} \sqrt{1+(f')^2}\, dx$$

$$= \int_0^{\pi/4} \sqrt{1+\left(\frac{1}{\sec x}\cdot \sec x \tan x\right)^2}\, dx$$

$$= \int_0^{\pi/4} \sqrt{1+\tan^2 x}\, dx$$

$$= \int_0^{\pi/4} \sec x\, dx$$

$$= \ln|\sec x + \tan x| \Big|_0^{\pi/4}$$

$$= \ln|\sec \pi/4 + \tan \pi/4| - \ln|\sec 0 + \tan 0|$$

$$= .8814\ldots$$

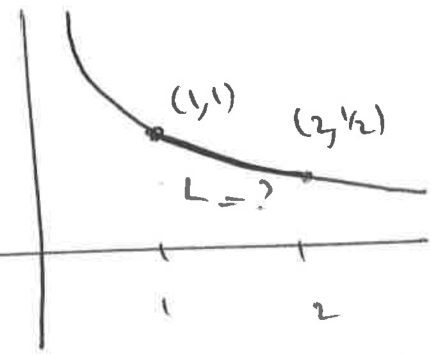
(6)

Sometimes exact evaluation of $\int_a^b \sqrt{1+(f')^2}\,dx$ is impossible, so we approximate.

<u>ex</u>: approximate length of hyperbola $xy=1$ between the points $(1,1)$ and $(2,1/2)$ using Simpson's rule w/ $n=10$.

$$y = \frac{1}{x} = f(x)$$

$$L = \int_1^2 \sqrt{(1+(f')^2)}\,dx$$



$$= \int_1^2 \sqrt{1+\left(\frac{-1}{x^2}\right)^2}\,dx$$

$$= \int_1^2 \sqrt{1+\frac{1}{x^4}}\,dx$$

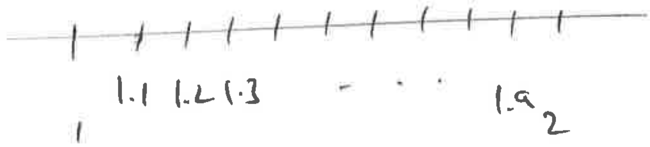
$\to F(x)$

want to approximate

$$\int_1^2 F(x)\,dx \quad \text{using Simpson}$$

w/ $n=10$.

7) In this case $\Delta x = \frac{(2-1)}{10} = .1$ 70



So $\int_1^2 F(x)\,dx \approx \frac{\Delta x}{3}\Big(F(1) + 4F(1.1) + 2F(1.2)$

$+ \cdots + 2F(1.8) + 4F(1.9)$

$+ F(2)\Big)$

$= \frac{.1}{3}\left(\sqrt{1+\frac{1}{1^4}} + 4\sqrt{1+\frac{1}{(1.1)^4}} + \cdots + \sqrt{1+\frac{1}{2^4}}\right)$

$= 1.1321\ldots$

(1)

9.3 Separable Differential Equations (71)

- Sps that $f(x)$ is a function whose identity we are trying to determine
- Sps that while we don't know $f(x)$, we know something about f's dependence on its derivatives
- e.g. $f(x)$ may be the population of a colony of bacteria at time $t = x$, and while we don't know f, we know that the growth rate of the colony (i.e. $f'(x)$) is always twice the current population size.

or, expressed mathematically, we know:

$$f'(x) = 2f(x)$$

[if we let $y = f(x)$ can rewrite eq'n as: $\frac{dy}{dx} = 2y$
or $y' = 2y$

→ such an equation is called a <u>differential equation</u>

other eg's: $y' + y'' = y$ ... $\frac{dy}{dx} - x^2 = y + \sin$ ...

② Q: can we use this eq/n to determine $f(x)$? ⑦②

<u>key idea</u>: if we rearrange:

$$\frac{1}{f(x)} f'(x) = 2$$

we can integrate both sides (wrt x)

$$\int \frac{1}{f(x)} f'(x)\, dx = \int 2\, dx$$

use chain rule! $\qquad = 2x + C$

$$= \ln|f(x)|$$

So we get

$$\ln|f(x)| = 2x + C$$

note: don't need a $+C$ on the left

exponentiating both sides:

$$|f(x)| = e^{2x+C} = e^{2x} e^{C}$$

e^C → call this A

$$= A e^{2x}$$

$$f(x) = \pm A e^{2x} = A e^{2x}$$

→ we've determined $f(x)$ up to a constant A. If we knew initial population is S, i.e. $f(0) = S$, can solve for A:

③

$$f(0) = Ae^{2\cdot 0} = 5$$

$$\Rightarrow A = A$$

so $f(x) = 5e^{2x}$ ✓

notice:

$$\frac{d}{dx}(5e^{2x}) = 10e^{2x} = 2(5e^{2x})$$

i.e. $f(x) = 5e^{2x}$ satisfies $f' = 2f$ ✓

Observe: our integration step is more transparent if we use differential notation:

$$\frac{dy}{dx} = 2y$$

$$\Rightarrow \frac{1}{y}\,dy = 2\,dx$$

$$\Rightarrow \int \frac{1}{y}\,dy = \int 2\,dx$$

$$\Rightarrow \ln|y| = 2x + C$$

$$\Rightarrow y = e^{2x+C} = Ae^{2x}$$ ✓

we say: $y = Ae^{2x}$ is the family of solutions to the equation $\frac{dy}{dx} = 2y$

sometimes write $y' = 2y$

this notational trick really helps! under the hood it's just the chain rule.

↳ can use similar idea to solve other diff eq'ns.

4

A separable eq'n is one of the form

$$\frac{dy}{dx} = g(y)\,f(x)$$

(in our ex: $g(y) = 2y$ $f(x) = 1$)
(here: $y = F(x)$ is the function we're trying to determine)

to solve; rearrange: (grap y's, grap x's)

$$\frac{1}{g(y)}\,dy = f(x)\,dx$$

now integrate both sides:

$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$$

↳ this will give an eq'n implicitly defining y in terms of x that can then be solved for y.

ex: determine the class of functions $y = f(x)$ satisfying the eq'n:

$$y' = x^2 y$$

(5)

Sol'n : we write

$$\frac{dy}{dx} = x^2 y$$

$$\Rightarrow \frac{1}{y}\,dy = x^2\,dx$$

$$\Rightarrow \ln|y| = \tfrac{1}{3}x^3 + C$$

$$\Rightarrow |y| = e^{\frac{1}{3}x^3 + C} = e^{1/3\,x^3} e^C$$

$$\Rightarrow y = \pm e^C e^{1/3\,x^3}$$

$$= A e^{1/3\,x^3}$$ ← general solution

and indeed, for any function of the form $y = Ae^{1/3\,x^3}$

we have $y' = Ae^{1/3\,x^3} \cdot x^2$

$= x^2 y$, so it satisfies

original eq'n

(possible to prove: these are the only sol'ns)

6 ~~ex~~: Find a function satisfying $y' = x e^y$ and the initial condition $y(0) = 0$.

Sol'n:

$$\frac{dy}{dx} = x e^y$$

$$\Rightarrow e^{-y}\, dy = x\, dx$$

$$\Rightarrow -e^{-y} = \frac{1}{2}x^2 + C$$

$$\Rightarrow e^{-y} = -\frac{1}{2}x^2 + C$$

$$\Rightarrow -y = \ln\left(-\frac{1}{2}x^2 + C\right)$$

$$\Rightarrow y = -\ln\left(-\frac{1}{2}x^2 + C\right)$$

so if $y(0) = 0 = -\ln\left(-\frac{1}{2}0^2 + C\right)$

$$\Rightarrow 0 = -\ln(C)$$

$$\Rightarrow C = 1$$

So: $y = f(x) = -\ln\left(-\frac{1}{2}x^2 + 1\right)$

is our solution.

(7) <u>a mixing problem</u>:

A tank contains 20 kg of salt dissolved in 5000 L of water. Water containing .03 kg of salt per liter is pumped into the tank @ a rate of 25 L/min. The solution is kept mixed and also drains at a rate of 25 L/min. How much salt is in the tank @ 30 min?

<u>Sol'n</u>: Let $y(t)$ = amount of salt in tank at time t (in kg).

amount of salt entering/min is:

$$(25\ L/min)(.03\ kg/L) = .75\ kg/min$$

amount of salt leaving/min is:

(25 L/min)(current concentration of salt)

$$= (25\ L/min)\left(\frac{y(t)}{5000}\ kg/L\right)$$

8 So overall rate of change of salt w: = rate in − rate out

$$= .75 - \frac{y(t)}{200}$$

i.e. $\frac{dy}{dt} = .75 - \frac{y}{200}$

we can solve:

$$\frac{dy}{dt} = \frac{150-y}{200}$$

$$\Rightarrow \frac{200}{150-y}\,dy = dt$$

$$\Rightarrow -200 \ln|150-y| = t + C$$

$$\Rightarrow \ln|150-y| = -\frac{t}{200} + C$$

but we know $y(0) = 20$ so:

$$\ln|150-y(0)| = -\frac{0}{200} + C$$

$$\ln|130| = C$$

9

So eq'n becomes:

$$\ln|150-y(t)| = -\frac{t}{200} + \ln(130)$$

Solving:

$$150 - y(t) = e^{-t/200} \cdot 130$$

$$\Rightarrow \quad y(t) = 150 - 130\,e^{-t/200}$$

gives amount of salt at time t min.

$$\Rightarrow y(30) = 150 - 130\,e^{-30/200}$$

$$\approx 38.1 \text{ kg.}$$

9.2 Direction fields

- last time, how solve diff'eqns of form

$$y' = f(x)\,g(y)$$

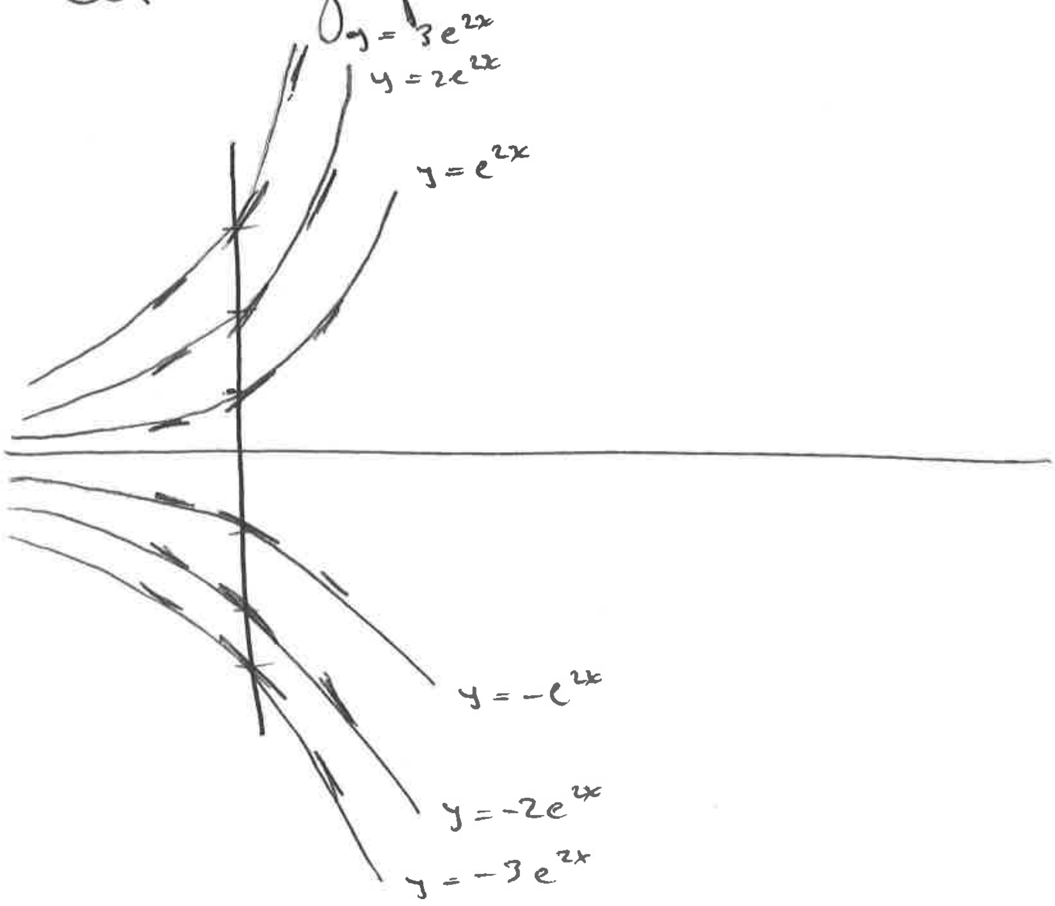
- called <u>separable</u> eq'ns.
- e.g. we showed that eq'n

$$y' = 2y \qquad (f = 1,\ g = 2y)$$

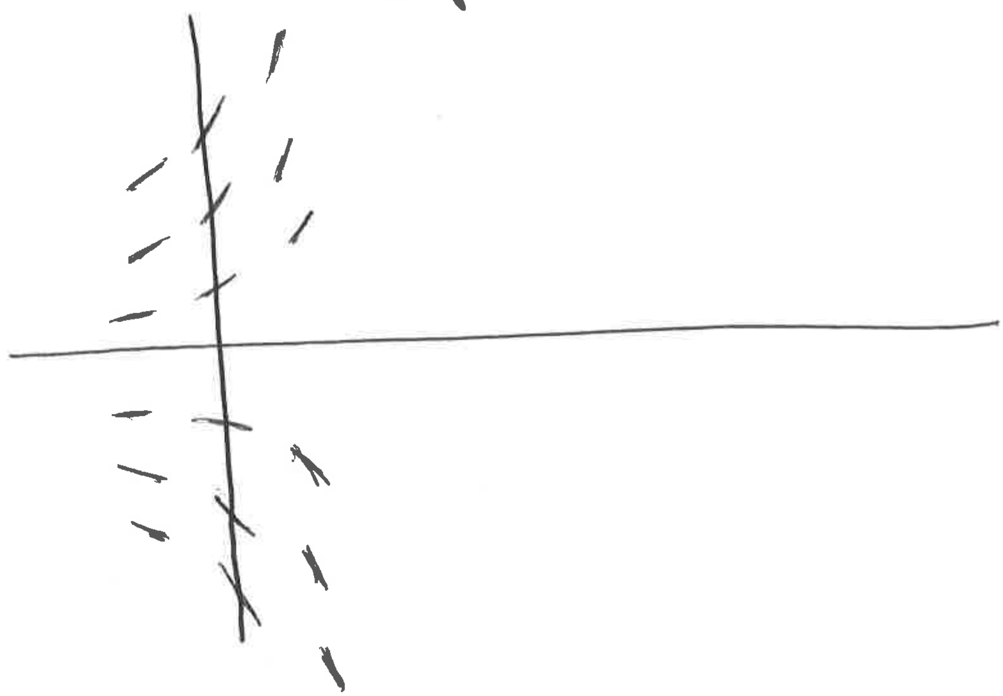
has family of sol'ns:

$$y = Ae^{2x}$$

- Let's graph sol'ns for $A = -3, \cancel{-2}, -1, 1, 2, 3$:



if we draw some mini tan lines at various points along curve, and then erase curves, retain a ~~[illegible]~~ our family: "slope field" for



- can determine slopes of these tan lines from <u>original</u> <u>eq'n</u>
- will use this idea to sketch solutions to diff'eqs we can't solve explicitly

<u>Ex</u>: Consider eq'n

$$y' = x + y$$

(3)

- not a separable eq'n
- instead of solving explicitly for family of sol'ns, we sketch their graphs using a slope field
- the point: if $y = f(x)$ is sol'n passing thru (x_0, y_0), eq'n tells tan line to f there has slope $x_0 + y_0$.

$x + y = y'$

- so e.g., sol'n thru $(0,1)$ has slope $0+1 = 1$
  " " $(0,2)$ " " $0+2 = 2$
  " " $(1,1)$ " " $1+1 = 2$

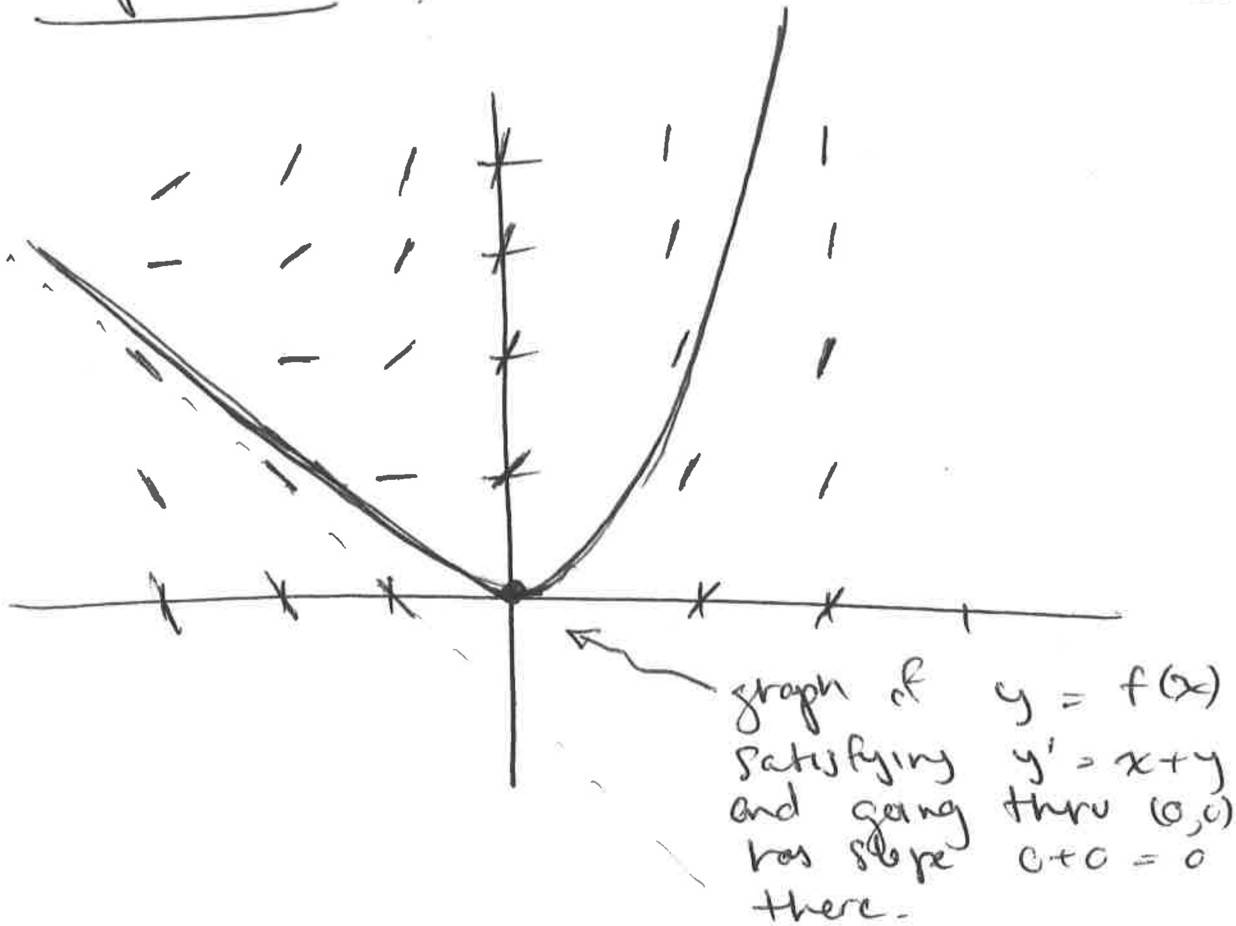
Table of slopes:

x

y \ x	-3	-2	-1	0	1	2
1	-2	-1	0	1	2	3
2	-1	0	1	2	3	4
3	0	1	2	3	4	5
4	1	2	3	4	5	6

$\leftarrow y'$

4 Slope field.



- can get rough sketch for a particular solution $y = f(x)$ going thru $(0,0)$ (e.g.) by "following field lines"

(actually family is given by:

$$y = ce^x - x - 1)$$

(check: $y' = ce^x - 1 = ce^x - x - 1 + x$
$= y + x$ ✓)

So sol'n thru $(0,0)$ is $y = e^x - x - 1$

(8) ex: a) Sketch slope field for eq'n (F4)

$y' = \cos^2(y)$ for $-\pi/2 \le y \le \pi/2$

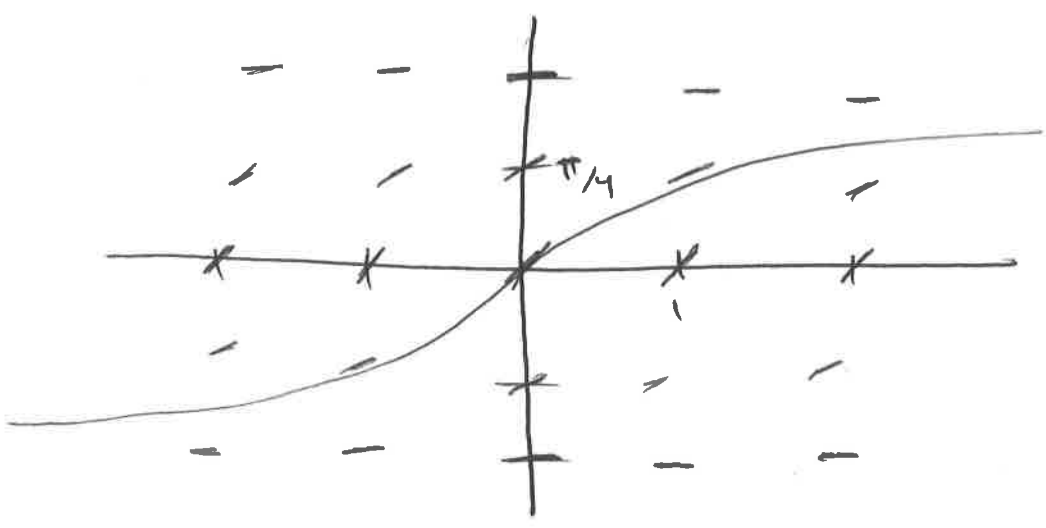
$-2 \le x \le 2$

b) Solve eq'n explicitly for sol'n through $(0,0)$

(9)

y \ x	-2	-1	0	1	2
$-\pi/2$	0	0	0	0	0
$-\pi/4$	$1/2$	$1/2$	$1/2$	$1/2$	$1/2$
0	1	1	1	1	1
$\pi/4$	$1/2$	$1/2$	$1/2$	$1/2$	$1/2$
$\pi/2$	0	0	0	0	0

notice: y' depends only on y



⑥ (b) eq'n is separable:

$$\frac{dy}{dx} = \cos^2 y$$

$$\Rightarrow \frac{1}{\cos^2 y}\, dy = dx$$

$$\Rightarrow \sec^2 y\, dy = dx$$

$$\Rightarrow \int \sec^2 y\, dy = \int dx$$

$$\Rightarrow \tan(y) = x + C$$

If $x = y = 0$ then $\tan(0) = 0 + C$

$$\Rightarrow 0 = C$$

So sol'n through $(0, 0)$ is given by $\tan(y) = x$

$$y = \tan^{-1}(x)$$ ✓