Partial Fractions

- it's easy to integrate a polynomial

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

- but what about a quotient of polynomials?

$$\frac{P(x)}{Q(x)}$$

aka a rational function?

ex Find $\displaystyle\int \frac{x^3+x^2+1}{x^2-1}\,dx$

Sol'n - need to somehow simplify expression before integrating

- first step always: if deg of numerator $\geq$ deg of denominator, divide!

$$\begin{array}{r@{}l} & \quad x + 1 \\ x^2-1 \; \big) & \; x^3 + x^2 + 0x + 1 \\ & -(x^3 + 0x^2 - x) \\ & \qquad\quad x^2 + x + 1 \\ & \quad\;\; -(x^2 + 0x - 1) \\ & \qquad\qquad x + 2 \end{array}$$

So: $$\frac{x^3+x^2+1}{x^2-1} = x+1+\frac{x+2}{x^2-1}$$

So: $$\int \frac{x^3+x^2+1}{x^2-1}\,dx = \int (x+1)\,dx + \int \frac{x+2}{x^2-1}\,dx$$

can handle this

but this?

General question: how to integrate

$$\int \frac{P(x)}{Q(x)}\,dx$$

when $\deg P < \deg Q$?

Answer: partial fraction decomposition

↳ First: factor $Q(x)$ into linear factors $(ax+b)$ and irreducible quadratic factors (ax^2+bx+c with $b^2-4ac<0$)

— always possible by FTA.

easiest case: when $Q(x)$ factors into distinct linear factors

$$Q(x) = l_1(x)\, l_2(x) \cdots l_n(x)$$

– then we set:

$$\frac{P(x)}{Q(x)} = \frac{A_1}{l_1(x)} + \frac{A_2}{l_2(x)} + \cdots + \frac{A_n}{l_n(x)}$$

and solve for the A_i.

<u>ex</u>: Consider:

$$\frac{x+2}{x^2-1} = \frac{x+2}{(x-1)(x+1)}$$ distinct linear factors

so we set $= \dfrac{A}{x-1} + \dfrac{B}{x+1}$

$$\Rightarrow \quad x+2 = A(x+1) + B(x-1)$$

<u>neat idea</u>: eq'n holds for <u>all</u> values of x, so letting $x = -1$ gives

$$1 = A \cdot 0 + B(-2)$$

$$\Rightarrow B = -\tfrac{1}{2}$$

and letting $x = 1$ gives

$$3 = A \cdot 2 + B \cdot 0$$

$$\Rightarrow A = \frac{3}{2}$$

so: $$\frac{x+2}{x^2-1} = \frac{x+2}{(x-1)(x+1)} = \frac{\frac{3}{2}}{x-1} + \frac{-\frac{1}{2}}{x+1}$$

so e.g. if we wanted to find:

$$\int \frac{x^3+x^2+1}{x^2-1}\,dx = \int x+1+\frac{x+2}{x^2-1}\,dx$$

$$= \int x+1+\frac{\frac{3}{2}}{x-1}+\frac{-\frac{1}{2}}{x+1}\,dx$$

$$= \frac{1}{2}x^2 + x + \frac{3}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C \checkmark$$

<u>another way</u> to solve for A, B in:

$$x+2 = A(x+1) + B(x-1)$$

$$\Rightarrow x+2 = Ax + A + Bx - B$$
$$= (A+B)x + (A-B)$$

must match

must match

$$A+B = 1 \qquad \Rightarrow 2A = 3 \Rightarrow A = \frac{3}{2}$$
$$A-B = 2 \qquad \Rightarrow 2B = -1 \Rightarrow B = -\frac{1}{2}$$

this method of equating coefficients <u>always</u> works, whereas choosing x's like before sometimes doesn't get you there.

What if there are repeated linear factors?

__ex__: Find $\int \frac{x^2+1}{x^3-x^2-x+1}\,dx$

__Sol'n__ deg num < deg denom so we factor denom:

$$\begin{aligned} x^3-x^2-x+1 &= (x^2-1)(x-1) \\ &= (x+1)(x-1)(x-1) \\ &= (x+1)\underbrace{(x-1)^2}_{\text{repeated factor}} \end{aligned}$$

p.f.d. now has three terms:
- one for $x+1$
- one for each factor of $(x-1)$

$$\frac{x^2+1}{x^3-x^2-x+1} = \frac{x^2+1}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$\Rightarrow x^2+1 = A(x-1)^2 + B(x+1)(x-1) + C(x+1)$$

$x=1 \Rightarrow \quad 2 = C\cdot 2 \Rightarrow C=1$

$x=-1 \Rightarrow \quad 2 = A\cdot 4 \Rightarrow A = \frac{1}{2}$

now can solve for B setting e.g. $x=0$

$x=0 \Rightarrow \quad 1 = 1\cdot A - 1\cdot B + 1\cdot C$

$= 1\cdot\frac{1}{2} - 1\cdot B + 1\cdot 1$

$\Rightarrow B = 1/2$

so: $\displaystyle\int \frac{x^2+1}{x^3-x^2-x+1}\,dx = \int \frac{\frac{1}{2}}{x+1} + \frac{\frac{1}{2}}{x-1} + \frac{1}{(x-1)^2}\,dx$

$= \frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1|$

$- \frac{1}{x-1} + C$ ✓

note: if we had higher powers of our factors, would add a term for each one:

e.g. for $\dfrac{x^2+1}{(x-1)^3(x+1)^2}$ would set

$= \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{(x-1)^3} + \dfrac{D}{x+1} + \dfrac{E}{(x+1)^2}$

and solve.

What about irred. quad. factors?

ex Find $\int \frac{x+2}{x(x^2+1)(x^2+4)}\,dx$

distinct irred. quad. factors

sol: $\frac{x+2}{x(x^2+1)(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{x^2+4}$

$$\Rightarrow x+2 = A(x^2+1)(x^2+4) + (Bx+C)(x)(x^2+4) + (Dx+E)(x)(x^2+1)$$

if $x = 0 \Rightarrow 2 = A\cdot 1\cdot 4$

$\Rightarrow A = \frac{1}{2}$

if $x = i = \sqrt{-1} \Rightarrow i+2 = (Bi+C)(i)(3)$

$= -3B + 3Ci$

$\Rightarrow 3C = 1 \qquad -3B = 2$

$\Rightarrow C = \frac{1}{3} \qquad B = -\frac{2}{3}$

if $x = 2i \Rightarrow (D2i+E)(2i)(-3) = 2i+2$

$\Rightarrow \quad 12D - 6Ci = 2 + 2i$

$\Rightarrow \quad 12D = 2 \qquad -6C = 2$

$\Rightarrow \quad D = 1/6 \qquad C = -1/3$

so: $\displaystyle\int \frac{x+2}{x(x^2+1)(x^2+4)}\,dx = \int \frac{1/2}{x} + \frac{(-2/3)x + 1/3}{x^2+1}$

$\displaystyle + \frac{\frac{1}{6}x - 1/3}{x^2+4}\,dx$

$\displaystyle = \frac{1}{2}\ln|x| - \frac{2}{3}\int \frac{x}{x^2+1}\,dx + \frac{1}{3}\int \frac{1}{x^2+1}\,dx$

$\displaystyle + \frac{1}{6}\int \frac{x}{x^2+4}\,dx - \frac{1}{3}\int \frac{1}{x^2+4}\,dx$

u-sub

use arctan

$\displaystyle = \frac{1}{2}\ln|x| - \frac{1}{3}\ln|x^2+1| + \frac{1}{3}\tan^{-1}(x)$

$\displaystyle + \frac{1}{12}\ln|x^2+4| - \frac{1}{6}\tan^{-1}\left(\frac{x}{2}\right) + C$ ✓

repeated quad factors handled sim to linear case:

e.g. to find p.f.d. for

$$\frac{x^2+1}{x(x^2+1)^2(x^2+4)} \quad \text{we set}$$

$$= \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2} + \frac{Fx+G}{x^2+4}$$

things get ugly...

7.5 More Integral Practice

#5 $$\int \frac{t}{t^4+2}\,dt$$

$u = t^2$

$du = 2t\,dt$

$\frac{1}{2}du = t\,dt$

$$\frac{1}{2}\int \frac{1}{u^2+2}\,du$$

$u = \sqrt{2}\tan\theta$

$du = \sqrt{2}\sec^2\theta\,d\theta$

$$\frac{1}{2}\int \frac{1}{2\tan^2\theta+2}\sqrt{2}\sec^2\theta\,d\theta$$

$$\frac{1}{2}\int \frac{\sqrt{2}}{2}\cancel{\sec^2\theta}\,\frac{1}{\cancel{\sec^2\theta}}\,d\theta$$

$$= \frac{\sqrt{2}}{4}\theta = \frac{\sqrt{2}}{4}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right)$$

$$= \frac{\sqrt{2}}{4}\tan^{-1}\left(\frac{t^2}{\sqrt{2}}\right) + C$$

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#8 $\int t \sin t \cos t \, dt$

$= \frac{1}{2}\int t \sin(2t)\, dt$

$u = t \qquad dv = \sin 2t$

$du = dt \qquad v = \frac{-1}{2}\cos 2t$

$= \frac{1}{2}\left(-\frac{1}{2}t\cos 2t + \frac{1}{2}\int \cos 2t\, dt\right)$

$= \frac{1}{2}\left(-\frac{1}{2}t\cos 2t + \frac{1}{2}\frac{1}{2}\sin 2t\right)$

$= -\frac{1}{4}t\cos 2t + \frac{1}{8}\sin 2t + C$

#12 $\int \frac{2x-3}{x^3+3x}\, dx$

$\Rightarrow \frac{2x-3}{x^3+3x} = \frac{2x-3}{x(x^2+1)}$

$= \frac{A}{x} + \frac{Bx+C}{x^2+1}$

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$\boxed{\#14}$ $\int \ln(1+x^2)\,dx$

Spicy!

$x = \tan\theta \qquad dx = \sec^2\theta\, d\theta$

$\int \ln(1+\tan^2\theta)\sec^2\theta\, d\theta$

$= \int \ln(\sec^2\theta)\sec^2\theta\, d\theta$

$u = \ln(\sec^2\theta) \qquad dv = \sec^2\theta\, d\theta$

$du = \frac{1}{\cancel{\sec^2\theta}} \cdot 2\cancel{\sec\theta} \cdot \cancel{\sec\theta}\tan\theta\, d\theta \qquad v = \tan\theta$

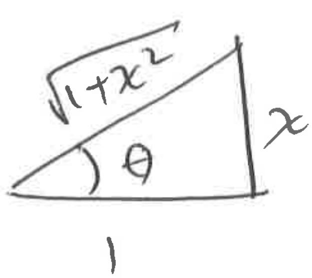
$= 2\tan\theta$

$\overset{IBP}{=} \tan\theta\ln(\sec^2\theta) - 2\int \tan^2\theta\, d\theta$

$- 2\int \sec^2\theta - 1\, d\theta$

$- 2(\tan\theta - \theta)$

$= \tan\theta\ln(\sec^2\theta) - 2\tan\theta + 2\theta$



$\cos\theta = \frac{1}{\sqrt{1+x^2}} \qquad \sec\theta = \sqrt{1+x^2}$

so

$= \boxed{x\ln\left(\sqrt{1+x^2}\right) - 2x + 2\tan^{-1}(x) + C}$

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#19 $\int e^{x+e^x}\,dx$

$= \int e^x e^{e^x}\,dx$

$u = e^x$

$du = e^x\,dx$

$= \int e^u\,du$

$\vdots$ ✓

#23 $\int_0^1 (1+\sqrt{x})^8$

$u = 1+\sqrt{x}$ $\qquad du = \frac{1}{2}x^{-1/2}\,dx$

$\sqrt{x} = (u-1)$ $\qquad \sqrt{x}\,du = \frac{1}{2}\,dx$

$(u-1)\,du = \frac{1}{2}\,dx$

$2(u-1)\,du = dx$

$2\int_1^2 (u-1)u^8 = 2\left(\frac{1}{10}u^{10} - \frac{1}{9}u^9\right)\Big|_1^2 \ldots$ ✓