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Analytical Model of the Time Developing Turbulent Boundary Layer¶

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An analytical model for the time-developing turbulent boundary layer (TD TBL) over a flat plate is presented. The model provides explicit formulae for the temporal behavior of the wall-shear stress and both the temporal and spatial distributions of the mean streamwise velocity, the turbulence kinetic energy and Reynolds shear stress. The resulting profiles are in good agreement with the DNS results of spatially-developing turbulent boundary layers at momentum thickness Reynolds numbers equal to 1430 and 2900 [5–7]. Our analytical model is, to the best of our knowledge, the first of its kind for TD TBL.

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Formulation of the problem. In this paper, we derive an analytical model for the time-developing turbulent boundary layer (TD TBL) over a flat plate. We consider a flat plate ($z = 0$) submerged in an incompressible viscous fluid at rest for time $t < 0$. At time $t = 0$, the fluid moves as a whole in the x direction with velocity V_∞. This motion creates a boundary layer near the plate. We assume that this boundary layer is turbulent. Note that, in the case of TD TBL, all the statistical characteristics of the flow depend only on the time t and distance z from the wall [e.g., the mean streamwise velocity is $V(z, t)$]. In contrast, in a spatially-developing TBL (SD TBL), the statistical characteristics depend on two space variables, z and x, and the mean velocity also has the V_z component normal to the wall. Nevertheless, in the limit of large Reynolds numbers, both the TD TBL and SD TBL become asymptotically equivalent (with the replacement $x \longleftrightarrow V_\infty t$) [1]. Therefore, it is reasonable to first consider the simpler simple case of TD TBL.

Definitions and model equations. We start from the Navier–Stokes equations for an incompressible fluid. The velocity $\mathbf{U}(\mathbf{r}, t)$ is decomposed into the sum of its mean value $\mathbf{V}(z, t) \equiv \langle \mathbf{U}(\mathbf{r}, t)\rangle$ and a fluctuating part $\mathbf{u}(\mathbf{r}, t)$, $\mathbf{U}(\mathbf{r}, t) = \hat{\mathbf{x}}\, V(z, t) + \mathbf{u}(\mathbf{r}, t)$. Here, $\hat{\mathbf{x}}$ is the unit vector in the x direction, $\mathbf{r} = \{x, y, z\}$ is the three-dimensional coordinate, and $\langle \ldots \rangle$ denotes averaging in time and in the span-wise direction y.

The three main quantities in the model are the mean shear $S(z, t)$; the tangential Reynolds stress $\tau(z, t)$; and the turbulence kinetic energy per unit of mass $K(z, t)$, which is defined as

$$S(z, t) \equiv \frac{\partial V}{\partial z}, \quad \tau(z, t) \equiv -\langle u_x u_z \rangle, \qquad K(z, t) = \frac{1}{2}\langle |\mathbf{u}|^2 \rangle. \tag{1}$$

The mean momentum equation (e.g., Eq. (4.12) in [2]) after integration over z has the form

$$\nu S(z, t) + \tau(z, t) = \tau_*(t) + \frac{\partial}{\partial t}\int_0^z V(z', t)dz'. \tag{2}$$

Here, ν is the kinematic viscosity and the right-hand side (RHS) is the momentum flux toward the wall. The integration constant $\tau_*(t) = \nu S(0, t)$ is the wall shear stress.

The turbulence kinetic energy conservation equation for a TD TBL (see, e.g., Eq. 5.132 in [2]) can be written as

$$\partial K(z, t)/\partial t + \mathcal{E}(z, t) + \nabla \cdot \mathbf{T}(z, t) = \tau(z, t)S(z, t). \tag{3}$$

The RHS of this equation represents the production of turbulence kinetic energy by the mean shear. The two terms in the left hand side (LHS)—the rate of energy dissipation, $\mathcal{E}$, and the spatial energy flux, $\mathbf{T}$—require modeling via S, τ, and K.

Equations (2) and (3) for S and K are exact. In order to solve them, we need to add a third equation for $\tau(z, t)$ and model both $\mathcal{E}(z, t)$ and $\mathbf{T}(z, t)$. It is reasonable to assume that, in the log layer (the region where the

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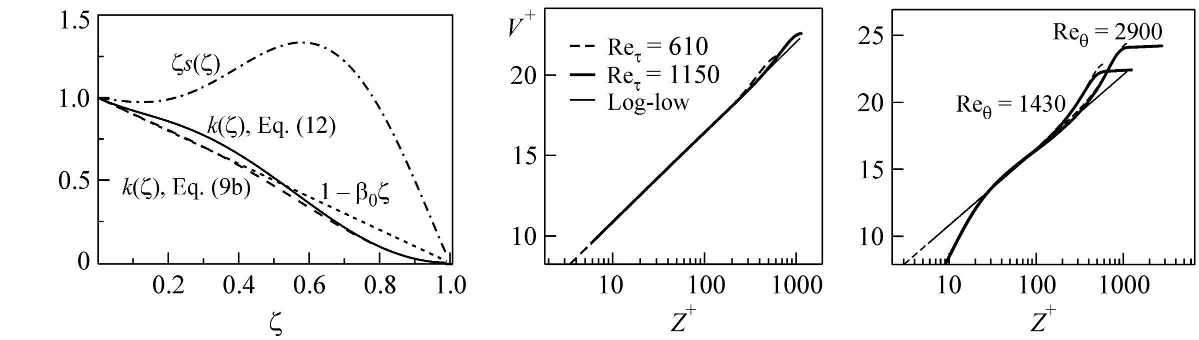

Fig. 1. Color online. Left panel: profiles of the approximation of the kinetic energy and compensated shear. For the discussion, see the text. Middle panel: log plots of the mean velocity profiles $V^+(z^+)$ in the min model for $\mathrm{Re}_\tau \approx 610$ and 1150. Right panel: comparison of the mean velocity profiles given by the improved min model (dashed lines) and DNS data for $\mathrm{Re}_\theta = 1430$ giving $\mathrm{Re}_\tau \approx 610$ and, for $\mathrm{Re}_\theta = 2900$, giving $\mathrm{Re}_\tau \approx 1150$. The von Kármán log law is also shown in the middle and right panels as a straight solid line.

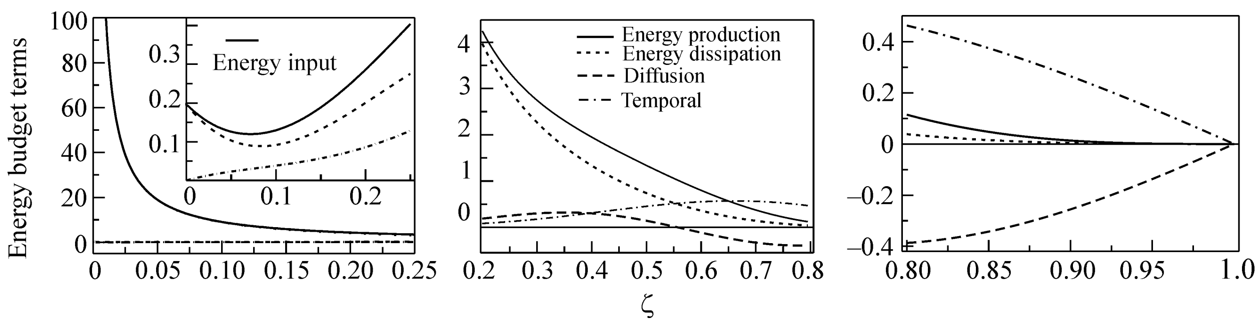

Fig. 2. Color online. Energy balance in Eq. (9a). The energy input (the difference between the energy production and dissipation) is shown in the insert in the left panel. The line types are the same in all the panels. The different panels display different regions of the TBL.

von Kármán log law holds), the shear and normal Reynolds stresses have the same z and t dependence; thus, their ratio is constant: $\tau(z, t)/K(z, t) = c^2$.

As suggested by [2], we write the rate of turbulence kinetic energy dissipation as $\mathcal{E}(z, t) = b[K(z, t)]^{3/2}/z$, where b is a positive constant, and the spatial energy flux in the z direction as $T(z, t) = -D(z, t)\partial K(z, t)/\partial z$, where $D(z, t) = dz\sqrt{K(z, t)}$ is the turbulence diffusivity and d is a positive coefficient.

Finally, we summarize the equations of the present model (the minimalist model (min model)) for TD TBL as

$$\nu S(z, t) + \tau(z, t) = \tau_*(t) + \int_0^z dz' \frac{\partial}{\partial t} V(z', t), \quad (4a)$$

$$\left[\frac{\partial}{\partial t} + \frac{b\sqrt{K(z, t)}}{z} - d\frac{\partial}{\partial z} z\sqrt{K(z, t)}\frac{\partial}{\partial z}\right] K(z, t) = \tau(z, t) S(z, t), \quad (4b)$$

$$\tau(z, t) = c^2 K(z, t). \quad (4c)$$

The boundary conditions at the wall (z = 0) and at the edge of the TBL ($z = \mathcal{L}(t)$) are the following:

$$\text{at the wall } z = 0\text{:} \quad V(0, t) = 0, \quad K(0, t) = 0, \quad (5a)$$

$$\text{at } z = \mathcal{L}(\dot{t})\text{:} \quad V(\mathcal{L}, t) = V_\infty, \quad K(\mathcal{L}(t), \dot{t}) = 0. \quad (5b)$$

The numerical values of the model coefficients b and c are prescribed to be 0.34 and 0.53, respectively, to obtain results in agreement with the experiments (e.g., [2]) and with the DNS data (see Fig. 3 in [3]). The third coefficient d is fixed equal to 0.07 in order to match the role of the turbulent diffusivity in fully-developed turbulent channel flows [4].

We normalize the variables of the TD TBL in "wall units" using the friction velocity $u_*(t)$, viscous length scale $l_*(t)$, and viscous time scale $t_*(t)$ defined as

$$u_* \equiv \sqrt{\tau_*(t)}, \quad l_* \equiv \nu/u_*, \quad t_* \equiv \nu/\tau_*(t) \quad (6)$$

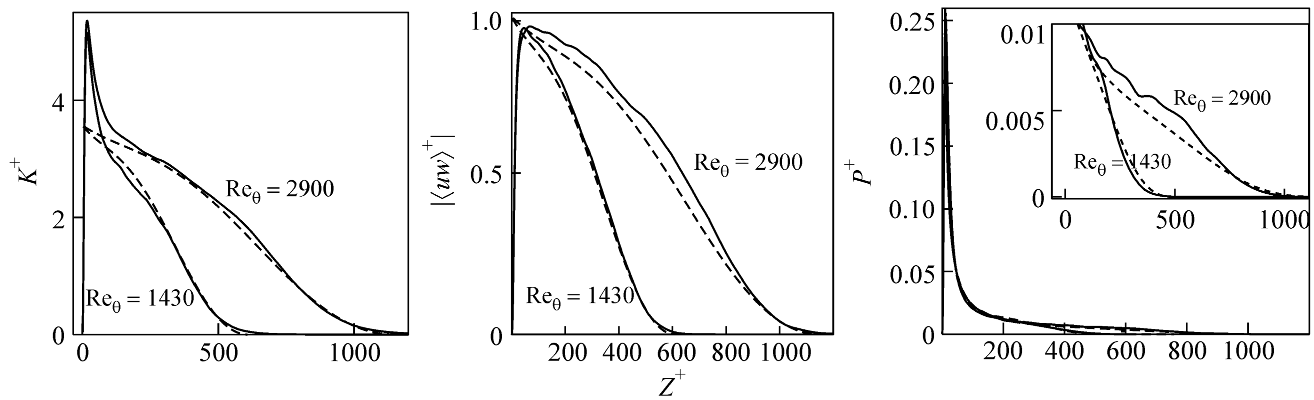

Fig. 3. Color online. Comparison of (dashed lines) the analytical predictions and (solid lines) the DNS results for (left) the turbulent kinetic energy, (middle) Reynolds stress, and (right) production term. The lines corresponding to $\mathrm{Re}_\theta = 1430$ (giving $\mathrm{Re}_\tau \approx 610$) and to $\mathrm{Re}_\theta = 2900$ (giving $\mathrm{Re}_\tau \approx 1150$) are marked in the figure. The insert in the right panel shows details of the large z^+ tails.

and denote the normalized variables $V^+ \equiv V/u_*$, $K^+ \equiv K/u_*^2$, $z^+ \equiv z/l_*$, $t^+ \equiv t/t_*$. The friction velocity Reynolds number is defined as $\mathrm{Re}_\tau(t) \equiv u_*\mathcal{L}(t)/\nu = \mathcal{L}(t)/l_* = \mathcal{L}^+(t)$, which is the width of the TD TBL in wall units. It should be noted that a similar normalization is usually employed for the spatially developing TBL (SD TBL) except that the dependent variables, in that case, are functions of the streamwise x location instead of time.

The main advantage of the above normalization is that, in wall units, the stationary SD TBL demonstrates the universal behavior. We assume that the time-developing TBL exhibits similar universality in the limit of large times, as will be clarified later. We will also show that the time-developing TBL has properties very similar to that of the stationary TBL and its subregions can be classified in the same manner.

"Logarithmic accuracy" for "asymptotically large times." For asymptotically large times we mean that $\ln t^+ \gg 1$. We assume that, in this limit, there is a region in the TBL called the log layer (with z^+ larger than the upper boundary of the buffer sublayer z^+_{buf}), where the mean streamwise velocity profile is described by the von Kármán law:

$$V^+(z^+) = \kappa^{-1}\ln z^+ + B, \quad z^+ > z^+_{\mathrm{buf}} \approx 50, \qquad (7)$$

where $\kappa \approx 0.41$ is the von Kármán constant, and $B \approx 5.2$. We define the edge of the TD TBL, $\mathcal{L}(t)$, as the normal distance from the wall where $V = V_\infty$. Thus, the von Kármán law at $\mathcal{L}^+(t)$ becomes $V^+_\infty \equiv V^+(\mathcal{L}^+) = \kappa^{-1}\ln\mathcal{L}^+ + B$. Moreover, in the limit $\ln t^+ \gg 1$, the width of the TBL, $\mathcal{L}(t)$, is large enough such that $\ln\mathcal{L}^+ \gg \kappa B \simeq 1$; thus, $\ln\mathrm{Re}_\tau \equiv \ln\mathcal{L}^+ \gg 1$. In the next section where we solve Eq. (4), this inequality will allow us to neglect terms of order unity with respect to terms of order $\ln\mathcal{L}^+$. We thus denote the accuracy of our results as the logarithmic accuracy. For example, with the logarithmic accuracy, Eq. (7) for V^+_∞ becomes $\kappa V^+_\infty = \ln\mathcal{L}^+$.

Self-consistent factorized solution. In order to reduce the system of partial differential Eqs. (4) for the unknown functions $S(z, t)$ and $K(z, t)$ of two variables z and t to a system of ordinary differential equations, it is convenient to introduce the following dimensionless functions

$$k(\zeta) \equiv \frac{\tau^+(z,t)}{\tau_*(t)}, \quad s(\zeta) \equiv \frac{c^3\mathcal{L}(t)S(z,t)}{b\tau_*(t)}, \qquad (8a)$$

which, in the turbulent region $z^+ > z^+_{\mathrm{buf}}$, are assumed to be only functions of the "outer variable" $\zeta = \zeta(t) \equiv z/\mathcal{L}(t)$. Thus, Eq. (4c) can be written as

$$K(z,t) = \tau_*(t)k(\zeta)/c^2. \qquad (8b)$$

We also assume that the temporal growth of the boundary layer is proportional to the friction velocity,

$$d\mathcal{L}(t)/dt = \alpha\sqrt{\tau_*(t)}, \qquad (8c)$$

with the dimensionless constant α.

We now substitute $K(z, t)$, $S(z, t)$, and $\tau(z, t)$ expressed by Eqs. (8) in terms of $k(\zeta)$, $s(\zeta)$, and $\tau_*(t)$ into the momentum and turbulence energy balance Eqs. (4). We also (i) neglect the viscous contribution of νS in Eq. (4a) in the region $z^+ > z^+_{\mathrm{buf}}$ and (ii) account only for the leading contribution to $\partial K/\partial t$ (that originates from $d\mathcal{L}/dt$) and neglect the contribution of $d\tau_*/dt$. Similarly, in Eq. (2), we only account for the leading contribution in $\ln\mathrm{Re}_\tau \gg 1$ terms. The result is two ordinary differential equations for $k(\zeta)$ and $s(\zeta)$ with an explicit expression for $\tau_*(t)$:

$$-\alpha c\zeta\frac{dk}{d\zeta}+\frac{bk^{3/2}}{\zeta}-d\frac{d}{d\zeta}\zeta\sqrt{k}\frac{dk}{d\zeta} = bks, \quad (9a)$$

$$\kappa\frac{dk}{d\zeta}+\alpha\zeta s = 0, \quad \tau_*(t) = \left[\frac{\kappa V_\infty}{\ln(t/t_*)}\right]^2. \quad (9b)$$

In (9a), the RHS again represent the kinetic energy production, and the three terms in the LHS describe the temporal dependence, the energy dissipation, and the diffusion, respectively. The above system of equations for the functions $k(\zeta)$ and $s(\zeta)$ indicates that the factorization (8a) is consistent with the momentum and turbulent energy balance (4) within the logarithmic accuracy; i.e., $k(\zeta)$ and $s(\zeta)$ depend only on one variable ζ.

The boundary conditions for these equations at the edge of the TBL, where $\zeta = 1$, are the following:

$$s(1) = 0, \quad k(1) = 0. \quad (10a)$$

To formulate the boundary condition near the wall, it is noted that Eqs. (9) are valid only for $z^+ \geq z^+_{\text{buf}} \approx 50$, which corresponds to $\zeta \geq \zeta_{\text{buf}} \equiv z^+_{\text{buf}}/\text{Re}_\tau$. In this region, the mean velocity satisfies the von Kármán law (7), which gives $S^+(z^+_{\text{buf}}) = 1/\kappa z^+_{\text{buf}}$. Noting that the full min model (4) leads, in the stationary case, to $K^+(z^+_{\text{buf}}) = c^{-2}$ and to Eq. (7) with $\kappa = c^3/b$, we obtain with the help of Eq. (8a)

$$\zeta_{\text{buf}}s(\zeta_{\text{buf}}) = 1, \quad k(\zeta_{\text{buf}}) = 1. \quad (10b)$$

Since $\zeta_{\text{buf}} \equiv z^+_{\text{buf}}/\text{Re}_\tau$, we take the limit of Eq. (10b) as $\zeta_{\text{buf}} \longrightarrow 0$ for asymptotically large times when $\ln\text{Re}_\tau \gg 1$. It should be noted that boundary conditions (10b) are formulated near the wall (in the log-law region) but not at the wall in the viscous layer, where $S^+ \approx 1$.

With boundary conditions (10b) and Eqs. (8a), we obtain the mean velocity $V(z, t)$ by simple integration:

$$V(z,t) = \sqrt{\tau_*(t)}\left[B+\frac{1}{\kappa}\int_{\text{Re}_\tau^{-1}}^{z/\mathcal{L}} s(\zeta')d\zeta'\right]. \quad (11)$$

Here, the lower limit of the integration and the term B are chosen to satisfy the von Kármán law, Eq. (7), in the turbulent log-law region (for details, see the Appendix).

In order to analytically solve Eq. (8), we now introduce a polynomial form of $k(\zeta)$:

$$k(\zeta) = (1-\zeta)^2(1+\beta_1\zeta+\beta_2\zeta^2+\beta_3\zeta^3+\beta_4\zeta^4)^2, \quad (12)$$

which satisfies the boundary conditions of Eq. (10). Substituting (12) into (8) and (10) and neglecting the terms of the third-order and higher in the resulting equations gives the five constants $\alpha, \beta_1, \ldots, \beta_4$ as

$$\beta_1 = 1-\beta_0/2, \quad \beta_0 \equiv \alpha/\kappa,$$

$$\beta_2 = \beta_0(1+c\kappa/d)/2-(2+\beta_3+\beta_4),$$
$$\beta_3 = 3-\beta_0(1+13c\kappa\beta_0/6)/2-2\beta_4, \quad (13)$$

$$\beta_4 = -4+\beta_0[1+c^4(c^4+bd)/24+371c\kappa/(108d)]/2.$$

Substituting $\kappa = \alpha\int_0^1\zeta\, s(\zeta)d\zeta$ (which follows from integrating Eq. (9b)) into Eq. (13), the constants $\beta_0, \ldots, \beta_4$ can be expressed in terms of the model's three parameters b, c and d. For the fixed values of $b = 0.36$, $c = 0.53$, and $d = 0.07$, the resulting values of β_i are $\beta_0 \approx 0.996$, $\beta_1 \approx 0.502$, $\beta_2 \approx 1.99$, $\beta_3 \approx -3.05$, $\beta_4 \approx 1.10$.

Figure 1 (left) displays the profile of (solid line) $k(\zeta)$ given by Eq. (12) and the above values of the constants $\beta_0 \ldots \beta_4$. Substituting this profile in the energy balance Eq. (9a), we obtain the profile of $\zeta s(\zeta)$ shown by the dot–dashed line. Substitution of the resulting profile $\zeta s(\zeta)$ into Eq. (9b) gives the profile of $\tilde{k}(\zeta)$ shown in Fig. 1 (left) by the blue dashed line. If the initial function $k(\zeta)$ was the exact solution of balance equations (8), then the functions $k(\zeta)$ and $\tilde{k}(\zeta)$ would coincide. Figure 1 shows that these functions are reasonably close. Thus, we conclude that the simple polynomial form (12) approximates the solution of the balance Eqs. (8) in the entire TBL, $0 < \zeta < 1$, with good accuracy.

Comparison of the min model results with the DNS. In the present section, we compare the results of our min model for TD TBL with those obtained by DNS of SD TBL [5–7]. This comparison is justified by the fact that TD TBL and SD TBL are asymptotically equivalent in the limit of large Re_τ [1].

(a) Mean streamwise velocity. In our solution, the unsteady and diffusion terms in the turbulent energy balance vanish near the wall for $\zeta \longrightarrow 0$. Accordingly, in this region, the mean streamwise velocity $V(z, t)$ satisfies the von Kármán law (7), which gives (in our normalization Eq. (8a)) $s(\zeta) \longrightarrow 1/\zeta$. Indeed, in Fig. 1 (left), $\zeta s(\zeta)$ become constant for small ζ.

The figure also shows that the product $\zeta s(\zeta)$ in the wide region $0 < \zeta < 0.6$ exceeds unity, which is the value of $\zeta s(\zeta)$ in the log-law layer. Accordingly, after integration $s(\zeta)$ over ζ, the resulting mean velocity profile $V_3(\zeta)$, which is shown in Fig. 1 (middle), exceeds the log-law level. Thus, the min model clearly demonstrates the wake contribution described by Coles [8] for a stationary channel flow. The wake contribution is also seen in our DNS results for the spatially-developing TBL. However, the size of the wake given by the min model in Fig. 1 (middle) is smaller than that in the DNS; see Fig. 1 (right). This discrepancy originates from the estimate of the outer scale of the turbulence l.

In the min model, l is estimated, following von Kátmán, as the distance to the solid wall z. This assumption is valid only for $z \ll \mathcal{L}$; otherwise, the scale l is affected by the free upper boundary and saturates. Recently [9], the effect of saturation of l (at a level of 0.3 of the channel half-width) on the mean velocity was studied in the channel flow. Accounting for the saturation of l in TD TBL at the level $\approx\mathcal{L}/2$ (which affects only the shear) improves the calculated velocity profile; Fig. 1 (right). The size of the wake given by the improved min model compares well with that in DNS. This correction does not affect the other results.

(b) Turbulent kinetic energy. Figure 2 displays the ζ dependence of various terms in the energy balance. One can see in the left panel that, close to the wall, where $\zeta < 0.25$, the energy production is well balanced by the energy dissipation; the diffusion and unsteady terms play a minor role. They become relevant only for $\zeta > 0.3$, as shown in the middle panel. Thus, the region $\zeta < 0.25$ can be considered as "equilibrium" TBL with a local spatial energy balance. For further clarification, we plot, in the insert, the difference between the energy production and dissipation. The difference is denoted as the energy input. This is the part of the energy flux that is required for the temporal development of TBL. Notice that the energy input is finite, while the energy production and energy dissipation themselves diverge as $1/\zeta$ for $\zeta \longrightarrow 0$, as shown in Fig. 2 (left).

The energy input is essential only in the first half of the TBL, where $\zeta < 1/2$. As both the energy production and energy dissipation become smaller (see Fig. 2 (middle and right)) their difference vanishes. The energy input can be totally neglected in the outer tenth of the TBL, where $\zeta > 0.9$. The only source of energy here is the turbulent diffusion, which transports energy from the region of $\zeta \lesssim 0.5$ to the region of $\zeta \gtrsim 0.5$. Thus, the turbulent diffusion leads to an increase the width of the TBL in time. Figure 3 (middle) shows good agreement between the analytical (dashed lines) and DNS (solid lines) profiles of the turbulent kinetic energy. The observed discrepancy between the analytical and DNS results for $z^+ < 50$ is expected since our min model was not designed to predict the buffer sublayer flow.

(c) The Reynolds stress is one of the most important characteristics of TBL, being responsible for the mechanical balance in the turbulent log-law and outer regions. In stationary regimes, the Reynolds stress is prescribed by the outer conditions maintaining the flow. For example, in the pressure driven channel flow (of half-width L), $\tau(z) \propto (1 - z/L)$, while, in the zero pressure-gradient plane Couette flow, $\tau(z) = \text{const}$. In developing the TBL, $\tau(z)$ is not known a priory; thus, the Reynolds stress has to be determined self-consistently. Our model (Fig. 1 (left), $k(\zeta)$) predicts that, in the first half of the TBL, near the wall, $\tau(z) \approx [1 - z/\mathcal{L}]$ as in the channel flow, while, in the last quarter of the TBL, near the free boundary, it decays quadratically, $\tau(z) \propto [1 - z/\mathcal{L}]^2$. This prediction is in good agreement with the DNS data (see Fig. 3, middle panel). The production term is also well described by our model (see Fig. 3, right panel).

Summary. We have presented a simple analytical model (min model) of the physics of time-developing TBL in a Newtonian fluid. The model is based on the exact equation for the momentum flux and on the model equation for the balance of the turbulent kinetic energy with the production, dissipation, and turbulent diffusion terms. The min model results in a partial differential equation for the kinetic energy, which, in the limit of large evolution time, was reduced to a relatively simple ordinary differential equation. We obtained an asymptotic formula for the time dependence of the width of the TBL and approximate analytical solutions for the profiles of the mean velocity, the turbulent velocity fluctuations, and the Reynolds stress as a function of the time and distance from the wall. These profiles are in good agreement with the DNS observations. In our future work, we will demonstrate the asymptotic equivalence of temporally- and spatially-developing turbulent boundary layers by the relationship $x \Leftrightarrow V_\infty t$ (here, x is the distance from the front edge and V_∞ is the free-stream velocity) and will clarify the difference between these two regimes in the preasymptotic region, where $\ln t^+$ or $\ln x^+$ are not very large with respect to unity.

APPENDIX

Here, we describe the derivation of Eq. (11). From the definition of the mean shear (1), the mean velocity is written as

$$V(z, t) = V(z_{\text{buf}}, t) + \int_{z_{\text{buf}}}^{z} S(z', t)dz'. \quad (14)$$

Substituting $S(z, t)$ from (8a) into (14) gives

$$V(z, t) = V(z_{\text{buf}}, t) + \frac{\sqrt{\tau_*(t)}}{\kappa}\int_{\zeta_{\text{buf}}}^{\zeta} s(\zeta')d\zeta', \quad (15)$$

where $\zeta \equiv z/\mathcal{L}(t)$, $\zeta_{\text{buf}} \equiv z_{\text{buf}}/\mathcal{L}(t)$. The mean velocity $V(z_{\text{buf}})$ at the edge, (z_{buf}), of the log-law region is given by the von Kármán law as

$$V(z_{\text{buf}}) = \sqrt{\tau_*}[B + \ln z^+_{\text{buf}}]. \quad (16)$$

Noting that $s(\zeta) = 1/\zeta$ for $\zeta \ll 1$, Eq. (15) becomes

$$V(z, t) = \sqrt{\tau_*(t)}\left[B + \frac{1}{\kappa}\int_{\zeta_0}^{\zeta} s(\zeta')d\zeta'\right], \quad (17)$$

where $\zeta_0 = l_*/Z(t) = 1/\text{Re}_\tau$.

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