From the probabilistic marginal problem to the causal marginal problem (beyond conditional independences)

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(probabilistic) marginal problem

Let $S := \{X_1, \ldots, X_d\}$ be random variables and $S_j \subset S$

Question:

Given the marginal distributions P_{S_j}, is there a joint distribution $P_{X_1,\ldots,X_d}$ having P_{S_j} as marginal distributions?

Vorobev: Consistent families of measures and their extensions (1962), Kellerer: Maßtheoretische Marginalprobleme (1964)

Simple negative example

- $P_{X,Y}$ such that $cor(X, Y) \approx 1$
- $P_{Y,Z}$ such that $cor(Y, Z) \approx 1$
- $P_{X,Z}$ such that $cor(X, Z) \approx -1$

$P_{X,Y,Z}$ cannot exist because covariance matrix would not be positive

Causal marginal problem

(vague formulation)

Given distributions P_{S_j} with causal model M_j, is there a distribution $P_{X_1,\ldots,X_d}$ with causal model M whose marginals on S_j coincide with M_j?

Vague because

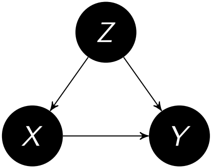
- 'causal model' not defined (DAG, MAG, PAG, structural equation model, semi-Markovian model, potential outcome, chain graph...)

- definition of 'marginalization of a causal model' not obvious

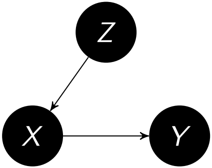
Note: marginal problems with focus on conditional independences have been considered elsewhere, e.g. Tsamardinos, Triantafillou, Lagani: Towards Integrative Causal Analysis of Heterogeneous Data Sets and Studies, JMLR 2012

Examples instead of definitions...

$Z \rightarrow Y$ can be the marginalization of

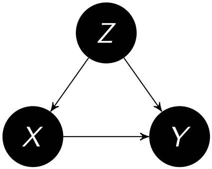


or



DAGs not closed under marginalization

usually, marginalization of



to X, Y is no longer a DAG because X, Y are confounded

then, marginal is a semi-Markovian model, for instance

Defining marginalization of a causal model

(for a special case)

Definition:

Given a causal DAG with nodes $X_1, \dots, X_d$ and joint distribution $P_{X_1,\dots,X_d}$. The DAG $X_i \to X_j$ is said to be a valid marginalization if $p(x_j|do(x_i)) = p(x_j|x_i)$.

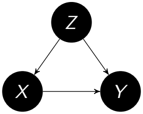
Note: here the trivial case $p(x_j|do(x_i)) = p(x_j) = p(x_j|do(x_i))$ is also allowed

Example for a causal marginal problem

Given:

- $P_{X,Y}$ with causal DAG $X \to Y$
- $P_{Z,X}$ with causal DAG $Z \to X$
- $P_{Z,Y}$ wih causal DAG $Z \to Y$

can they be obtained from marginalizing some $P_{X,Y,Z}$ with DAG

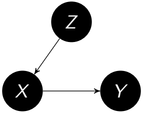


?

1) a natural solution

Let $P_{X,Y}, P_{Z,X}, P_{Z,Y}$ be Gaussians such that their covariance matrices $\Sigma_{XY}, \Sigma_{ZY}, \Sigma_{Z,X}$ yield a positive matrix $\Sigma_{X,Y,Z}$ (solving the probabilistic marginal problem)

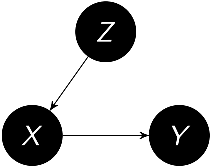
If the partial correlation $\rho_{XY|Z}$ vanishes (can be checked from pairwise covariances alone), the DAG



with Gaussian $P_{X,Y,Z}$ solves the causal marginal problem

Generalization to non-Gaussians?

The DAG



is only possible if

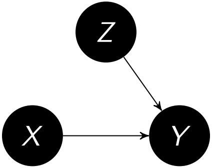
$$I(Z : Y) \leq \min\{I(X : Y), I(Z : X)\}$$

(data processing inequality, follows from $I(Z : Y|X) = 0$)

necessary condition, I'm pretty sure it's not sufficient

2) a boring solution

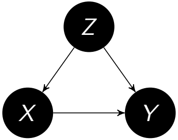
If $P_{ZX} = P_Z P_X$ the DAG



solves our causal marginal problem

3) a paradox solution

'improper confounding'



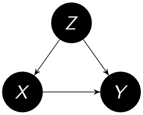
- let Z contain 2 independent bits, one influencing X and one influencing Y.
- relabel the states $Z = 00, 01, 10, 11$ as $Z = 1, 2, 3, 4$ to make the example more serious.
- clearly $p(y|do(x)) = p(y|x)$.

Z is a confounder in the sense that it really influences both X and Y, but it does not really *confound* the influence of X on Y

Inconsistent causal marginals

Let $P_{X,Y}, P_{Z,X}, P_{Z,Y}$ be Gaussians such that their covariance matrices $\Sigma_{XY}, \Sigma_{ZY}, \Sigma_{Z,X}$ yield a positive matrix $\Sigma_{X,Y,Z}$ (solving the probabilistic marginal problem)

If the partial correlation $\rho_{XY|Z}$ does not vanish, the DAG



with Gaussian $P_{X,Y,Z}$ does *not* solve the causal marginal problem because $p(y|do(x)) \neq p(y|x)$

Simple marginal problem

- let X, Y, Z be binary
- define $P_{X,Y}$ by

$$p(x,y) = \begin{cases} \frac{1}{2} - \epsilon & \text{for } x = y \\ \epsilon & \text{for } x \neq y \end{cases}$$

- $p(y,z)$ and $p(x,z)$ defined similarly with the same ϵ
- causal models $Z \to X$, $X \to Y$, and $Z \to Y$

Conjecture: inconsistent marginals, at least for small ϵ

Question:

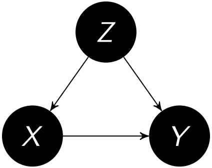
What are necessary and sufficient conditions in the above tripartite case?

Structural causal marginal problem

Given:

- $P_{X,Y}$ with structural equation $Y = f_Y(X, N_Y)$
- $P_{Z,X}$ with structural equation $X = f_X(Z, N_X)$
- $P_{Z,Y}$ with structural equation $Y = f_Y(Z, N_Y)$

Are there structural equations for the DAG



that are consistent with the above (even regarding counterfactual statements)?

Example with binaries

1. $Y = X \oplus N_X$ with unbiased N_X for $X \to Y$ (hence $X \perp\!\!\!\perp Y$)
2. $X = Z \oplus N_Z$ with unbiased N_Z for $Z \to X$ (hence $Z \perp\!\!\!\perp X$)
3. $Y = N_Y$ with unbiased N_Y for $Z \to Y$ (hence $Z \perp\!\!\!\perp Y$)

counterfactual statements inconsistent:

- for fixed values of N_X, N_Y, N_Z, inverting Z has no effect on Y according to (3)
- according to (1) and (2), inverting Z inverts Y

Applications of the causal marginal problems

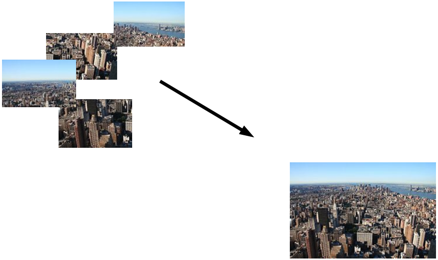
Given observations from different data sets with overlapping variables, do they correspond to the same background conditions?

- if the marginal problem is not solvable they correspond to different conditions
- causal information can render the extension of marginals to a joint unique or *less ambiguous*

causal information helps in putting pieces of the world together

Science as a puzzle:

putting pieces together without causal information:



Science as a puzzle...

with causal information: