Discovering Dynamical Kinds

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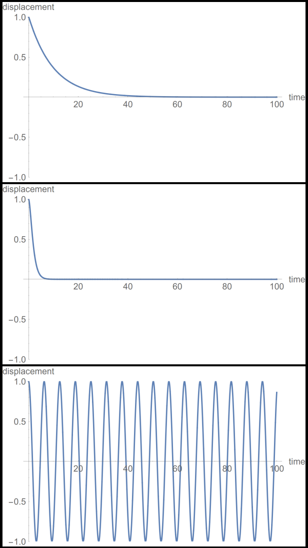
Outline

Dynamical form

What do these systems have in common?



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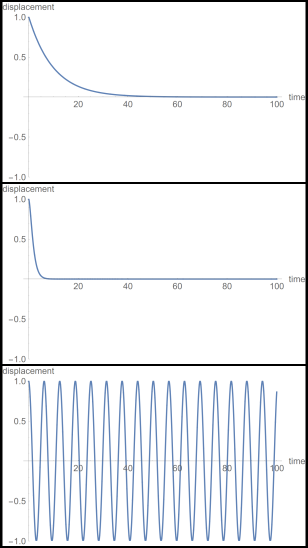

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = 0,$$

$$\omega_0 = \sqrt{\frac{k}{m}},\ \zeta = \frac{c}{2\sqrt{mk}}$$

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- (choose a model to parameterize the system)
- fit a function to particular trajectories or fit a transfer function
- only after the fact, consider classifying dynamical form

The advantages of directly discerning dynamical kinds

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Helpful to know if two or more systems of causally connected variables have the same dynamical form:

- could tell if a system exhibits distinct dynamical regimes
- could validate complex computer models
- data from multiple experiments can be pooled prior to model selection

What it takes to find kinds

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1. a rigorous definition of dynamical kind

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1. a rigorous definition of dynamical kind
2. an empirical test for sameness of dynamical kind

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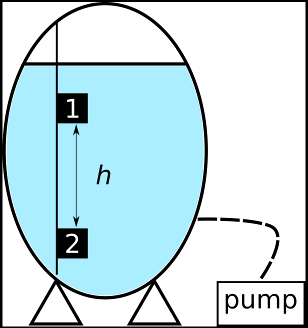
Dynamical symmetry

Definition (Dynamical symmetry)

Let V be a set of variables. Let σ be an intervention on the variables in $Int \subset V$. The transformation σ is a dynamical symmetry with respect to some index variable $X \in V - Int$ if and only if σ has the following property: for all x_i and x_f, the final state of the system is the same whether σ is applied when $X = x_i$ and then an intervention on X makes it such that $X = x_f$, or the intervention on X is applied first, changing its value from x_i to x_f, and then σ is applied.

Example: Pressure and additive symmetry



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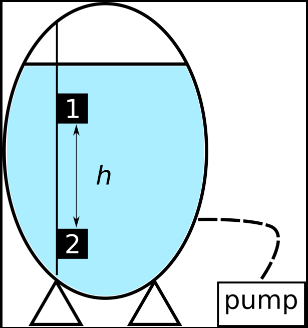

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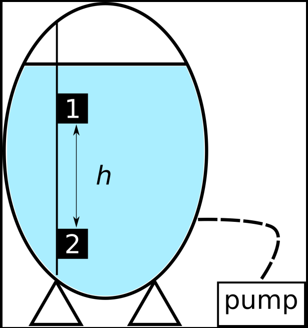

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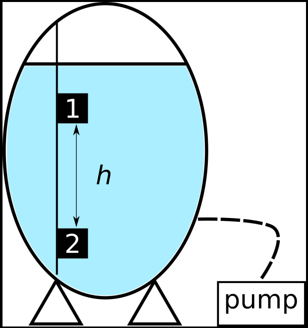

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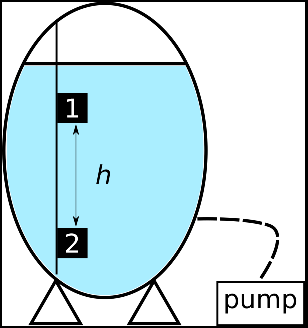

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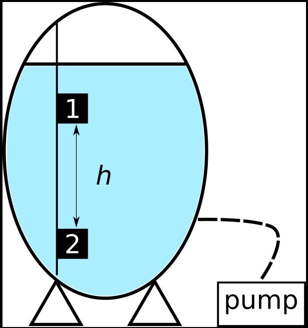

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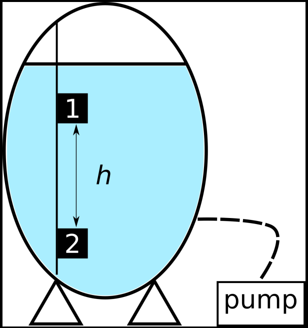

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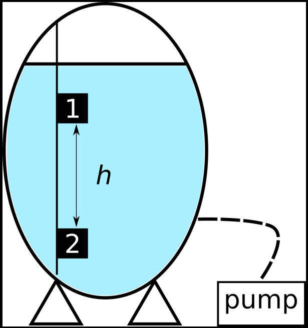

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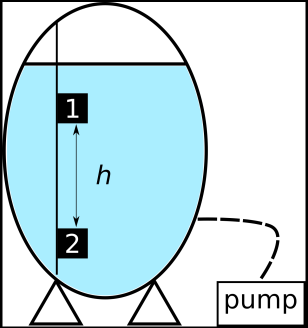

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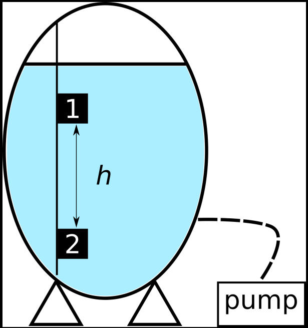

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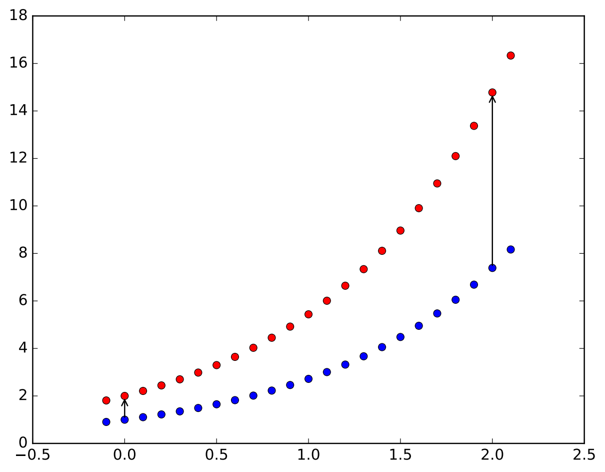
Dynamical symmetry with respect to time

Definition (Dynamical symmetry with respect to time)

Let t be the variable representing time, and let V be a set of additional dynamical variables such that $t \notin V$. Let σ be an intervention on the variables in $Int \subset V$. The transformation σ is a dynamical symmetry with respect to time if and only if for all intervals Δt, the final state of the system is the same whether σ is applied at some time t_0 and the system evolved until $t_0 + \Delta t$, or the system first allowed to evolve from t_0 to $t_0 + \Delta t$ and then σ is applied.

Example: Scaling and exponential growth



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- Symmetries of differential equations in time familiar from physics.
- Can be relaxed – nothing special about this sort of dynamics.

Symmetry structure

Definition (Symmetry structure:)

The *symmetry structure* of a collection of dynamical symmetries, $\Sigma = \{\sigma_i | i = 1, 2, \ldots\}$ is given by the composition function $\circ : \Sigma \times \Sigma \to \Sigma$.

Dynamical kind

Definition (Dynamical kind)

Two systems are of the same *dynamical kind* (same dynamical form) iff they have the same symmetry structure.

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1. *collect* data for target variable(s) as a function of time, before and after intervention

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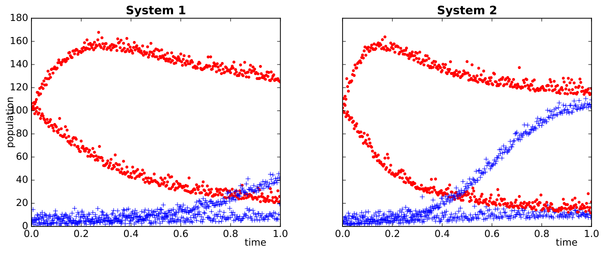
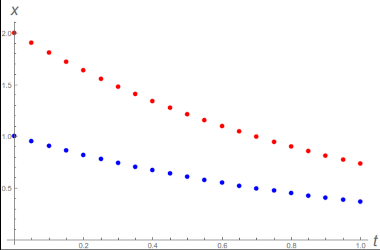
3. *compare* symmetries

Phase 1: Sampling



Phase 2: Transformation



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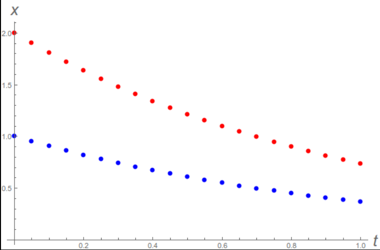

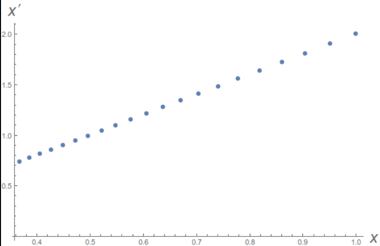

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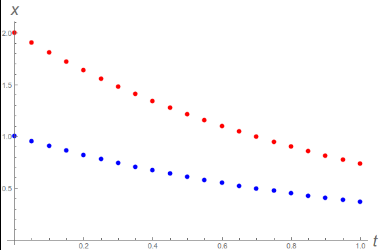

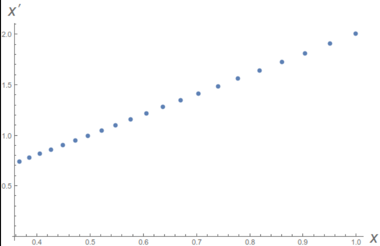

$$\begin{array}{cc} t & x \end{array} \qquad \begin{array}{cc} t & \tilde{x} \end{array}$$

$$\left[\begin{array}{c|c} t_0 & a_{00} \\ t_1 & a_{10} \\ t_2 & a_{20} \\ \vdots & \vdots \end{array}\right] \quad \left[\begin{array}{c|c} t_0 & b_{00} \\ t_1 & b_{10} \\ t_2 & b_{20} \\ \vdots & \vdots \end{array}\right]$$

$$\Downarrow$$

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\end{array}\right]
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\left\langle
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   - Else, conclude they are the same dynamical kind.

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$$\dot{x} = rx\left(1 - \frac{x}{K}\right) \text{ vs. } \dot{x} = rx\left(1 - \left(\frac{x}{K}\right)^{\beta}\right)$$

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2. Two-species Lotka-Volterra competition:

$$\dot{x}_1 = r_1 x_1 \left(1 - (x_1 + \alpha_{12} x_2)/K_1\right)$$
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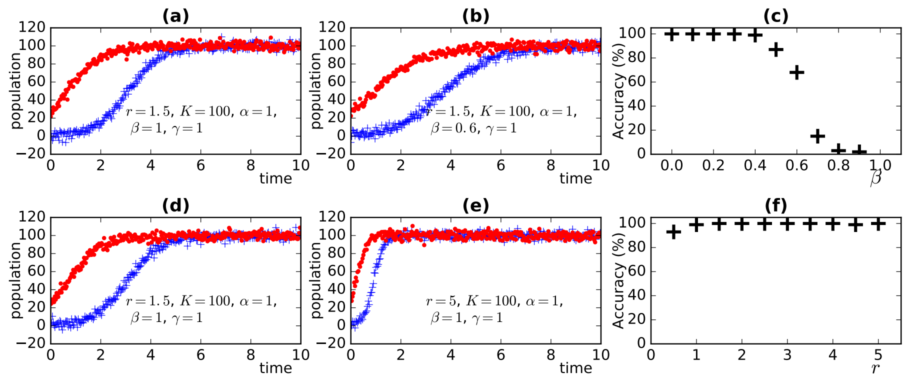
   Symmetries: $f(r_2/r_1)$

Accuracy: single dependent variable



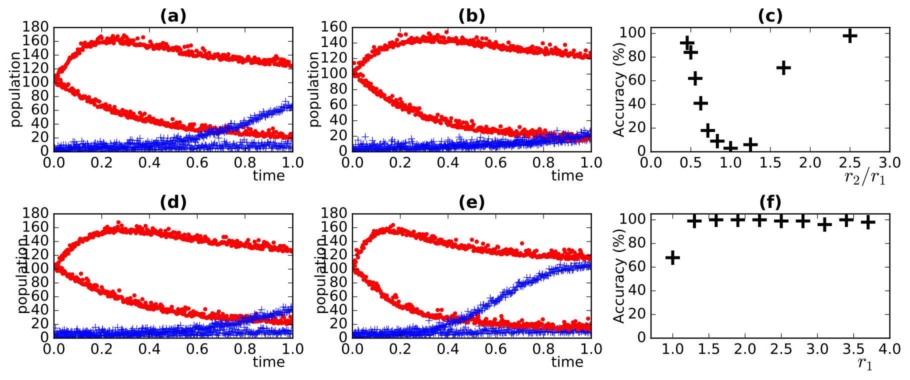
(a), (b) generalized logistic growth, *different* dynamical kinds

(c) accuracy discerning different kinds

(d), (e) generalized logistic growth, *same* dynamical kind

(f) accuracy detecting similarity of kind

Accuracy: two dependent variables



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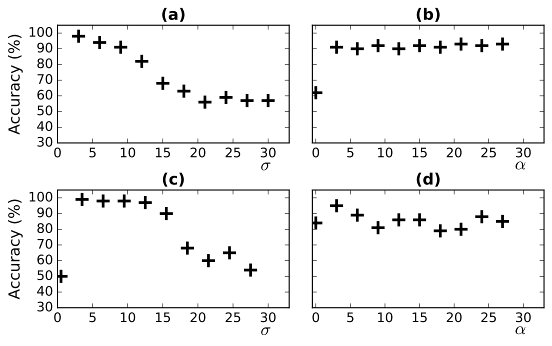
Noise and normality



(a) Accuracy as a function of standard deviation of normally distributed noise for logistic growth models.
(b) Accuracy as a function of the α-parameter of the skew normal distribution for logistic growth systems.
(c) Accuracy versus standard deviation of normally distributed noise for two-species Lotka-Volterra systems.
(d) Accuracy versus α for Lotka-Volterra systems.

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The two variable case

Suppose

$$
\begin{aligned}
x &:= x + \epsilon \\
y &:= f(x; y_0) + \eta
\end{aligned}
$$

where

$$f(x_0; y_0) = y_0$$

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$$p(x, y) = p_x(x)p_\eta(y|x) = p_x(x)p_\eta(y - f(x; y_0))$$

To satisfy the symmetry condition for transformation, σ, must have:

$$p(x_0 + \delta x)p_\eta(y - f(x_0 + \delta; \sigma(y_0))) = p(x_0 + \delta)p_\eta(y - \sigma(f(x_0 + \delta; y_0)$$

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$$\boxed{f(x_0 + \delta; \sigma(y_0))) = \sigma(f(x_0 + \delta; y_0)}$$

Recasting the logistic growth example

$$
\begin{aligned}
x :=& x + \epsilon \\
x_{\Delta t}(x; y_0) :=& \frac{(K - x_0) y_0 x}{(K - y_0) x_0 + (y_0 - x_0) x}
\end{aligned}
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where $y_0 = x_{\Delta t}(x_0)$.

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Summary

- It is possible to directly assess sameness of dynamical kind.
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- The algorithm presented can be extended to stochastic causation.

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- Comparing sameness of dynamical kind is critical for automatically choosing a domain for scientific investigation.
- The EUGENE project is aimed at automating this and other components of scientific inference that have resisted algorithmic solution.

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Chaotic circuits in phase space