ONLINE APPENDIX TO "RECOVERING PREFERENCES FROM FINITE DATA"

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Abstract. This supplemental material contains results omitted from the main paper.

S.1. Estimation and closed preference sets

This section illustrates that having a closed set of preferences is critical for estimation, while it is not needed for identification.

Throughout, the set of alternatives is $X \equiv [0,1]$, and the set of preferences $\mathcal{P}$ is the set of all locally strict and transitive preferences on X. The argument extends to other sets of alternatives, but using the unit interval makes it particularly simple. Here, X meets Assumption (1) but $\mathcal{P}$ violates Assumption (2) because it is not closed, as we show below.

Denote by $\succeq^I$ the preference that corresponds to complete indifference, defined by $x \succeq^I y$ for all $x, y \in X$. Note that $\succeq^I$ is transitive but not locally strict. We measure the distance between preference relations by the Hausdorff distance between the corresponding subsets of $X \times X$:

$$\rho(\succeq, \succeq') = \max \Big\{ \sup_{x \succeq y} \inf_{x' \succeq' y'} \|(x,y) - (x',y')\|, \\ \sup_{x' \succeq' y'} \inf_{x \succeq y} \|(x',y') - (x,y)\| \Big\},$$

where $\|\cdot\|$ is the Euclidean norm. Because X is compact, the Hausdorff metric is compatible with the topology of closed convergence.

Consider a subject who has a preference in $\mathcal{P}$, denoted $\succeq^*$, and makes choices accordingly without error: if $x \succ^* y$, the subject always chooses x over y. Using

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the formalism of Section 2, decisions without error mean that the subject's choice function over binary choice problems rationalizes $\succeq^*$.

Proposition S.1. *Suppose $\{\Sigma_n\}$ is an exhaustive set of experiments, and let Σ_∞ be the collection of all the binary choice problems used in this set of experiments.*

(1) If a preference $\succeq \in \mathcal{P}$ rationalizes the observed choices on every binary choice problem in Σ_∞, then $\succeq = \succeq^*$.
(2) There exists, for every n, a preference $\succeq_n \in \mathcal{P}$ that rationalizes the observed choices on Σ_n, and such that $\rho(\succeq_n, \succeq^I)$ converges to zero as n goes to infinity.

In particular, $\mathcal{P}$ is not closed. Neither is the set of all locally strict preferences.

The first part of Proposition S.1 asserts that, if the experimenter *could* observe the behavior of the subject over all experiments of an exhaustive set, she would be able to infer exactly the subject's true preference. Thus, a countably infinite set of data points that samples well enough the set of alternatives is sufficient to uniquely pin down the subject's true preference. This identification result owes to the fact that the subject's true preference is assumed to be locally strict. It is implied by Lemma 1.

The second part of Proposition S.1 contrasts with the interpretation of the first part of the proposition. On any experiment in an exhaustive set, the experimenter can find a preference in $\mathcal{P}$ that perfectly rationalizes the observed behavior, and yet remains uninformative about the true preference of the subject—no matter how many data points have been collected—in the sense that the estimated preference converges to the same preference relation independently of the subject's true preference. The proof of the second part of the proposition relies on constructing sequences of rationalizations that behave increasingly erratically as experiments grow in size. This result stresses the importance of the assumption that the class of preferences considered be closed.

Proof of Proposition S.1. The first part of the proposition is a special case of Lemma 1 by setting $\succeq_B = \succeq$. To prove the second part, let Z_n be the set of alternatives used in experiment Σ_n. Let us write Z_n as $\{z_1, \dots, z_{m_n}\}$ for some m_n, with $z_i < z_j$ if $i < j$.

Denote by $v_n : Z_n \to [-1/2, +1/2]$ a utility representation of $\succeq^*$ restricted to Z_n. Such utility representation is guaranteed to exist because $\succeq^*$ is transitive and Z_n is finite. We define a utility function $u_n : X \to [-1, +1]$ that extends v_n as follows. First, if $z_1 \neq 0$, let $u_n(0) = 0$, if $z_{m_n} \neq 1$, let $u_n(1) = 0$, and for all $z \in Z_n$, let

$u_n(z) = v_n(z)$. Second, for $i = 1, \ldots, m_n - 1$, let

$$u_n\left(\frac{2}{3}z_i + \frac{1}{3}z_{i+1}\right) = +1$$
$$u_n\left(\frac{1}{3}z_i + \frac{2}{3}z_{i+1}\right) = -1.$$

Third, we complete the definition of u_n on X by linear interpolation between the points just defined.

Let $\succeq_n$ be the preference relation that u_n produces on the full set of alternatives. Of course, $\succeq_n$ is transitive. It is also locally strict because u_n is never constant on any open interval. Thus, $\succeq_n$ belongs to $\mathcal{P}$. Since it agrees with $\succeq^*$ on the alternatives used in experiment Σ_n, it rationalizes the observed choices on Σ_n. Finally, $\succeq_n$ converges to $\succeq^I$ as n goes to infinity. Indeed, recall that the convergence of preferences in the closed convergence topology can be defined by the two properties detailed in Section 3. The first property holds because, no matter the choice of $x, y \in [0, 1]$, for every $\varepsilon > 0$ one can always find n large enough and $x_n, y_n \in [0, 1]$ with $|x_n - x| < \varepsilon$, $|y_n - y| < \varepsilon$ so that $u_n(x_n) \geq u_n(y_n)$, which means $x_n \succeq_n y_n$. The second property is immediately satisfied. □

S.2. Convergence rates in commodity-space environments

In this section, we compute explicit convergence rates for the statistical preference model in the commodity-space environment of Section 5.1.

In this environment, the set of alternatives X is the positive orthant $\mathbf{R}^d_{++}$. We use the Euclidean norm (and metric) on X and the L^∞ product norm on the product space $X \times X$. For a subset S of X or $X \times X$, let S^ε denote the set of all points within distance ε of S.

To enable the computation of convergence rates, we require that $\mathcal{P}$ be identified on a compact set. Given a subset of alternatives from X, we say that the class $\mathcal{P}$ is *identified* on the subset if, whenever two preferences coincide on this subset, they must be identical on X. We also ask that $\mathcal{P}$ have finite VC dimension. These requirements are satisfied by a number of common models; for example, the class of preferences with a constant elasticity of substitution (CES) utility representation, or, when $\{1, \ldots, d\}$ is interpreted as a state space, and the set of alternatives X is interpreted as a space of monetary acts, preferences with a CARA subjective expected utility representation.[S.1] Throughout, we fix a compact set $K \subset X$ with nonempty interior (without loss of generality), and we let $\theta > 0$ be small enough so that $K^\theta \in X$. We refer to θ as a "fudge parameter." In effect, K^θ is a slightly enlarged version of K.

Similarly to Section 4.2, we focus on error probability functions that are polynomially bounded. However, since we do not impose that the preferences in $\mathcal{P}$ have a utility representation, we use the Euclidean distance between alternatives instead of the difference of utilities. Specifically, we assume that there exists $C > 0$ and $k > 0$ such that, if $x \in K^\theta$ is strictly preferred to $y \in K^\theta$ according to preference $\succeq$, then the error probability function q satisfies

$$q(\succeq; x, y) \geq \frac{1}{2} + C\|x - y\|^k. \tag{S.1}$$

Observe that, as in Section 4.2, Equation (S.1) only bites as the distance between x and y vanishes. The reason is that K^θ is bounded and C can be set to be arbitrarily small.

The metric we use on preferences is a "fudged metric" based on the Hausdorff distance.[S.2] It is defined as follows:

$$\begin{aligned}\rho(\succeq, \succeq') = \max\Big\{ &\sup\big\{\rho\big((x,y), \succeq' \cap (K \times K)^\theta\big) : x \succeq|_K y\big\},\\ &\sup\big\{\rho\big((x,y), \succeq \cap (K \times K)^\theta\big) : x \succeq'|_K y\big\}\Big\},\end{aligned}$$

where $\succeq|_K$ is the restriction of $\succeq$ to K and, for $A \subseteq X \times X$,

$$\rho((x,y), A) = \inf\{\|(x,y) - (x',y')\| : (x',y') \in A\}.$$

Note that the distance between two preferences weakly increases with the fudge parameter, and as θ becomes small, the fudged metric becomes equal to the usual Hausdorff metric restricted to $K \times K$. The reasons for adding a fudge to the Hausdorff distance are technical and unsubstantive.

The above conditions enable us to derive explicit convergence rates as a corollary to Theorem 3.

Corollary S.1. *Suppose the statistical preference model for commodity spaces* $(X, \mathcal{P}, \lambda, q)$ *meets the following conditions:*

[S.1] See Basu and Echenique (2018) for a discussion of other uses of the VC dimension for choice under uncertainty. The results in Basu and Echenique (2018) provide convergence rates for learning preferences in a revealed preference model, different from the one under consideration here.

[S.2] We do not need to show that the fudged metric is a compatible metric, because Theorem 3 applies to any metric, not just metrics compatible with the topology on preferences.

(1) $\mathcal{P}$ has a finite VC-dimension and is identified on K;
(2) each preference in $\mathcal{P}$ is transitive and strictly monotone with respect to $\gg$;
(3) λ is the uniform distribution over on K^θ;
(4) q satisfies Equation (S.1).

Then the Kemeny-minimizing estimator is consistent and, as $\eta \to 0$ and $\delta \to 0$,

$$N(\eta, \delta) = O\left(\frac{1}{\eta^{4d+2k}} \ln \frac{1}{\delta}\right).$$

Proof. The proof proceeds similarly to the proof of Corollary 4 on expected utility preferences: we compute an asymptotic lower bound on the value of $r(\eta)$ defined in Section 3.2, and then we apply Theorem 3.

For $x \in \mathbf{R}^d_{++}$ and $\varepsilon > 0$, we let $\mathcal{B}_\varepsilon(x)$ be the open ball of radius ε and center x. We also let

$$\mathcal{B}^+_\varepsilon(x) = \big\{z \in \mathcal{B}_\varepsilon(x) : z \gg x\big\},$$
$$\mathcal{B}^-_\varepsilon(x) = \big\{z \in \mathcal{B}_\varepsilon(x) : x \gg z\big\}.$$

The proof makes use of the following lemma.

Lemma S.1. *Let $0 < \eta < \theta$, and $\succeq_A$ and $\succeq_B$ be preferences in $\mathcal{P}$. Suppose that there exist $x_0, y_0 \in X$ with $x_0 \succeq_A y_0$ and such that for all $x, y \in \mathbf{R}^d_{++}$ with $\|(x_0, y_0) - (x, y)\| < \eta$, we have $y \succ_B x$. Then, for all $(x, y) \in \mathcal{B}^+_{\eta/2}(x_0) \times \mathcal{B}^-_{\eta/2}(y_0)$, we have* (i) *$x \succeq_A y$ and $y \succ_B x$,* (ii) *$(x, y) \in (K \times K)^\theta$, and* (iii) *$\|x - y\| \geq \eta/2$.*

Proof. Let $(x, y) \in \mathcal{B}^+_{\eta/2}(x_0) \times \mathcal{B}^-_{\eta/2}(y_0)$. We have $x \succ_A x_0 \succeq_A y_0 \succ_A y$, by monotonicity of the preference $\succeq_A$. Hence, by transitivity, $x \succeq_A y$. And since $\|(x_0, y_0) - (x, y)\| < \eta$, we have $y \succ_B x$. Because $\eta/2 < \theta$, $x \in K^\theta$ and $y \in K^\theta$, therefore $(x, y) \in (K \times K)^\theta = K^\theta \times K^\theta$. Finally, let us show that $\|x - y\| \geq \eta/2$. We have

$$\|(x_0, y_0) - (y, x)\| \leq \|(x_0, y_0) - (x, y)\| + \|(x, y) - (y, x)\|,$$

and by choice of (x, y),

$$\|(x_0, y_0) - (x, y)\| \geq \frac{\eta}{2}.$$

If $(y, x) \in (K \times K)^\theta$, then using that $y \succeq_B x$, we get

$$\|(x_0, y_0) - (y, x)\| \geq \eta.$$

If $(y, x) \notin (K \times K)^\theta$, then since $(x_0, y_0) \in K \times K$,

$$\|(x_0, y_0) - (y, x)\| \geq \theta \geq \eta.$$

In both cases, we get

$$\eta \leq \frac{\eta}{2} + \|(x,y) - (y,x)\|$$

and hence, $\|x - y\| = \|(x,y) - (y,x)\| \geq \eta/2$. □

We now return to the main proof. Let us fix the subject's preference $\succeq^*$, and let $\succeq$ be any preference of $\mathcal{P}$ with $\rho(\succeq^*, \succeq) \geq \eta$, with $0 < \eta < \theta$. As in the proofs of our main results, we continue to use $q(x,y)$ as a short notation for $q(\succeq^*; x, y)$, and for a binary relation R, we let $\mathbf{1}_R(x,y) = 1$ if and only if $(x,y) \in R$.

We established in the proof of Theorem 2 that

$$\mu(\succeq^*) - \mu(\succeq) = \int_{X \times X} \mathbf{1}_{\succ^* \setminus \succ}(x,y) \big[q(x,y) - q(y,x)\big] \, \mathrm{d}\lambda(x,y).$$

There are two cases to consider.

First, suppose that there exist $x_0, y_0 \in K$ with $x_0 \succeq^* y_0$ such that, if $x, y \in \mathbf{R}^d_{++}$ and $\|(x_0, y_0) - (x,y)\| < \eta$, then $(x,y) \notin \succeq \cap (K \times K)^\theta$—which implies that $y \succ x$ by completeness. We always have $q(x,y) - q(y,x) \geq 0$ if $x \succ^* y$, and by Lemma S.1, $\mathcal{B}^+_{\eta/2}(x_0) \times \mathcal{B}^-_{\eta/2}(y_0) \subset X \times X$, so

$$\mu(\succeq^*) - \mu(\succeq) \geq \int_{\mathcal{B}^+_{\eta/2}(x_0) \times \mathcal{B}^-_{\eta/2}(y_0)} \mathbf{1}_{\succ^* \setminus \succ}(x,y) \big[q(x,y) - q(y,x)\big] \, \mathrm{d}\lambda(x,y).$$

By Lemma S.1, if $(x,y) \in \mathcal{B}^+_{\eta/2}(x_0) \times \mathcal{B}^-_{\eta/2}(y_0)$, $x \succeq^* y$ while $y \succ x$, and since $\succeq^*$ is locally strict, the set $\{(x,y) : x \sim^* y\}$ has λ-probability zero, so

$$\begin{aligned}
\mu(\succeq^*) - \mu(\succeq) &\geq \int_{\mathcal{B}^+_{\eta/2}(x_0) \times \mathcal{B}^-_{\eta/2}(y_0)} \big[q(x,y) - q(y,x)\big] \, \mathrm{d}\lambda(x,y) \\
&\geq \inf \big\{ q(x,y) - q(y,x) : (x,y) \in \mathcal{B}^+_{\eta/2}(x_0) \times \mathcal{B}^-_{\eta/2}(y_0) \big\} \\
&\quad \times \lambda\big(\mathcal{B}^+_{\eta/2}(x_0) \times \mathcal{B}^-_{\eta/2}(y_0)\big).
\end{aligned}$$

Recall that $\mathcal{B}_{\eta/2}(x_0)$ and $\mathcal{B}_{\eta/2}(y_0)$ are d-dimensional balls of radius $\eta/2$, and so each of $\mathcal{B}^+_{\eta/2}(x_0)$ and $\mathcal{B}^-_{\eta/2}(y_0)$ has a Lebesgue measure equal to the volume of a d-dimensional ball of radius $\eta/2$ divided by 2^d, which is equal to

$$\frac{\pi^{d/2}}{4^d \cdot \Gamma\left(\frac{d}{2} + 1\right)} \eta^d.$$

where Γ is the Gamma function. Since λ is the uniform probability measure on $(K \times K)^\theta$, $\lambda\big(\mathcal{B}^+_{\eta/2}(x_0) \times \mathcal{B}^-_{\eta/2}(y_0)\big)$ is directly proportional to η^{2d}.

In addition, by Lemma S.1, $\|x-y\| \geq \eta/2$, so by Equation (S.1),

$$\begin{aligned} q(x,y)-q(y,x) &\geq C\|x-y\|^k \\ &\geq \frac{C\eta^k}{2^k}, \end{aligned}$$

and

$$\inf\left\{q(x,y)-q(y,x) : (x,y) \in \mathcal{B}^+_{\eta/2}(x_0) \times \mathcal{B}^-_{\eta/2}(y_0)\right\} \geq \frac{C\eta^k}{2^k}.$$

Second, suppose that there exist $x_0, y_0 \in K$ with $x_0 \succeq y_0$ such that, if $x, y \in \mathbf{R}^d_{++}$ with $\|(x_0,y_0)-(x,y)\| < \eta$, then $(x,y) \notin \succeq^* \cap (K \times K)^\theta$. By a symmetric argument, we get that

$$\begin{aligned} \mu(\succeq^*) - \mu(\succeq) \geq &\inf\left\{q(y,x)-q(x,y) : (x,y) \in \mathcal{B}^+_{\eta/2}(x_0) \times \mathcal{B}^-_{\eta/2}(y_0)\right\} \\ &\times \lambda\big(\mathcal{B}^+_{\eta/2}(x_0) \times \mathcal{B}^-_{\eta/2}(y_0)\big), \end{aligned}$$

with

$$\inf\left\{q(y,x)-q(x,y) : (x,y) \in \mathcal{B}^+_{\eta/2}(x_0) \times \mathcal{B}^-_{\eta/2}(y_0)\right\} \geq \frac{C\eta^k}{2^k}.$$

In both cases, as $\eta \to 0$,

$$\mu(\succeq^*) - \mu(\succeq) = \Omega(\eta^{2d+k}).$$

where the big Omega notation refers to the asymptotic lower bound, and hence,

$$r(\eta) = \Omega\left(\eta^{2d+k}\right).$$

Corollary S.1 then follows from Theorem 3. Note that λ does not have full support on X, and we have not required that $\mathcal{P}$ be closed, so Assumptions (2) and (3) may be violated. Although the statement of Theorem 3 asks that Assumptions (2) and (3) be satisfied to ensure consistency of the Kemeny-minimizing estimator, this condition is not needed to obtain the asymptotic upper bound of the theorem: when $r(\eta) > 0$ for η close enough to zero, as in this case, $N(\eta,\delta)$ is guaranteed to be finite, so the estimator is consistent and the bound obtains. □

S.3. Subjective expected utility preferences

Subjective expected utility preferences are yet another case where we can ground the analysis in a family of utility representations. Specifically, we consider environments of choice under uncertainty, and study preferences that have a subjective expected utility representation.

Let $\Pi = \{\pi_1, \ldots, \pi_d\}$ be a set of d prizes (or outcomes) and $S = \{\omega_1, \ldots, \omega_s\}$ be a set of s states. The set of alternatives X is the set of Anscombe-Aumann acts; that is, the set of all mappings that send each state to a lottery over prizes.

As in Section 4.2, a lottery is represented by an element of the $(d-1)$-dimensional simplex Δ^{d-1}. It will be convenient to represent an act f as an s-by-d matrix $\{f_{ij}\}_{i,j}$ interpreted as follows: f sends state ω_i to lottery $(f_{i1}, \ldots, f_{id}) \in \Delta^{d-1}$. Throughout this section, all the finite-dimensional spaces are endowed with the Euclidean norm denoted $\|\cdot\|$.

A *subjective expected utility preference* or SEU preference for short is a preference $\succeq$ on X that complies with subjective expected utility theory: there exists a vector of subjective state probabilities $(p_1, \ldots, p_s)$ and a vector of utilities $(u_1, \ldots, u_d)$ such that $f \succeq g$ if and only if the subjective expected utility of f, which equals $p \cdot (fu)$, is not less than the subjective expected utility of g, $p \cdot (gu)$. Let $\mathcal{P}$ be the set of SEU preferences that are non-constant, which means that for every $\succeq \in \mathcal{P}$, there exists $f, g \in X$ for which $f \succ g$; or, equivalently, the corresponding vector of utilities u satisfies $u_i \neq u_j$ for some i, j.

By an analogous argument to that in Section 4.2, it can be shown that each nonconstant SEU preference is locally strict, and that $\mathcal{P}$ is closed. Therefore, the SEU environment satisfies Assumptions (1) and (2).

Now, analogously to the expected-utility environment, we provide explicit convergence rates, under some mild conditions on the error probability function.

Each nonconstant SEU preference is captured by a $(s+d)$-dimensional parameter consisting of the state probabilities and utilities. We normalize nonconstant vectors of utilities u by requiring that $u \in U^{d-1}$ with

$$U^{d-1} = \left\{ u \in \mathbf{R}^d : \sum_{j=1}^{d} u_j = 0 \text{ and } \|u\| = 1 \right\}.$$

Each preference in $\mathcal{P}$ is then associated with a unique pair $(p, u) \in \Delta^{s-1} \times U^{d-1}$, interpreted as "parameters" of the prefererence, with $\Delta^{s-1} \times U^{d-1}$ the finite-dimensional parameter space. We measure the distance $\rho(\succeq, \succeq')$ between nonconstant SEU preferences $\succeq$ and $\succeq'$ as the Euclidean distance between their respective parameters; one can follow the steps of Section 6.3 and show that ρ is a compatible metric.

We restrict error probability functions in the following way: we ask that there exists $C > 0$ and $k > 0$ such that, for all $\succeq \in \mathcal{P}$, if $f \succ g$,

$$q(\succeq; f, g) \geq \frac{1}{2} + C|\, EU(f) - EU(g)|^k, \tag{S.2}$$

where $EU(f)$ and $EU(g)$ are the expected utilities of f and g respectively.

Corollary S.2. *For the statistical preference model $(X, \mathcal{P}, \lambda, q)$, where $X \equiv (\Delta^{d-1})^s$, $\mathcal{P}$ is the set of all nonconstant SEU preferences, λ is the uniform distribution on $(\Delta^{d-1})^s$, and q satisfies Equation* (S.2)*, the Kemeny-minimizing estimator is consistent and, as $\eta \to 0$ and $\delta \to 0$,*

$$N(\eta, \delta) = O\left(\frac{1}{\eta^{8s(d-1)+4k}} \ln \frac{1}{\delta}\right).$$

The uniform distribution is chosen for simplicity, but not required. More generally, the above convergence rate continues to apply when λ is absolutely continuous with respect to the Lebesgue measure and its Radon–Nikodym derivative is bounded.

Proof of Corollary S.2. The proof is very similar to the proof of Corollary 4 in Section 4.2.

Let $\overline{\Delta}^{d-1}$ be the affine span of Δ^{d-1} in $\mathbf{R}^d$, and $\overline{X} = (\overline{\Delta}^{d-1})^s$. For $x \in X$ and $\varepsilon > 0$, we let $\mathcal{B}_\varepsilon(x)$ be the open ball of radius ε and center x in $\overline{X}$.

For a preference $\succeq \in \mathcal{P}$ associated with the pair $(p, u) \in \Theta$, and the subjective expected utility of act f is $p \cdot (fu)$.

Lemma S.2. *Let $0 < \eta < 1$ and $\succeq_A, \succeq_B \in \mathcal{P}$ with $\rho(\succeq_A, \succeq_B) \geq \eta$. There exists $f^\star, g^\star \in X$ such that, for all $f \in \mathcal{B}_{\eta'}(f^\star)$ and $g \in \mathcal{B}_{\eta'}(g^\star)$,*

$$p \cdot (fu) \geq p \cdot (gu) + \frac{\eta^2}{80d\sqrt{d}},$$
$$q \cdot (gv) \geq q \cdot (gv) + \frac{\eta^2}{80d\sqrt{d}},$$

where (p, u) and (q, v) are the parameters associated respectively with $\succeq_A$ and $\succeq_B$, and $\eta' \equiv \eta^2/(200d)$. In addition, $\mathcal{B}_{\eta'}(f^\star) \times \mathcal{B}_{\eta'}(g^\star) \subset X \times X$.

Proof. Let $\tilde{p} = p/\|p\|$ and $\tilde{q} = q/\|q\|$. Observe that $p = \tilde{p}/\sum_i \tilde{p}_i$ and $q = \tilde{q}/\sum_i \tilde{q}_i$. Then,

$$\begin{aligned}
\|p - q\| &= \|\frac{\tilde{p}}{\sum_i \tilde{p}_i} - \frac{\tilde{q}}{\sum_i \tilde{q}_i}\| \\
&= \frac{1}{\sum_i \tilde{p}_i} \|\tilde{p} - \tilde{q} + \tilde{q} - \frac{\sum_i \tilde{q}_i}{\sum_i \tilde{p}_i} \tilde{q}\| \\
&\leq \frac{1}{\sum_i \tilde{p}_i} \|\tilde{p} - \tilde{q}\| + \frac{1}{\sum_i \tilde{p}_i} \frac{1}{\sum_i \tilde{q}_i} \left| \sum_i (\tilde{p}_i - \tilde{q}_i) \right|,
\end{aligned}$$

where we use the triangle inequality and the fact that $\|\tilde{q}\| = 1$. Observe that $\sum_i \tilde{p}_i \geq 1$ and $\sum_i \tilde{q}_i \geq 1$, so

$$\begin{aligned}\|p-q\| &\leq \|\tilde{p}-\tilde{q}\| + \left|\sum_i (\tilde{p}_i - \tilde{q}_i)\right| \\ &\leq \|\tilde{p}-\tilde{q}\| + \sum_i |\tilde{p}_i - \tilde{q}_i| \\ &\leq 3\|\tilde{p}-\tilde{q}\|.\end{aligned}$$

Now suppose $\rho(\succeq_A, \succeq_B) \geq \eta$. Then, either $\|u-v\|^2 \geq \eta^2/2$, or $\|p-q\|^2 \geq \eta^2/2$ and $\|\tilde{p}-\tilde{q}\|^2 \geq \eta^2/18$. Therefore we have $u \cdot v \leq 1 - \eta^2/2$ or $\tilde{p} \cdot \tilde{q} \leq 1 - \eta^2/36$, and $(u \cdot v)(\tilde{p} \cdot \tilde{q}) \leq 1 - \eta^2/100$.

Next, let

$$f_i^\star = \frac{1}{d}\mathbf{1} + \left(\frac{1}{d} - \eta'\right)\tilde{p}_i u, \qquad \text{and} \qquad g_i^\star = \frac{1}{d}\mathbf{1} + \left(\frac{1}{d} - \eta'\right)\tilde{q}_i v.$$

(Abusing notation, since we are making a row vector equal to a column vector, to fix later.)

Let $f \in \mathcal{B}_{\eta'}(f^\star)$ and $g \in \mathcal{B}_{\eta'}(g^\star)$. The following sequence of inequalities obtains:

$$\begin{aligned}\tilde{p} \cdot (fu) &= \tilde{p} \cdot ((f - f^\star)u) + \tilde{p} \cdot (f^\star u) \\ &\geq \tilde{p} \cdot (f^\star u) - \eta' \\ &\geq \tilde{p} \cdot (g^\star u) + \left(\frac{1}{d} - \eta'\right)\frac{\eta^2}{40} - \eta' \\ &= \tilde{p} \cdot ((g_0 - g)u) + \tilde{p} \cdot (gu) + \left(\frac{1}{d} - \eta'\right)\frac{\eta^2}{40} - \eta' \\ &\geq \tilde{p} \cdot (gu) + \left(\frac{1}{d} - \eta'\right)\frac{\eta^2}{40} - 2\eta' \\ &\geq \tilde{p} \cdot (gu) + \frac{\eta^2}{80d}.\end{aligned}$$

To get the first inequality, observe that

$$|\tilde{p} \cdot ((f - f^\star)u)| \leq \|\tilde{p}\| \cdot \|f - f^\star\| \cdot \|u\| \leq \|f - f^\star\| \leq \eta'.$$

Similarly we have $|\tilde{p} \cdot ((g - g^\star)u)| \leq \eta'$, which yields the third inequality. The second inequality comes from $p \cdot (f^\star u) = 1/d - \eta'$ and

$$\tilde{p} \cdot (g^\star u) = \left(\frac{1}{d} - \eta'\right)(u \cdot v)(p \cdot q) \leq \left(\frac{1}{d} - \eta'\right) - \left(\frac{1}{d} - \eta'\right)\frac{\eta^2}{40}.$$

The fourth inequality comes from

$$\left(\frac{1}{d}-\eta'\right)\frac{\eta^2}{4}-2\eta'=\frac{\eta^2}{80d}+\frac{20\eta^2-\eta^4}{8000d}\geq\frac{\eta^2}{80d}.$$

Then,

$$\frac{1}{\sum_i \tilde{p}_i}\tilde{p}\cdot(fu)\geq\frac{1}{\sum_i \tilde{p}_i}\tilde{p}\cdot(gu)+\frac{1}{\sum_i \tilde{p}_i}\frac{\eta^2}{80d}$$
$$p\cdot(fu)\geq p\cdot(gu)+\frac{1}{\sum_i \tilde{p}_i}\frac{\eta^2}{80d}\geq\frac{\eta^2}{80d\sqrt{d}}.$$

observing that $\sum_i \tilde{p}_i \leq \sqrt{d}$.

By a symmetric argument we also have

$$q\cdot(gv)\geq q\cdot(gv)+\frac{\eta^2}{80d\sqrt{d}}.$$

Finally, observe that η' is chosen small enough to ensure that the balls $\mathcal{B}_{\eta'}(f^\star)$ and $\mathcal{B}_{\eta'}(g^\star)$ of $\overline{X}$ are included in X. □

We now return to the main proof. Let us fix the subject's preference $\succeq^*$, and let $\succeq$ be any preference of $\mathcal{P}$ with $\rho(\succeq^*,\succeq)\geq\eta$, with $0<\eta<1$. As in the proofs of Theorems 2 and 3, we use $q(f,g)$ as a short notation for $q(\succeq^*;f,g)$, and for a binary relation R, we let $\mathbf{1}_R(f,g)=1$ if and only if $(f,g)\in R$.

We established in the proof of Theorem 2 that

$$\mu(\succeq^*)-\mu(\succeq)=\int_{X\times X}\mathbf{1}_{\succ^*\setminus\succ}(x,y)\big[q(x,y)-q(y,x)\big]\,\mathrm{d}\lambda(x,y).$$

Let $\eta'=\eta^2/(200d)$. By Lemma S.2, there exists $f^\star,g^\star\in X$ such that $\mathcal{B}_{\eta'}(f^\star)\times\mathcal{B}_{\eta'}(g^\star)\subset X\times X$, and if $(f,g)\in\mathcal{B}_{\eta'}(f^\star)\times\mathcal{B}_{\eta'}(g^\star)$ then $f\succ^* g$ while $g\succ f$. Also, if $f\succ^* g$, then $q(f,g)-q(g,f)\geq 0$. Hence,

$$\begin{aligned}\mu(\succeq^*)-\mu(\succeq)&=\int_{\succ^*\setminus\succ}\big[q(f,g)-q(g,f)\big]\,\mathrm{d}\lambda(f,g)\\&\geq\int_{\mathcal{B}_{\eta'}(x_0)\times\mathcal{B}_{\eta'}(y_0)}\big[q(f,g)-q(g,f)\big]\,\mathrm{d}\lambda(f,g)\\&\geq\inf\big\{q(f,g)-q(g,f):(f,g)\in\mathcal{B}_{\eta'}(f^\star)\times\mathcal{B}_{\eta'}(g^\star)\big\}\\&\quad\times\lambda\big(\mathcal{B}_{\eta'}(f^\star)\big)\times\lambda\big(\mathcal{B}_{\eta'}(g^\star)\big).\end{aligned}$$

The Lebesgue measure of each of the $s \times (d-1)$-dimensional balls $\mathcal{B}_{\eta'}(f^\star)$ and $\mathcal{B}_{\eta'}(g^\star)$ is proportional to ${\eta'}^{s(d-1)}$, and so is proportional to $\eta^{2s(d-1)}$. So

$$\lambda\big(\mathcal{B}_{\eta'}(x_0)\big) \times \lambda\big(\mathcal{B}_{\eta'}(y_0)\big) = \Omega(\eta^{4s(d-1)})$$

as $\eta \to 0$, where the big Omega notation refers to the asymptotic lower bound.

Since $x \in \mathcal{B}_{\eta'}(x_0)$ and $y \in \mathcal{B}_{\eta'}(y_0)$ implies $x \succ^* y$, by Equation (S.2),

$$q(x,y) - q(y,x) \geq 2C|p \cdot (fu) - p \cdot (gu)|^k,$$

where (p,u) is paramater associated with $\succeq^*$. By Lemma S.2, we have

$$p \cdot (fu) - p \cdot (gu) \geq \frac{\eta^2}{80d\sqrt{d}}$$

and hence,

$$\inf\left\{q(x,y) - q(y,x) : (x,y) \in \mathcal{B}_{\eta'}(x_0) \times \mathcal{B}_{\eta'}(y_0)\right\} = \Omega(\eta^{2k})$$

as $\eta \to 0$.

Overall, we get $\mu(\succeq^*) - \mu(\succeq) = \Omega(\eta^{4(d-1)+2k})$, and thus $r(\eta) = \Omega(\eta^{4(d-1)+2k})$. Applying Theorem 3 and observing that the VC dimension of $\mathcal{P}$ is no greater than $d+1$ (and so finite) by Proposition 4.20 of Wainwright (2019), we have

$$N(\eta, \delta) = O\left(\frac{1}{\eta^{8s(d-1)+4k}} \ln \frac{1}{\delta}\right).$$

□

References

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