

# FAIRNESS AND EFFICIENCY FOR PROBABILISTIC ALLOCATIONS WITH ENDOWMENTS

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ABSTRACT. We propose a notion of fairness for allocation problems in which different agents may have different endowments. Fairness is usually understood as the absence of envy, but when agents differ in endowments it is impossible to rule out envy without violating property rights. Instead we seek to rule out *justified envy*, defined as envy for which the remedy would not violate any agent's property rights.

We show that fairness, meaning the absence of justified envy, can be achieved together with efficiency and individual rationality (respect for property rights). Our approach requires standard assumptions on agents' preferences, and is compatible with quantity constraints on allocations. The main application of our results is to school choice, where we can simultaneously achieve fairness, efficiency, and diversity-motivated quantity constraints.

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*Date:* August 2019.

We thank Eric Budish, Fuhito Kojima, Andy McLennan, Hervé Moulin, and Tayfun Sönmez for comments. Echenique thanks the National Science Foundation for its support through the grants SES-1558757 and CNS-1518941. Miralles acknowledges financial support from the Spanish Ministry of Economy and Competitiveness, through the Severo Ochoa Programme for Centres of Excellence in R&D (SEV-2015-0563.)

## 1. INTRODUCTION

We investigate the meaning of fairness among unequal agents. When two agents start from the same position, absence of envy is a natural notion of fairness; but when they start from different positions, what does it mean to treat them fairly? Our work focuses on discrete resource allocation, where the starting point of an agent is her endowment. When different agents have different endowments, it is impossible to rule out envy without violating some agents' property rights. In consequence, we shall say that an allocation of resources is fair if any remaining envy could not be remedied without violating someone's property rights.

Fairness, we show, can be achieved together with efficiency and respect for property rights. This fact is well known for economies where all agents have the same endowments (Varian, 1974; Hylland and Zeckhauser, 1979), but ours is the first such result for economies with different individual endowments. The contribution is threefold. The first contribution is to propose a notion of fairness among agents with different endowments, and prove the existence of fair, efficient, and individually rational allocations. Second, these fair and efficient outcomes can, under certain conditions, be viewed as *market outcomes* (as in Varian and Hylland-Zeckhauser). The third contribution is to accommodate quantitative constraints, such as those of course allocations (e.g all students must take at least two math courses), or controlled school choice (e.g a school seeks certain diversity objectives).

We understand fairness as the absence of “justified” envy, or as “ruling out envy that can be remedied within agents' property rights.” We do not want to say that an outcome is unfair if its unfairness can be traced to differences in initial endowments. For example, suppose that Alice and Bob are endowed with seats at two schools. The final allocation of seats depends on agents' preferences. If Alice envies Bob's final allocation, but her endowment was worse than Bob's, then we may be willing to tolerate her envy. Such inequity could simply be the product of an unequal starting points: Alice had a much worse endowment than Bob to start with. We say that envy, or inequality, among agents is justified if it can be traced to the agents' differing endowments.

So we define fairness as the absence of justified envy. Alice envies Bob at an allocation  $x$  if she would rather have Bob's assignment in  $x$  than hers. To decide whether this envy is justified, we invoke property rights. Property rights are defined from endowments: any agent has the right to be at least as well off as they would be from consuming their endowments. Now we say that Alice's envy is justified if

Bob could have received Alice's assignment without violating his property rights. In other words, if Bob ranks Alice's assignment as at least as good as his endowment.

Two remarks are in place. First, our notion of envy presumes that the obvious remedy for Alice's envy is for Bob and her to switch assignments. One might devise more complicated remedies, with a fuller reallocation that would seek to eliminate Alice's envy, but these would necessarily be complicated and require Alice's complaint to rely on a wealth of information. For example, if Alice envies Bob and wants to bring the matter to court, the most natural and plausible remedy she could offer is for the two of them to switch assignments. That said, our methods *do accommodate very general remedies*; see Section 7.1 for a detailed discussion and a significant strengthening of our main result.

Second, our fairness notion is analogous to the standard definition of fairness based on priorities in school choice. The reason is that priorities are often thought of as granting property rights. In the presence of priorities, Alice's envy towards Bob is regarded as not justified if Bob has a higher priority than Alice at the school he is assigned to. Let us consider the effect of switching Alice's and Bob's assignments in a model with priorities. If the switch makes them both better off, then the allocation in the market must not be efficient. However, recall that our no-justified envy is compatible with efficiency. So to make a proper analogy let us consider an efficient allocation. If Alice envies Bob, then efficiency demands that Bob must regard Alice's assignment as worse than his own. This means that Bob ranks his assignment, at which he has a higher priority than Alice, over Alice's. Think of Bob's higher priority over Alice as a property right that Bob has relative to Alice for his assignment. This means that the switch would give Bob an assignment that is worse than the school at which Bob has property rights. The notion of fairness with priorities is therefore analogous to our notion of fairness. We flesh out the school choice application further in Section 6.

Importantly, our notion of fairness is compatible with efficiency. In the standard model of school choice and matching, fairness and efficiency are generally incompatible; and a lot of the school choice literature has been devoted to the resulting trade-off. In our model, the trade-off between fairness and efficiency can be avoided altogether.

Finally, we show that, under some conditions, our solution can be achieved as a market outcome. The idea seeks to generalize Varian's and Hylland and Zeckhauser's competitive equilibrium from equal incomes. The obvious solution would

be to endogenize incomes as in the Walrasian model, where agents derive income from selling their endowments. When incomes equal the value of endowments at market prices, the resulting solution is obviously individually rational, and one can hope to obtain some notion of fairness in equilibrium. However, it has been shown by [Hylland and Zeckhauser \(1979\)](#) that Walrasian equilibrium may not exist in this model, even under strong assumptions on utility functions. So the Walrasian approach is a non-starter. Instead we construct specific price-dependent income functions. These income functions ensure individual rationality and fairness (efficiency is a matter of the same kind of first-welfare theorem as in Hylland and Zeckhauser; fairness and individual rationality require careful construction of income functions). The income functions also “close” the model adequately, so that market clearing can be obtained.

## 2. RELATED LITERATURE.

Our notion of justified envy is analogous to the fairness notion of [Yilmaz \(2010\)](#). Yilmaz uses ordinal preferences instead of utility functions, and says that agent  $i$  justifiably envies agent  $j$  if  $i$  does not regard her allocation as first-order stochastically dominating  $j$ 's, while any object that she obtains with positive probability in her allocation is regarded by  $j$  as acceptable. While our notion of justified envy is analogous to Yilmaz's, the exercise we carry out, and our results, are very different. Endowments are deterministic in Yilmaz's model and probabilistic in ours. Yilmaz focuses his analysis on the probabilistic serial rule ([Bogomolnaia and Moulin, 2001](#)); as a consequence, his results are simply unrelated to ours.

Efficiency and fairness can be achieved in models without endowments. Examples are the solutions of, [Varian \(1974\)](#), [Hylland and Zeckhauser \(1979\)](#) and [Bogomolnaia and Moulin \(2001\)](#). Our problem is complicated, both conceptually and technically, by the presence of endowments. Conceptually because the meaning of fairness among unequal agents is not obvious. Technically because market equilibrium may not exist in economies with endowments ([Hylland and Zeckhauser, 1979](#)). Under some conditions, our fair solution can be achieved through a market with price-dependent incomes. This requires careful construction of income functions, something we carry out in Section 8.

Our results are applicable to school choice (see Section 6), and in particular to controlled school choice, or school choice with constraints. School choice was first introduced as an application of resource allocation models by [Abdulkadiroğlu and](#)

Sönmez (2003). Hamada, Hsu, Kurata, Suzuki, Ueda, and Yokoo (2017) is the only paper we are aware of that emphasizes endowments in school choice. They assume that each child owns one seat of some school as endowment. Their goal is to design strategy-proof allocation mechanisms to meet the distributional constraint in the market and individual rationality constraint of each child. Since they consider deterministic endowments and ordinal preferences, and their fairness notions are based on priorities, their results are unrelated to ours. The constraints we analyze have been discussed extensively in the literature on controlled school choice: see Kojima (2012), Hafalir, Yenmez, and Yildirim (2013), Ehlers, Hafalir, Yenmez, and Yildirim (2014), and Echenique and Yenmez (2015).

More generally, distributional constraints (motivated not only by school choice, but also by geographic distributional considerations), have been introduced and studied by Kamada and Kojima (2015) and Kamada and Kojima (2017), among others. Our approach (see Section 5) of eliminating envy where it does not conflict with constraints, is common to those papers. For example, Kamada and Kojima consider matchings where no blocking pair that would not violate distributional constraints are present.

Balbuzanov and Kotowski (2019) explore the role of property rights for discrete allocation problems. Different from us, they interpret endowments as the rights to exclude others, and propose a new solution concept called exclusion core. Although they allow for public ownership, or collective ownership by subgroups, endowments in their model are deterministic. As a result, their results are unrelated to ours.

### 3. THE MODEL

Notation and preliminary definitions. The simplex  $\{x \in \mathbf{R}_+^n : \sum_{j=1}^n x_j = 1\}$  in  $\mathbf{R}^n$  is denoted by  $\Delta^n \subseteq \mathbf{R}^n$ , while the set  $\{x \in \mathbf{R}_+^n : \sum_{j=1}^n x_j \leq 1\}$  is denoted by  $\Delta_-^n \subseteq \mathbf{R}^n$ . When  $n$  is understood, we simply use the notation  $\Delta$  and  $\Delta_-$ .

A function  $u : \Delta_-^n \rightarrow \mathbf{R}$  is

- *quasi-concave* if  $\{z \in \Delta_-^n : u(z) \geq u(x)\}$  is a convex set, for all  $x \in \Delta_-^n$ .
- *concave* if, for any  $x, z \in \Delta_-^n$ , and  $\lambda \in (0, 1)$ ,  $\lambda u(z) + (1 - \lambda)u(x) \leq u(\lambda z + (1 - \lambda)x)$ ;
- *expected utility* if it is linear. In this case we identify  $u(\cdot)$  with a vector  $u \in \mathbf{R}^n$  and denote  $u(x)$  as  $u \cdot x$ .

**3.1. Model.** Our model subsumes the standard discrete allocation problem. A finite set of agents need to be assigned a finite set of indivisible objects. For example, the agents could be children and the objects could be seats at a school.

A *discrete allocation problem* is a tuple  $\Gamma = \{S, I, Q, (u^i, \omega^i)_{i \in I}\}$ , where:

- $S = \{s_k\}_{k=1}^L$  represents a set of indivisible objects.
- $I = \{1, \dots, N\}$  represents a set of agents, each of whom demands exactly one copy of an object.
- $Q = \{q_s\}_{s \in S}$  is a capacity vector, and  $q_s \in \mathbf{N}$  is the number of copies of object  $s$ . For simplicity, we assume that  $\sum_{s \in S} q_s = N$ , i.e., the number of copies of objects is equal to the number of agents.
- For each agent  $i$ ,  $u^i : \Delta_-^L \rightarrow \mathbf{R}$  is a continuous utility function defined on  $\Delta_-^L$ .
- For each agent  $i$ ,  $\omega^i \in \Delta^L$  is  $i$ 's endowment vector such that  $\omega_s^i$  is the fraction of object  $s$  owned by  $i$ . We assume that all objects are owned by agents. So  $\sum_{i=1}^N \omega^i = Q$ .

We say that  $\Gamma$  admits a *common favorite object* if there is object  $l$  such that for any  $i \in I$  and  $x^i \in \Delta_-^L$ , decreasing consumption of any object  $k \neq l$  in favor of  $l$  leads to an increase in  $u^i$ . For example, if all utility functions are differentiable and there is  $l \in S$  such that  $\frac{\partial u^i(x^i)}{\partial x_l^i} > \frac{\partial u^i(x^i)}{\partial x_k^i}$ ,  $\forall i \in I, x^i \in \Delta_-^L, k \neq l$ .

**3.2. Allocations.** It is impossible to even talk about fairness without introducing randomizations. Think of two identical agents who want the same object. We need to flip a coin if we are to be fair. So our model will from the beginning take randomized allocations as primitive. For this sake, in the above we assume that every agent  $i$  has a utility function  $u^i$  defined on  $\Delta_-^L$ .

An *allocation* in  $\Gamma$  is a vector  $x \in \mathbf{R}_+^{LN}$ , which we write as  $x = (x^i)_{i=1}^N$ , with  $x^i \in \Delta_-^L$ , such that

$$\sum_{i \in I} x_s^i = q_s$$

for all  $s \in S$ .  $x^i$  is the assignment obtained by each  $i \in I$ . When  $x_s^i \in \{0, 1\}$  for all  $i \in I$  and all  $s \in S$ ,  $x$  is a deterministic allocation. The Birkhoff-von Neumann theorem (Birkhoff, 1946; Von Neumann, 1953) implies that every allocation is a convex combination of deterministic allocations. Let  $\mathcal{A}$  be the set of all allocations. It is clear that  $\mathcal{A}$  is nonempty, closed and bounded.

**3.3. Individual rationality and Pareto optimality.** An allocation  $x$  is *acceptable* to agent  $i$  if  $u^i(x^i) \geq u^i(\omega^i)$ ;  $x$  is *individually rational* (IR) if it is acceptable to all agents. We also define a notion of approximate individual rationality: for any  $\epsilon > 0$ ,  $x$  is  *$\epsilon$ -individually rational* ( $\epsilon$ -IR) if  $u^i(x^i) \geq u^i(\omega^i) - \epsilon$  for all  $i \in I$ .

The notion of efficiency comes in three flavors: an allocation  $x$  is

- *weak Pareto optimal* (wPO) if there is no allocation  $y$  such that  $u^i(y^i) > u^i(x^i)$  for all  $i$ ;
- *$\epsilon$ -weak Pareto optimal* ( $\epsilon$ -PO), for any  $\epsilon > 0$ , if there is no allocation  $y$  such that  $u^i(y^i) > u^i(x^i) + \epsilon$  for all  $i \in I$ . Note that wPO is compatible with wasteful situations where one can use existing resources to make some agents strictly better off, but cannot construct an allocation that makes all agents strictly better off because there are agents who have obtained the best possible assignments.
- *Pareto optimal* (PO) if there is no allocation  $y$  such that  $u^i(y^i) \geq u^i(x^i)$  for all  $i$  with at least one strict inequality for one agent.

**3.4. Fairness.** As discussed in the introduction, we introduce a new notion of fairness that tries to parallel the standard definition of absence of justified envy in the model of school choice with priorities. Our notion of fairness relies on the idea of property rights, and rules out envy that cannot be justified through property rights.

We regard agents as having the right to consume their endowments, which means that agents have the right to be at least as well off as they would be by consuming their endowments. If an agent  $i$  envies another agent  $j$  at an allocation  $x$  (that is,  $i$  prefers  $x^j$  to  $x^i$ ), our fairness notion regards the envy as not justified if switching their allocations would violate the property rights of  $j$  (that is,  $j$  prefers  $\omega^j$  to  $x^i$ ).

Formally, we say that an agent  $i$  has a *justified envy* towards another agent  $j$  at an allocation  $x$  if

$$u^i(x^j) > u^i(x^i) \text{ and } u^j(x^i) \geq u^j(\omega^j).$$

In words,  $i$  justifiably envies  $j$  if she envies  $j$  and  $j$  could have received  $i$ 's assignment without violating any property rights of  $j$ . We say that  $x$  has *no justified envy* (NJE) if no agent has justified envy towards any other agent at  $x$ .

Fairness as NJE provides an argument for a social planner to defend against possible complaints. Suppose that  $x$  is individually rational. Faced with  $i$ 's complaint that she envies  $j$ , the planner's response is that  $i$  has received  $x^i$ , which is acceptable

to her but not to  $j$ . If  $i$  and  $j$  were to switch assignments, the planner would violate  $j$ 's property rights.

Of course, one can imagine complaints of envy that could be remedied through rearrangements more complicated than a pairwise switch. Such remedies may or may not be realistic, but in any case our methods easily accommodate much more general remedies. See Section 7.1. Specifically, one can devise cyclic rearrangements, where arbitrarily long sequences of agents collaborate in the satisfaction of an agent's envy, as long as the last agent's property rights are not violated. Theorem 4 extends our main result to cover this case.

**3.4.1. Implications of NJE.** In an IR and NJE allocation  $x$ , if  $u^i = u^j$  and  $u^i(\omega^i) \geq u^j(\omega^j)$ , then it must be that  $u^i(x^i) \geq u^j(x^j)$ .<sup>1</sup> That is, if two agents  $i$  and  $j$  have equal preferences and both agree that  $i$ 's endowment is weakly better than  $j$ 's, then both agree that  $i$ 's allocation in  $x$  is also weakly better than  $j$ 's. In particular, if  $u^i = u^j$  and  $u^i(\omega^i) = u^j(\omega^j)$ , then it must be that  $u^i(x^i) = u^j(x^j)$ . So NJE and IR imply *equal treatment of equals* (also called symmetry by Zhou (1990)).

Finally, our results are based on certain approximations that make use of two variants of justified envy: one is stronger than justified envy, while the other is weaker. We say that  $i$  has a *strong justified envy* (SJE) towards  $j$  at  $x$  if  $u^i(x^j) > u^i(x^i)$  and  $u^j(x^i) > u^j(\omega^j)$ . For any  $\epsilon > 0$ , we say  $i$  has an  $\epsilon$ -*justified envy* ( $\epsilon$ -JE) towards  $j$  at  $x$  if  $u^i(x^j) > u^i(x^i)$  and  $u^j(x^i) > u^j(\omega^j) - \epsilon$ . *No strong justified envy* (NSJE) and *no  $\epsilon$ -justified envy* (N $\epsilon$ JE) are defined similarly as before. It is easy to see that

$$\text{no } \epsilon\text{-justified envy} \implies \text{no justified envy} \implies \text{no strong justified envy}$$

## 4. MAIN RESULTS

Let  $\Gamma = \{S, I, Q, (u^i, \omega^i)_{i \in I}\}$  be a discrete allocation problem.

**Theorem 1.** *Suppose that agents' utility functions in  $\Gamma$  are concave.*

- (1) *For any  $\varepsilon > 0$ , there exists an allocation that is  $\varepsilon$ -individually rational,  $\varepsilon$ -Pareto optimal and has no  $\varepsilon$ -justified envy;*
- (2) *There exists an allocation that is individually rational, weak Pareto optimal and has no strong justified envy.*

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<sup>1</sup>If  $u^i(x^i) < u^j(x^j)$ , then  $i$ 's envy towards  $j$  is justified because  $u^j(x^i) = u^i(x^i) \geq u^i(\omega^i) \geq u^j(\omega^j)$ .



(3) Moreover, if each utility function in  $\Gamma$  is expected utility, there exists an allocation that is individually rational, Pareto optimal and has no strong justified envy.

The proof of Theorem 1 is in Section 8.

Under an additional hypothesis, we can provide a market foundation for our notion of fairness. It is possible to create a market, and price-dependent incomes, with fair, acceptable, and efficient equilibrium outcomes.

**Theorem 2.** *Suppose that agents' utility functions in  $\Gamma$  are quasi-concave, that  $\Gamma$  admits a common favorite object  $l$  and that  $\omega_i^l > 0$  for all  $i \in I$ . Then there exists continuous functions  $m^i : \Delta \rightarrow \mathbf{R}_+$  and  $(x, p) = ((x^i)_{i=1}^I, p) \in (\Delta_-^I) \times \Delta$ , such that*

- (1)  $\sum_i x^i = \sum_i \omega^i$  ( $x$  is an allocation; or, “supply equals demand”).
- (2)  $x$  is Pareto optimal, individually rational and has no justified envy.
- (3)  $x^i \in \operatorname{argmax}\{u^i(z^i) : z^i \in \Delta_- \text{ and } p \cdot z^i \leq m^i(p)\}$

The proof of Theorem 2 is in Section 9.

*Remark 1.* The role of the favorite common object is technical. It is there to ensure that incomes for all agents are strictly positive. The same objective may be achieved through other, very different, assumptions: for example by an Inada property of utility and assuming that  $\omega^i \gg 0$  for all  $i$ .

The contribution in Theorem 2 lies in the construction of the “income functions”  $m^i(p)$ . These ensure, in equilibrium, individual rationality as well as no justified envy. In fact, for any  $p'$ ,  $m^i(p')$  lies in the interval between the lowest income needed to achieve utility  $u^i(\omega^i)$  at prices  $p'$ , and the lowest income needed to achieve the maximum possible utility  $\sup u^i(\Delta_-^L)$  at prices  $p'$ . As a consequence, for any price vector  $p'$ , the vector  $x'_i$  solving the utility maximization problem (3) is acceptable for agent  $i$ .

*Remark 2.* The proof of Theorem 1 is based on weighted utilitarian maximization, and the choice of welfare weights that ensure fairness (in fact we prove the stronger Theorem 4 in this way in Section 8). But one can also show that the outcomes in Theorem 1 can be approximated by market equilibrium outcomes, along the lines of Theorem 2.

*Remark 3.* An important take-away from our result is that fairness and efficiency are compatible. In the standard model of school choice with priorities, these policy

objectives are incompatible (Abdulkadiroğlu and Sönmez, 2003), and a lot of work has been devoted to understanding the resulting tradeoff between efficiency and fairness.

## 5. EFFICIENT AND FAIR ASSIGNMENT UNDER CONSTRAINTS

It is easy to adapt our model and result to situations with quantity constraints. Throughout we focus on ex-ante constraints imposed on an allocation, but it is a simple corollary of the main result in Budish, Che, Kojima, and Milgrom (2013) that these can be achieved as randomized deterministic allocations that satisfy the relevant constraints.

**5.1. Constrained allocations.** We shall first introduce constraints through a restriction on the set of feasible allocations. We take as primitive a set  $\mathcal{A}^C \subseteq \mathcal{A}$ , and interpret it as the set of all allocations that comply with some given collection of constraints. For now, we do not describe the constraints explicitly; we leave this to Section 5.2. Throughout we suppose that  $\mathcal{A}^C$  is closed and convex.

The definitions of individual rationality and efficiency extend naturally to  $\mathcal{A}^C$ . An allocation  $x \in \mathcal{A}^C$  is *weak Pareto optimal* (wPO) if there is no allocation  $y \in \mathcal{A}^C$  such that  $u^i(y^i) > u^i(x^i)$  for all  $i \in I$ ; and  $x$  is  $\varepsilon$ -*weak Pareto optimal* ( $\varepsilon$ -PO), for any  $\varepsilon > 0$ , if there is no allocation  $y \in \mathcal{A}^C$  such that  $u^i(y^i) > u^i(x^i) + \varepsilon$  for all  $i \in I$ .

Previously, an agent  $i$ 's envy towards another agent  $j$  is negated if switching their assignments violates the property rights of  $j$ . Now constraints provide another source to negate  $i$ 's envy: switching the assignments of  $i$  and  $j$  may not be feasible because it violates some constraints. In other words, switching  $i$  and  $j$ 's assignment in  $x$  may result in an allocation outside of  $\mathcal{A}^C$ .

To formalize this idea, let  $x_{i \leftrightarrow j}$  denote the allocation obtained by switching the assignments of  $i$  and  $j$  in an allocation  $x$ . Formally,  $x_{i \leftrightarrow j}$  is an allocation such that

$$x_{i \leftrightarrow j}^i = x^j, x_{i \leftrightarrow j}^j = x^i, \text{ and } x_{i \leftrightarrow j}^k = x^k \text{ for all } k \in I \setminus \{i, j\}.$$

Now an agent  $i$  has an *justified envy* towards another agent  $j$  at an allocation  $x \in \mathcal{A}^C$  if

$$u^i(x^j) > u^i(x^i), u^j(x^i) \geq u^j(\omega^j) \text{ and } x_{i \leftrightarrow j} \in \mathcal{A}^C.$$

Under this definition, fairness (absence of justified envy) may no longer be compatible with efficiency and individual rationality as stated in Theorem 1. To overcome this difficulty, we group, or classify, agents into disjoint types. Intuitively, we say  $i$  and  $j$  are of equal type if the constraints behind  $\mathcal{A}^C$  do not distinguish between

them. We identify agents' types by checking whether switching their assignments in any feasible allocation is still feasible. We then prove that we can achieve fairness among agents of equal types.

Formally, we say two agents  $i, j \in I$  are of *equal type*, denoted by  $i \sim j$ , if

$$\text{for all } x \in \mathcal{A}^C, x_{i \leftrightarrow j} \in \mathcal{A}^C.$$

The binary relation  $\sim$  is reflexive and transitive.<sup>2</sup> Hence it partitions  $I$  into disjoint types.

Then we say  $i$  has an *equal-type justified envy* towards  $j$  at an allocation  $x \in \mathcal{A}^C$  if  $i$  has a justified envy towards  $j$  and  $i, j$  are of equal type. We say that  $x$  has *no equal-type justified envy* if no agent has an equal-type justified envy towards any other agent at  $x$ . *No strong equal-type justified envy* and *no equal-type  $\epsilon$ -justified envy* are defined in the obvious way, by stating that the relevant kind of envy is absent in the allocation.

With the preceding definitions out of the way, we are in a position to extend Theorem 1 to accommodate constraints.

**Theorem 3.** *Suppose agents' utility functions are concave and that  $\omega \in \mathcal{A}^C$ .*

- (1) *For any  $\varepsilon > 0$ , there exists an allocation that is  $\varepsilon$ -individually rational,  $\varepsilon$ -Pareto optimal and has no equal-type  $\varepsilon$ -justified envy;*
- (2) *There exists an allocation that is individually rational, weak Pareto optimal and has no strong equal-type justified envy.*

We say that the implicit constraints behind  $\mathcal{A}^C$  are *anonymous* if all agents are identified to be of equal type. The model we discussed in Section 3 is one where constraints are anonymous (since  $\mathcal{A}^C = \mathcal{A}$ ). Thus, Theorem 1 is a special case of Theorem 3.

**5.2. Constraint structures.** In our previous discussion,  $\mathcal{A}^C$  was exogenous and agents' types are identified from  $\mathcal{A}^C$ . It is often most useful to explicitly model the source of types and constraints. For example, types can arise from definitions of socio-economic status, or racial and ethnic classifications. Our approach is to follow [Budish, Che, Kojima, and Milgrom \(2013\)](#) and define a general constraint structure, then discuss how an exogenous collection of types gives rise to constrained allocations.

<sup>2</sup> Suppose  $i \sim j \sim k$ . For all  $x \in \mathcal{A}^C$ ,  $x_{i \leftrightarrow k} = [(x_{i \leftrightarrow j})_{j \leftrightarrow k}]_{i \leftrightarrow j}$ .  $i \sim j$  implies that  $x_{i \leftrightarrow j} \in \mathcal{A}^C$ ,  $j \sim k$  implies that  $(x_{i \leftrightarrow j})_{j \leftrightarrow k} \in \mathcal{A}^C$ , and  $i \sim j$  implies that  $[(x_{i \leftrightarrow j})_{j \leftrightarrow k}]_{i \leftrightarrow j} \in \mathcal{A}^C$ . So  $x_{i \leftrightarrow k} \in \mathcal{A}^C$ .

A constraint  $[H, (\underline{q}_H, \bar{q}_H)]$  consists of a set  $H \subseteq I \times S$  and a pair of integers  $(\underline{q}_H, \bar{q}_H)$  with  $\underline{q}_H \leq \bar{q}_H$ . Given a collection  $\mathcal{H}$  of constraints, we define

$$\mathcal{A}^{\mathcal{H}} = \left\{ x \in \mathcal{A} : \underline{q}_H \leq \sum_{(i,s) \in H} x_s^i \leq \bar{q}_H \text{ for all } [H, (\underline{q}_H, \bar{q}_H)] \in \mathcal{H} \right\}$$

as the set of feasible allocations satisfying  $\mathcal{H}$ .

In our model of Section 3, we have already encountered the  $N$  constraints:  $[\{i\} \times S, (0, 1)]$  for each  $i \in I$ ; but one can imagine many other possibilities. For example, in controlled school choice (Ehlers, Hafalir, Yenmez, and Yildirim, 2014; Echenique and Yenmez, 2015), the set of students  $I$  are partitioned into  $T_1, \dots, T_K$  (which are interpreted as types), and for each school  $s_\ell$ , the desirable number of type  $k$  students where  $k \in \{1, \dots, K\}$  is between  $\underline{q}_{\ell,k}$  and  $\bar{q}_{\ell,k}$ . So for each school  $s_\ell$  and each type  $k \in \{1, \dots, K\}$ , we have a constraint

$$[T_k \times \{s_\ell\}, (\underline{q}_{\ell,k}, \bar{q}_{\ell,k})].$$

Theorem 3 says that there is an efficient and individually rational allocation that achieves fairness within each type.

For distributional constraints studied by Kamada and Kojima (2015, 2017), every constraint is of the form

$$[I \times S', (0, \bar{q}_{S'})]$$

with  $S' \subseteq S$ . Here, the agents are doctors, and the objects are hospital positions. The set  $S'$  is the set of hospitals in a geographic region (a city or a prefecture). A collection of such constraints is anonymous because each constraint does not distinguish among the identities of individual doctors. In general, a collection  $\mathcal{H}$  is *anonymous* if for every constraint  $[H, (\underline{q}_H, \bar{q}_H)] \in \mathcal{H}$ ,  $H$  is of the form  $I \times S'$  for some  $S' \subseteq S$ . Theorem 3 implies that for anonymous constraints, there is an efficient and individually rational allocation that achieves fairness among *any* two agents.

In our last example,  $\mathcal{H}$  consists of *individual constraints*, which impose restrictions on each individual's assignment. Formally,  $\mathcal{H} = \{\mathcal{H}_i\}_{i \in I}$  where each  $\mathcal{H}_i$  consists of constraints of the form

$$[\{i\} \times S', (\underline{q}_{i,S'}, \bar{q}_{i,S'})]$$

with  $S' \subseteq S$ . In school choice, if we want a student  $i$  to be assigned a neighborhood school, we can use the constraint  $[\{i\} \times S', (1, 1)]$  where  $S'$  is the set of schools in  $i$ 's neighborhood. In course allocation, if we want a student  $i$  to take at least one math course but no more than three math courses, we can use the constraint

$\{i\} \times S', (1, 3)$  where  $S'$  is the set of math courses. Theorem 3 says that we can achieve fairness among agents of equal individual constraints.

## 6. APPLICATION TO SCHOOL CHOICE

School choice is the problem of allocating children to schools when we want to take into account children's (or their parents') preferences (Abdulkadiroğlu and Sönmez, 2003). Several large US school districts have in the last 15 years implemented school choice programs that follow economists' recommendation and are based on economic theory.<sup>3</sup> Practical implementation of school choice programs presents us with a number of lessons and challenges.

The first lesson is that school choice should be guided by fairness, or lack of justified envy. When given the choice of implementing either a fair or an efficient outcome, school districts have consistently chosen fairness (Abdulkadiroğlu, Pathak, Roth, and Sönmez, 2005; Abdulkadiroğlu, Pathak, and Roth, 2005). One reason could be that district administrators are concerned with litigation: if Alice envies Bob's school, then the district can invoke justified envy to argue as a defense that Bob had a higher priority than Alice at the school in question.<sup>4</sup> It is also likely that district administrators, and society as a whole, have an intrinsic preference for fairness. Such a preference for fairness is important enough to outweigh concerns over efficiency.

The second lesson is that school districts want to give children certain rights, like the right to attend a neighborhood school if they wish to, or the right to go to the same school as an older sibling. Rights are achieved by giving children different priorities. For example, Bob might have a high priority for admission in a neighborhood school, or in a school that his brother already attends. While priorities are common in practice, we argue that they are problematic. Priorities do *not* translate immediately into property rights. Alice may have a high priority in one school, but still not get in. Her chances of getting into a particular school depends on all agents' choices and priorities in the system, not only on her priority at a given school. Specifically:

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<sup>3</sup>Boston (Abdulkadiroğlu and Sönmez, 2003; Abdulkadiroğlu, Pathak, Roth, and Sönmez, 2005), New York (Abdulkadiroğlu, Pathak, and Roth, 2005), and Chicago (Pathak and Sönmez, 2013) are leading examples.

<sup>4</sup>Observe that this notion of justified envy is pairwise, as is ours. Then again this is generalized in Section 7.1.

*Observation 1.* In school choice with coarse priorities it is computationally hard to determine if a given student will be assigned a given school.

*Observation 2.* The ordering of students in a priority may not reflect their “rights” at the school. One can easily construct examples of students who switch places in the priority of one school, and where the student who climbed up in the priority ends with a worse outcome.

Given the absence of a direct connection between priorities and property rights, we propose the use of endowments to ensure property rights. Endowments provide transparent, and immediate, property rights. A child who is endowed with a seat at her neighborhood school can simply choose to attend that school. Her right to attend that school does not depend on other agents in any way.

The third lesson is that school districts have demonstrated a strong preference for controlling the racial and socio-economic composition of their schools: so-called *controlled school choice*. A common critique of existing school choice programs is that they have led to undesirable school compositions. For example, in Boston, schools have been left with too few neighborhood children, which has motivated a move away from the system recommended by economists (Dur, Kominers, Pathak, and Sönmez, 2017). In New York City, the new school choice system exhibits high degrees of racial segregation. Segregation in NYC schools is not new, but the complaint is that the new school choice program may have made it worse, and certainly has not helped. In the words of a recent New York Times article “. . . school choice has not delivered on a central promise: to give every student a real chance to attend a good school. Fourteen years into the system, black and Hispanic students are just as isolated in segregated high schools as they are in elementary schools — a situation that school choice was supposed to ease.<sup>5</sup>” The article points to a dissatisfaction with school composition, and access to the best schools.

The situation in NYC has reached a point where there are talks of doing away with school priorities, and instead instituting a lottery. In fact, Professor Eric Nadelstern at Columbia University, who served as deputy school chancellor when the new school choice system was implemented, has recently proposed that children be allowed to apply to any school, and have a lottery decide the allocations.<sup>6</sup>

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<sup>5</sup>“The Broken Promises of Choice in New York City Schools”, *New York Times*, May 5th, 2017.

<sup>6</sup>See “Confronting Segregation in New York City Schools”, *New York Times*, May 15th, 2017.

Our paper can make Nadelstern's approach compatible with school choice. We imagine that there is a lottery that gives an *initial* probabilistic allocation of children to schools. The lottery could be as simple as giving each child the same chance of attending any school. It could also reflect different objectives in controlled school choice, such as giving each child a higher chance of attending his or her neighborhood school, or giving each minority child a chance (literally, a positive probability) of attending the highest-ranked schools. Our model takes as primitive an *arbitrary and given* probabilistic allocation of children to schools.

The initial allocation is typically not the final allocation, because we want preferences to play a role. So one can use quantity constraints, and our results in Section 5, to capture the desired bounds on the composition of a school. Subject to constraints, then, our solution achieves all the desirable properties. The final allocation will be fair, efficient, and individually rational.

## 7. DISCUSSION

**7.1. Envy justified by exchange.** Our notion of fairness relies on pairwise exchanges, or swaps, as being the remedy for envy. We think of such swaps as natural, and (for reasons explained in the Introduction) as the counterpart to the definition of justified envy in the model with priorities. But if swaps are seen as limited, it is important to note that our result is, in fact, easily generalized to allow for much more general remedies.

So let us think of envy that can be addressed by carrying out a chain of exchanges, each agent giving up their allocation in favor of an agent who envies them, and the last agent in the exchange being given the assignment of the first. If this reallocation does not violate the last agent's property rights, then the envy is justified.

Formally, agent  $i$  has  $\varepsilon$ -justified envy by exchange towards  $j$  at allocation  $x$  if there exists a sequence  $(i_k)_{k=1}^K$  with

- $i_1 = i$  and  $i_2 = j$ ;
- $i_k$  envies  $i_{k+1}$ ,  $1 \leq k \leq K - 1$
- and  $u^{i_K}(x^{i_1}) > u^{i_K}(\omega^{i_K}) - \varepsilon$

The idea is that  $i$  could conjure a remedy for his envy of  $j$  by proposing a coalition of agents, and a reallocation of their assignments, such that all are made better off, with the possible exception of one agent whose property rights are not  $\varepsilon$ -violated. It is possible that, were  $i$  and  $j$  to swap their assignments, then  $j$ 's property rights

would be violated; but  $j$  could be given  $h$ 's assignment without violating  $h$ 's property rights.

**Theorem 4.** *Suppose that all agents' utility functions are concave. Then, for any  $\varepsilon > 0$ , there exists an allocation that is  $\varepsilon$ -individually rational,  $\varepsilon$ -Pareto optimal, and has no  $\varepsilon$ -justified envy by exchange.*

**7.2. An example of envy between agents with identical endowment in an allocation of no justified envy.** We present an example of a discrete allocation problem in which all agents have expected utility preferences, together with an allocation that is individually rational, Pareto optimal, and satisfies no strong justified envy. In the example, one agent envies another agent even though they have equal endowments.

The example matters because one may think that no-envy among agents with equal endowments is intrinsically desirable. After all, we have tied the notion of justified envy to endowments; we have insisted on fairness by “controlling for endowments.” The idea behind the example, and the explanation for what makes the example work, is straightforward. The punchline is that endowments are not the end of the story: The two agents in question have equal endowments, but they have different preferences. Through their preferences, the two agents play very different roles in the economy. Other agents “trade” with the two agents in question, and as a result one of the agents ends up being more useful to the rest of the agents than the other. The outcome implies the presence of envy. Put differently, an agent can be valuable to others because she has a very desirable endowment, or because she is willing to trade in ways that enhance the welfare of others. The example we present in this section illustrates the role of preferences in generating value.

Another reason for why the example is important is that it suggests that our notion of fairness may fail to be incentive compatible. We have not specified a selection mechanism, and opted not to discuss incentives and strategy-proofness, but the example conveys some insights. One agent envies another even though they have equal endowments. This fact suggests that one agent may want to pretend to be the agent that he envies. In a large economy, in which the number of agents who report each type of preference does not change very much after a misreport, it stands to reason that such a misreport would not be profitable. Of course, the example we present here falls short of proving that if we were to define a fair mechanism it would not be strategy proof.



*Example 1.* Suppose that there are five agents, labeled  $i = 1, \dots, 5$ , and three schools,  $s_1, s_2$  and  $s_3$ . There are two copies (seats) of schools  $s_2$  and  $s_3$ . There is only one copy of school  $s_1$ . In the example, all the “action” involves agents 1 and 2. The remaining three agents are, in a sense, residual; they are also identical.

The agents’ von-Neumann-Morgenstern utilities, and endowments, are as described in the following table:

$i$	$u_{s_1}^i$	$u_{s_2}^i$	$u_{s_3}^i$	$i$	$\omega_{s_1}^i$	$\omega_{s_2}^i$	$\omega_{s_3}^i$
1	3	1	2	1	0	1	0
2	3	2	1	2	0	1	0
3	2	3	1	3	1/3	0	2/3
4	2	3	1	4	1/3	0	2/3
5	2	3	1	5	1/3	0	2/3.

Observe that agents 1 and 2 have identical endowments.

Finally, consider the following allocation  $x$ :

$i$	$x_{s_1}^i$	$x_{s_2}^i$	$x_{s_3}^i$
1	0	0	1
2	1/2	0	1/2
3	1/6	2/3	1/6
4	1/6	2/3	1/6
5	1/6	2/3	1/6

Observe that agent 1 envies agent 2, as

$$u^1 \cdot x^1 = 2 < 3/2 + 2/2 = u^1 \cdot x^2.$$

The envy is not justified, however, as

$$u^2 \cdot x^1 = 1 < 2 = u^2 \cdot \omega^2.$$

In fact, it is easy to see that  $x$  has no justified envy.

It is also easy to see that the allocation  $x$  is individually rational and Pareto optimal. In any PO allocation  $y$ , we cannot have  $y_{s_2}^1 > 0$ , as agent 1 and any agent  $j \in \{3, 4, 5\}$  are willing to trade school 2 for any other school. So  $y^1$  must be a convex combination of  $(1, 0, 0)$  and  $(0, 0, 1)$ . To make agent 1 better off then we would need to give agent 1 some shares in school 1, but these can only come at the expense of agent 2. To make agent 2 better off, she would need to get more shares in school 1, but these can only come at the expense of agents 3,4 and 5. These agents could only exchange shares in school 1 for shares in school 2, which agent 2 does not have. All agents 2, 3, 4 and 5 rank schools 3 and 1 in the same way.

**7.3. Extension.** It is possible to generalize our model to an environment where the total amount of consumption of an agent is bounded above by some arbitrary  $T^i > 0$ , and where  $i$ 's endowment also sums up to  $T^i$ . This allows us to capture the phenomenon of *time banks*, where agents exchange labor (see [Andersson, Cseh, Ehlers, Erlanson, et al. \(2018\)](#)). One example of time banks in market design is child care cooperatives.

## 8. PROOF OF THEOREMS 1, 3 AND 4

**8.1. Proof of Theorem 1.** The first two statements in the Theorem follow from Theorem 4. The last statement needs some preparatory work, and uses Theorem 2 as it is obtained as the limit of market equilibria. The proof is in Section 10.

**8.2. Proof of Theorem 3.** For given  $\varepsilon > 0$ , define

$$\mathcal{A}^* = \{x \in \mathcal{A}^C \text{ is } \varepsilon\text{-individually rational and } \varepsilon\text{-Pareto optimal}\}.$$

It is easy to see that  $\mathcal{A}^*$  is nonempty and compact.<sup>7</sup>

For any  $\lambda \in \Delta$ , define

$$\phi(\lambda) = \operatorname{argmax}\left\{\sum_{i \in I} \lambda^i u^i(x^i) - \delta \sum_{i \in I} \|x^i - (1, \dots, 1)\| : (x^i)_{i \in I} \in \mathcal{A}^*\right\},$$

where  $\delta > 0$  is small enough such that

$$\delta \max_{x \in \mathcal{A}^*} \sum_{i \in I} \|x^i - (1, \dots, 1)\| < \varepsilon.$$

Since all  $u^i$  are continuous and concave and  $\sum_{i \in I} \|x^i - (1, \dots, 1)\|$  is continuous and strictly convex, the objective function  $\sum_{i \in I} \lambda^i u^i(x^i) - \delta \sum_{i \in I} \|x^i - (1, \dots, 1)\|$  is continuous and strictly concave. Moreover,  $\mathcal{A}^*$  is compact. Thus,  $\phi : \Delta \rightarrow \mathcal{A}^*$  is a function (meaning it is singleton-valued), and, by the Maximum Theorem, continuous.

For any agent  $i$ , define

$$C^i = \{\lambda \in \Delta : \nexists j \in I \text{ s.t. } i \text{ has an equal-type } \varepsilon\text{-justified envy towards } j \text{ at } \phi(\lambda)\}.$$

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<sup>7</sup> $\mathcal{A}^*$  is nonempty because  $\omega \in \mathcal{A}^C$  is individually rational and any feasible allocation strictly Pareto dominating  $\omega$  is individually rational.  $\mathcal{A}^*$  is also closed. Let  $\{x_n\} \subseteq \mathcal{A}^*$  and  $x_n \rightarrow x$ . We have  $x \in \mathcal{A}^C$  because  $\mathcal{A}^C$  is closed. Since  $u^i(x_n^i) \geq u^i(\omega^i) - \varepsilon$  for all  $n$ , in the limit  $u^i(x^i) \geq u^i(\omega^i) - \varepsilon$ . So  $x$  is  $\varepsilon$ -individually rational. Suppose  $x$  is not  $\varepsilon$ -Pareto optimal. Then there exists  $y \in \mathcal{A}^C$  such that  $u^i(y^i) > u^i(x^i) + \varepsilon$  for all  $i \in I$ . For big enough  $n$ , it must be that  $u^i(y^i) > u^i(x_n^i) + \varepsilon$  for all  $i \in I$ , which contradicts the  $\varepsilon$ -Pareto optimality of  $x_n$ .  $\mathcal{A}^*$  is obviously bounded.

The proof relies on an application of the so-called KKM Lemma (the lemma is due to Knaster, Kuratowski and Mazurkiewicz; see Theorem 5.1 in [Border \(1989\)](#)). In the following two lemmas we prove that  $\{C^i\}_{i=1}^N$  is a *KKM covering* of the simplex  $\Delta$ . This means that every  $C^i$  is closed and that for any  $\lambda \in \Delta$  there is at least one  $C^i$  such that  $\lambda^i > 0$  and  $\lambda \in C^i$ .

**Lemma 1.** *For every  $i \in I$ ,  $C^i$  is closed.*

*Proof.* Let  $\lambda_n$  be a sequence in  $C^i$  such that  $\lambda_n \rightarrow \lambda \in \Delta$ . Let  $x_n = \phi(\lambda_n)$ . By continuity of  $\phi$ ,  $x_n \rightarrow x = \phi(\lambda) \in \mathcal{A}^*$ . Now we prove that  $\lambda \in C^i$ , that is,  $i$  does not have an equal-type  $\varepsilon$ -justified envy towards any other agent. Suppose that there is an agent  $j$  of equal type with  $i$  such that  $u^i(x^j) > u^i(x^i)$  and  $w^j(x^i) > w^j(\omega^j) - \varepsilon$ . Since  $i$  and  $j$  are of equal type,  $x_{i \leftrightarrow j} \in \mathcal{A}^C$ , and  $(x_n)_{i \leftrightarrow j} \in \mathcal{A}^C$  for every  $n$ . By continuity of  $u^i$  and  $w^j$ , for  $n$  large enough we have  $u^i(x_n^j) > u^i(x_n^i)$  and  $w^j(x_n^i) > w^j(\omega^j) - \varepsilon$ . These mean that  $i$  has an equal-type  $\varepsilon$ -justified envy towards  $j$  at  $x_n$ , which is a contradiction. Therefore,  $\lambda \in C^i$  and  $C^i$  is closed.  $\square$

**Lemma 2.** *For every  $\lambda \in \Delta$ ,  $\lambda \in \cup_{i \in \text{supp}(\lambda)} C^i$ .*

*Proof.* Suppose, towards a contradiction, that for some  $\lambda \in \Delta$ ,  $\lambda \notin \cup_{i \in \text{supp}(\lambda)} C^i$ . Let  $x = \phi(\lambda)$ . Then for every  $i \in \text{supp}(\lambda)$  there exists some  $j$  of equal type with  $i$  such that  $u^i(x^j) > u^i(x^i)$  and  $w^j(x^i) > w^j(\omega^j) - \varepsilon$ .

Suppose first that there exists some  $i$  and  $j$  in the aforementioned situation such that  $j \notin \text{supp}(\lambda)$ . Then consider the allocation  $y = x_{i \leftrightarrow j} \in \mathcal{A}^C$ .  $y$  is  $\varepsilon$ -individually rational as  $x$  was  $\varepsilon$ -individually rational and  $w^j(x^i) > w^j(\omega^j) - \varepsilon$ . Note that  $\lambda^j = 0$  and  $u^i(x^j) > u^i(x^i)$  imply that  $\sum_{h \in I} \lambda^h u^h(x^h) < \sum_{h \in I} \lambda^h u^h(y^h)$ . We also have that  $\sum_{h \in I} \|x^h - \mathbf{1}\| = \sum_{h \in I} \|y^h - \mathbf{1}\|$ , hence

$$\sum_{h \in I} \lambda^h u^h(x^h) - \delta \sum_{h \in I} \|x^h - \mathbf{1}\| < \sum_{h \in I} \lambda^h u^h(y^h) - \delta \sum_{h \in I} \|y^h - \mathbf{1}\|.$$

By definition of  $\phi$ , then,  $y \notin \mathcal{A}^*$ . Since  $y$  is an  $\varepsilon$ -individually rational allocation, it cannot be  $\varepsilon$ -Pareto optimal. So there is an  $\varepsilon$ -Pareto optimal feasible allocation  $z \in \mathcal{A}^*$  such that  $u^h(z^h) > u^h(y^h) + \varepsilon$  for all  $h \in I$ . Then  $z$  must be  $\varepsilon$ -individually rational and belong to  $\mathcal{A}^*$ . By our choice of  $\delta$ ,

$$\sum_{h \in I} \lambda^h u^h(y^h) - \delta \sum_{h \in I} \|y^h - \mathbf{1}\| < \sum_{h \in I} \lambda^h u^h(z^h) - \delta \max_{x \in \mathcal{A}^*} \sum_{h \in I} \|x^h - \mathbf{1}\| \leq \sum_{h \in I} \lambda^h u^h(z^h) - \delta \sum_{h \in I} \|z^h - \mathbf{1}\|.$$

Therefore,

$$\sum_{h \in I} \lambda^h u^h(x^h) - \delta \sum_{h \in I} \|x^h - \mathbf{1}\| < \sum_{h \in I} \lambda^h u^h(z^h) - \delta \sum_{h \in I} \|z^h - \mathbf{1}\|,$$

which contradicts the definition of  $x = \phi(\lambda)$ .

The above argument means that every  $i \in \text{supp}(\lambda)$  has an equal-type  $\varepsilon$ -justified envy towards some  $j \in \text{supp}(\lambda)$ . Then, since the set of agents in  $\text{supp}(\lambda)$  is finite, there must exist a subset of distinct agents  $\{i_1, \dots, i_K\} \subseteq \text{supp}(\lambda)$  such that  $i_1$  has an equal-type  $\varepsilon$ -justified envy towards  $i_2$ ,  $i_2$  has an equal-type  $\varepsilon$ -justified envy towards  $i_3$ , and so on until  $i_K$  has an equal-type  $\varepsilon$ -justified envy towards  $i_1$ . Then we can construct a new allocation  $y$  by letting agents in the cycle exchange their allocations. Since the agents in the cycle are of equal type,  $y$  must be feasible, that is,  $y \in \mathcal{A}^c$ .<sup>8</sup> As before, we have that  $\sum_{h \in I} \|x^h - \mathbf{1}\| = \sum_{h \in I} \|y^h - \mathbf{1}\|$  because  $y$  is obtained from  $x$  by permutating the assignments of agents in the cycle. Then we have

$$\sum_{h \in I} \lambda^h u^h(x^h) - \delta \sum_{h \in I} \|x^h - \mathbf{1}\| < \sum_{h \in I} \lambda^h u^h(y^h) - \delta \sum_{h \in I} \|y^h - \mathbf{1}\|.$$

As before,  $y$  is  $\varepsilon$ -individually rational but cannot be  $\varepsilon$ -Pareto optimal. Then as before we can find an allocation  $z \in \mathcal{A}^*$  that results in a contradiction.  $\square$

Now we are ready to prove Theorem 3.

*Proof of Theorem 3.* The proof is an application of the KKM lemma: see Theorem 5.1 in Border (1989).

By Lemmas 1 and 2,  $\{C^i\}_{i=1}^n$  is a KKM covering of  $\Delta$ . So there exists  $\lambda_\varepsilon^* \in \cap_{i=1}^n C^i$ . Let  $x_\varepsilon^* = \phi(\lambda_\varepsilon^*)$ . Then  $x_\varepsilon^*$  is  $\varepsilon$ -individually rational,  $\varepsilon$ -Pareto optimal and has no equal-type  $\varepsilon$ -justified envy.

Now let  $\{\varepsilon_n\}$  be a sequence such that  $\varepsilon_n > 0$  for all  $n$  and  $\varepsilon_n \rightarrow 0$ . Let  $x_n^*$  be the allocation found above for each  $\varepsilon_n$ . Since the sequence  $\{x_n^*\}$  is bounded, it has a subsequence  $\{x_{n_k}^*\}$  that converges to some  $x^*$ . Since the set of feasible allocations is closed,  $x^*$  is a feasible allocation. We prove that  $x^*$  is individually rational, weak Pareto optimal and has no strong equal-type justified envy.

Since  $u^i(x_{n_k}^*) > u^i(\omega^i) - \varepsilon_{n_k}$  for all  $n_k$  and all  $i$ , in the limit  $u^i(x^*) \geq u^i(\omega^i)$  for all  $i$ . So  $x^*$  is individually rational. Suppose  $x^*$  is not weak Pareto optimal, then there exists a feasible allocation  $y$  such that  $u^i(y^i) > u^i(x^{*i})$  for all  $i$ . For big enough

<sup>8</sup>We can consider a sequence of allocations  $\{x(k)\}_{k=0}^{K-1}$  with  $x(0) = x$  and  $x(k) = x_{i_k \leftrightarrow i_{k+1}}(k-1)$  for each  $1 \leq k \leq K-1$ . Since all agents in the cycle are of equal type, each  $x(k) \in \mathcal{A}^c$ . We let  $y = x(K-1)$ .

$n_k$ ,  $u^i(y^i) > u^i(x_{n_k}^{*i}) + \varepsilon_{n_k}$  for all  $i$ , which contradicts the  $\varepsilon_{n_k}$ -Pareto optimality of  $x_{n_k}^*$ . Suppose some agent  $i$  has a strong equal-type justified-envy towards another agent  $j$  in  $x^*$ ; that is,  $u^i(x^{*j}) > u^i(x^{*i})$ ,  $u^j(x^{*i}) > u^j(\omega)$ . Then for big enough  $n_k$ ,  $u^i(x_{n_k}^{*j}) > u^i(x_{n_k}^{*i})$ ,  $u^j(x_{n_k}^{*i}) > u^j(\omega) - \varepsilon_{n_k}$ . But given that  $i$  and  $j$  are of equal type, this contradicts the property of no equal-type  $\varepsilon_{n_k}$ -justified envy of  $x_{n_k}^*$ .  $\square$

**8.3. Proof of Theorem 4.** The proof follows along the lines of the proof of Theorem 3. We omit the steps in common and highlight the differences. Let  $C^i$  be the set of all  $\lambda \in \Delta$  at which  $i$  has no  $\varepsilon$ -justified envy by exchange towards any agent.

**Lemma 3.** *For every  $i \in I$ ,  $C^i$  is closed.*

*Proof.* Let  $\lambda_n$  be a sequence in  $C^i$  such that  $\lambda_n \rightarrow \lambda \in \Delta$ . Let  $x_n = \phi(\lambda_n)$ . By continuity of  $\phi$ ,  $x_n \rightarrow x = \phi(\lambda) \in \mathcal{A}^*$ . Now we prove that  $\lambda \in C^i$ , that is,  $i$  does not have  $\varepsilon$ -justified envy by exchange towards any other agent. Suppose that this is not the case, and there exists  $j$  with  $i$  having  $\varepsilon$ -justified envy by exchange towards  $j$ , with the sequence  $(i_k)_{k=1}^K$  being as in the definition of such envy. By continuity of utility, and since the sequence  $(i_k)_{k=1}^K$  is finite, for  $n$  large enough we have  $u^{i_k}(x_n^{i_{k+1}}) > u^{i_k}(x_n^{i_k})$  for  $1 \leq k \leq K-1$  while  $u^{i_K}(x_n^{i_1}) > u^{i_K}(\omega^{i_K}) - \varepsilon$ . The sequence  $(i_k)_{k=1}^K$  satisfies the rest of the requirements of  $i$  having  $\varepsilon$ -justified envy by exchange towards  $j$  at  $x_n$ , which is a contradiction. Therefore,  $\lambda \in C^i$  and  $C^i$  is closed.  $\square$

**Lemma 4.** *For every  $\lambda \in \Delta$ ,  $\lambda \in \cup_{i \in \text{supp}(\lambda)} C^i$ .*

*Proof.* Suppose, towards a contradiction, that for some  $\lambda \in \Delta$ ,  $\lambda \notin \cup_{i \in \text{supp}(\lambda)} C^i$ . Let  $x = \phi(\lambda)$ . Then for every  $i \in \text{supp}(\lambda)$ , there exists some  $j$  such that  $i$  has  $\varepsilon$ -justified envy by exchange towards  $j$ . Suppose first that there exists such envy, with corresponding sequence  $(i_k)_{k=1}^K$ , in which  $\lambda^{i_K} = 0$ . Let  $y$  be the allocation in  $\mathcal{A}$  obtained from  $x$  by letting each  $i_k$  get  $x^{i_{k+1}}$  ( $1 \leq k \leq K-1$ ) and  $i_K$  get  $x^i$ . Clearly  $y$  is feasible and  $\varepsilon$ -IR, as  $x$  was  $\varepsilon$ -individually rational and  $u^{i_K}(x^i) > u^{i_K}(\omega^{i_K}) - \varepsilon$  by definition of justified envy by exchange. Note that  $\lambda^{i_K} = 0$  and  $u^{i_k}(y^{i_k}) > u^{i_k}(x^{i_k})$  for all  $1 \leq k \leq K-1$  imply that  $\sum_{h \in I} \lambda^h u^h(x^h) < \sum_{h \in I} \lambda^h u^h(y^h)$ . We also have that  $\sum_{h \in I} \|x^h - \mathbf{1}\| = \sum_{h \in I} \|y^h - \mathbf{1}\|$ , hence

$$\sum_{h \in I} \lambda^h u^h(x^h) - \delta \sum_{h \in I} \|x^h - \mathbf{1}\| < \sum_{h \in I} \lambda^h u^h(y^h) - \delta \sum_{h \in I} \|y^h - \mathbf{1}\|.$$

By definition of  $\phi$ , then,  $y \notin \mathcal{A}^*$ . Since  $y$  is an  $\varepsilon$ -individually rational allocation, it cannot be  $\varepsilon$ -Pareto optimal. So there is an  $\varepsilon$ -Pareto optimal allocation  $z \in \mathcal{A}^*$

such that  $u^h(z^h) > u^h(y^h) + \varepsilon$  for all  $h \in I$ . Then  $z$  must be  $\varepsilon$ -individually rational and belong to  $\mathcal{A}^*$ . By our choice of  $\delta$ ,

$$\sum_{h \in I} \lambda^h u^h(y^h) - \delta \sum_{h \in I} \|y^h - \mathbf{1}\| < \sum_{h \in I} \lambda^h u^h(z^h) - \delta \max_{x \in \mathcal{A}^*} \sum_{h \in I} \|x^h - \mathbf{1}\| \leq \sum_{h \in I} \lambda^h u^h(z^h) - \delta \sum_{h \in I} \|z^h - \mathbf{1}\|.$$

Therefore,

$$\sum_{h \in I} \lambda^h u^h(x^h) - \delta \sum_{h \in I} \|x^h - \mathbf{1}\| < \sum_{h \in I} \lambda^h u^h(z^h) - \delta \sum_{h \in I} \|z^h - \mathbf{1}\|,$$

which contradicts the definition of  $x = \phi(\lambda)$ .

The above argument means that every  $i \in \text{supp}(\lambda)$  has  $\varepsilon$ -justified envy by exchange towards some agent  $j$ , with corresponding sequence  $(i_k)_{k=1}^K$  in which  $\lambda^{i_k} > 0$ . Thus,  $i_K \in \text{supp}(\lambda)$ . But it means that  $i_K$  also has  $\varepsilon$ -justified envy by exchange towards some agent  $j'$ , with corresponding sequence  $(i'_k)_{k=1}^K$  in which  $\lambda^{i'_k} > 0$ . Since the set of agents in  $\text{supp}(\lambda)$  is finite, there must exist a subset of agents  $\{h_1, \dots, h_M\} \subseteq \text{supp}(\lambda)$  such that  $h_1$  has  $\varepsilon$ -justified envy by exchange towards some agent with  $h_2$  being the end of the corresponding sequence,  $h_2$  has  $\varepsilon$ -justified envy by exchange towards some agent with  $h_3$  being the end of the corresponding sequence, and so on until  $h_M$  has  $\varepsilon$ -justified envy by exchange towards some agent with  $h_1$  being the end of the corresponding sequence. We write this situation as the following cycle

$$h_1 \rightarrow \dots \rightarrow h_2 \rightarrow \dots \rightarrow h_3 \rightarrow \dots \rightarrow \dots \rightarrow h_M \rightarrow \dots \rightarrow h_1,$$

where  $a \rightarrow b$  means that  $a$  envies  $b$ , and  $h_k \rightarrow \dots \rightarrow h_{k+1}$  is the corresponding sequence of  $h_k$ 's  $\varepsilon$ -justified envy by exchange towards some agent. Now note that if an agent  $h$  appears more than once in the above cycle, we can shorten the cycle by skipping the agents between any two consecutive positions of  $h$  in the cycle. So we can, without loss of generality, focus on the cycle in which each agent appears once. If we carry out the exchange in the cycle as in the proof of Lemma 2, then we obtain an improvement on the objective that defines  $\phi$ . This is a contradiction.  $\square$

## 9. PROOF OF THEOREM 2

We let the  $L$ -dimensional simplex  $\Delta^L$  be the domain of prices.

**9.1. Incomes.** The key to the theorem is to carefully construct price-dependent “income functions.” For each consumer  $i$ , define  $i$ 's *expenditure function* as

$$e^i(v, p) = \inf\{p \cdot x : u^i(x) \geq v\},$$

for  $p \in \Delta^L$  and  $v \in \mathbf{R}$ .

Let  $v^i = \sup u^i(\Delta^L)$  be the utility of agent  $i$  when she is satiated.

For any scalar  $m \geq 0$  and  $p \in \Delta^L$ , let

$$\mu^i(m, p) = \text{median}(\{e^i(u^i(\omega^i), p), m, e^i(v^i, p)\}).$$

Consider the function

$$\varphi(m, p) = \sum_i \mu^i(m, p) - \sum_i p \cdot \omega^i.$$

Observe that

- $e^i(u^i(\omega^i), p) \leq e^i(v^i, p)$ .
- $\mu^i$  is continuous and  $m \mapsto \mu^i(m, p)$  weakly monotone increasing.
- $\varphi$  is continuous and  $m \mapsto \varphi(m, p)$  weakly monotone increasing.
- $\varphi(m, p) \leq 0$  for  $m \geq 0$  small enough as  $e^i(u^i(\omega^i), p) \leq p \cdot \omega^i$ .

We shall define  $m^i(p)$ . First, in the case that  $\sum_i e^i(v^i, p) < \sum_i p \cdot \omega^i$ , we let  $m^i(p) = e^i(v^i, p)$ . Second, in the case that  $\sum_i e^i(v^i, p) \geq \sum_i p \cdot \omega^i$ , we have that  $\varphi(m, p) \leq 0$  for  $m \geq 0$  small enough, and  $\varphi(m, p) \geq 0$  for  $m \geq 0$  large enough. Therefore there exists  $m^* \geq 0$  with  $\varphi(m^*, p) = 0$ .

Now let  $m^i(p) = \mu^i(m^*, p)$ . To show that this is well defined, we need to prove that  $m^i(p)$  is independent of the choice of  $m^*$ . To that end, suppose that there are  $m_1, m_2 \in \mathbf{R}_+$  with  $m_1 \neq m_2$  and  $0 = \varphi(m_1, p) = \varphi(m_2, p)$ . Suppose wlog that  $m_1 < m_2$ . Now, since each  $\mu^i$  is weakly monotone increasing as a function of  $m$  we must have  $\mu^i(m_1, p) = \mu^i(m_2, p)$  for all  $i$ . Then the definition of  $m^i(p)$  is the same regardless of whether we choose  $m_1$  or  $m_2$ .

**Lemma 5.**  $m^i$  is continuous.

*Proof.* Let  $p^n \rightarrow p \in \Delta^L$ . Note that if  $\sum_i e^i(v^i, p) - \sum_i p \cdot \omega^i < 0$ , then for  $n$  large enough we will have  $\sum_i e^i(v^i, p^n) - \sum_i p^n \cdot \omega^i < 0$ . Then  $m^i(p^n) = e^i(v^i, p^n) \rightarrow e^i(v^i, p) = m^i(p)$  by continuity of the expenditure function.

So suppose that  $\sum_i e^i(v^i, p) - \sum_i p \cdot \omega^i \geq 0$ , and let  $m$  be such that  $\varphi(m, p) = 0$ . We shall discuss two cases.

Case1: Consider the case that  $\sum_i e^i(v^i, p^{n_k}) - \sum_i p^{n_k} \cdot \omega^i < 0$  for some subsequence  $p^{n_k}$ . Then  $\sum_i e^i(v^i, p) - \sum_i p \cdot \omega^i = 0$ . This means that if  $\varphi(m, p) = 0$  then  $m \geq e^i(v^i, p)$  for all  $i$ . Hence  $m^i(p) = e^i(v^i, p)$ . But since  $m^i(p^{n_k}) = e^i(v^i, p^{n_k})$  we get that  $m^i(p^{n_k}) \rightarrow m^i(p)$ .

Case 2: Now turn to a subsequence  $p^{n_k}$  with  $\sum_i e^i(v^i, p^{n_k}) - \sum_i p^{n_k} \cdot \omega^i \geq 0$ . Then there is  $m^{n_k}$  with  $\varphi(m^{n_k}, p^{n_k}) = 0$ . We can take this sequence to be bounded: consider any further convergent subsequence  $m^{n'_k}$  and say that  $m^{n'_k} \rightarrow m'$ . Then  $0 = \varphi(m^{n'_k}, p^{n'_k}) \rightarrow \varphi(m', p)$ . Thus  $m^i(p^{n'_k}) = \mu^i(m^{n'_k}, p^{n'_k}) \rightarrow \mu^i(m', p)$ , as  $\mu^i$  is continuous. Since the sequence  $\{m^{n_k}\}$  is bounded, this implies that  $m^i(p^{n_k}) \rightarrow m^i(p)$ .

Cases 1 and 2 exhaust all possible subsequences of  $p^n$ .  $\square$

The role of the following lemma will be clear towards the end of the proof.

**Lemma 6.** *If  $m^i(p) < \min\{m^j(p), e^i(v^i, p)\}$  then  $m^j(p) = e^j(u^j(\omega^j, p))$ .*

*Proof.* Since  $m^i(p) < e^i(v^i, p)$ , we must be in the case  $\sum_i e^i(v^i, p) \geq \sum_i p \cdot \omega^i$  of the definition of income functions. So let  $m^* \geq 0$  with  $\varphi(m^*, p) = 0$ .

Since  $m^i(p) = \mu^i(m^*, p) < e^i(v^i, p)$ , we must have  $m^* \leq m^i(p)$ . By hypothesis,  $m^* < m^j(p)$ . Then  $m^j(p) = \mu^j(m^*, p)$  implies that  $m^j(p) = e^j(u^j(\omega^j, p))$ .  $\square$

**9.2. Existence of quasi-equilibrium.** We first establish the existence of a quasiequilibrium with  $p^* \neq 0$ . The argument is similar to [Gale and Mas-Colell \(1975\)](#). See also [Mas-Colell, Whinston, Green, et al. \(1995\)](#) (Chapter 17, Appendix B).

For any  $p \in \Delta^L$ , let  $\underline{d}^i(p)$  be the set of vectors  $x^{i'} \in \Delta^L_-$  that satisfy the following properties:

$$\begin{aligned} p \cdot x^{i'} &\leq m^i(p) \\ u^i(x^{i'}) &\geq u^i(\hat{x}^i) \text{ for all } \hat{x}^i \in \Delta^L_- \text{ with } p \cdot \hat{x}^i < m^i(p). \end{aligned}$$

We consider the correspondence  $p \mapsto \underline{d}^i(p)$  with domain in  $\Delta^L$ .

Observe that  $\emptyset \neq \arg \max_{x^{i'} \in \Delta^L_-} \{u^i(x^{i'}) : p \cdot x^{i'} \leq m^i(p)\} \subseteq \underline{d}^i(x, p)$ . So  $\underline{d}^i$  takes non-empty values.

Observe also that  $\underline{d}^i$  is convex valued. To see this, let  $z^i, y^i \in \underline{d}^i(p)$  and define  $x^i(\alpha) = \alpha z^i + (1 - \alpha)y^i$ , for  $\alpha \in [0, 1]$ . It is obvious that  $x^i(\alpha) \in \Delta^L_-$  and that  $p \cdot x^i(\alpha) \leq m^i(p)$ . For any  $\hat{x}^i \in \Delta^L_-$  with  $p \cdot \hat{x}^i < m^i(p)$ ,  $\min\{u^i(z^i), u^i(y^i)\} \geq u^i(\hat{x}^i)$  and quasi-concavity of  $u^i$  imply that  $u^i(x^i(\alpha)) \geq u^i(\hat{x}^i)$ . Thus  $x^i(\alpha) \in \underline{d}^i(p)$ .

A third observation is that  $\underline{d}^i(p)$  is upper-hemicontinuous. To this end, consider a sequence  $p_n$  in  $\Delta^L$  with  $p_n \rightarrow p \in \Delta^L$ . Consider  $z_n^i \in \underline{d}^i(p_n)$  such that  $z_n^i \rightarrow z^i$ . Clearly,  $z^i \in \Delta^L_-$  and  $p \cdot z^i \leq m^i(p)$  as  $m^i$  is continuous (Lemma 5). Moreover, for any  $\hat{x}^i \in \Delta^L_-$  with  $p \cdot \hat{x}^i < m^i(p)$ , we have that  $p_n \cdot \hat{x}^i < m^i(p_n)$  for  $n$  large enough



(again by Lemma 5). Thus  $u^i(z_n^i) \geq u^i(\hat{x}^i)$  for  $n$  large enough, which by continuity of  $u^i$  implies that  $u(z^i) \geq u^i(\hat{x}^i)$ . Hence  $z^i \in \underline{d}^i(p)$ .

For  $x \in (\Delta_-^L)^I$  and  $p \in \Delta^L$ , let

$$\bar{\pi}(x, p) = \operatorname{argmax}\left\{p \cdot \left(\sum_i x^i - \sum_i \omega^i\right) : p \in \Delta^L\right\},$$

and consider the correspondence

$$\xi : (\Delta_-^L)^N \times \Delta^L \rightarrow (\Delta_-^L)^N \times \Delta^L$$

defined by  $\xi(x^1, \dots, x^I, p) = (\times_i \underline{d}^i(p)) \times \bar{\pi}(x, p)$ .

By the previous observations, and the maximum theorem,  $\xi$  is in the hypotheses of Kakutani's fixed point theorem. Let  $(x^*, p^*)$  be a fixed point of  $\xi$ .

We argue that  $(x^*, p^*)$  is a Walrasian quasiequilibrium. We have that  $p^* \cdot x^{i*} \leq m^i(p^*)$  for every  $i$ , by construction of  $\xi$ . By definition of  $m^i$ , we have  $\sum_i m^i(p^*) \leq p^* \cdot \sum_i \omega^i$ . Hence,  $p^* \cdot (\sum_i x^{i*} - \sum_i \omega^i) \leq 0$ . This implies  $\sum_i x^{i*} - \sum_i \omega^i \leq 0$  since otherwise, by definition of  $\bar{\pi}$ , we would have

$$p^* \cdot \left(\sum_i x^{i*} - \sum_i \omega^i\right) = \max_{p' \in \Delta^L} \left\{p' \cdot \left(\sum_i x^{i*} - \sum_i \omega^i\right)\right\} > 0.$$

We also claim that  $p^* \cdot x^{i*} = m^i(p^*)$  for every  $i$ , since by definition of  $m^i$  we have  $m^i(p^*) \leq e^i(v^i, p^*)$ . Indeed, suppose that  $p^* \cdot x^{i*} < m^i(p^*) \leq e^i(v^i, p^*)$ . Since  $x^{i*}$  does not satiate the agent, for an arbitrarily small ball  $B$  around  $x^{i*}$  there is  $x^{i'} \in B$  with  $u^i(x^{i'}) > u^i(x^{i*})$  and  $p^* \cdot x^{i'} < m^i(p^*)$ , contradicting  $x^{i*} \in \underline{d}^i(x^*, p^*)$ . Observe that, as a consequence of the above,

$$(1) \quad \sum_i p^* \cdot \omega^i = \sum_i p^* \cdot x^{i*} = \sum_i m^i(p^*).$$

We have shown that  $p^* \cdot (\sum_i x^{i*} - \sum_i \omega^i) = 0$ . Since  $\sum_i x^{i*} - \sum_i \omega^i \leq 0$ , we obtain  $p_l^* = 0$  for any  $l$  with  $\sum_i x_l^{i*} - \sum_i \omega_l^i < 0$  (underdemanded objects). Then, since preferences are monotonic, it is wlog to assume that  $\sum_i x^{i*} - \sum_i \omega^i = 0$  by consuming the remaining units of underdemanded objects for free. (Notice that  $\sum_{i \in I} x_i - \sum_{i \in I} \omega_i = 0$  particularly implies  $x_i^* \in \Delta^L$  for all  $i$ .) This proves that  $(x^*, p^*)$  is a Walrasian quasiequilibrium.

**9.3. Existence of equilibrium.** We prove now that  $(x^*, p^*)$  is a Walrasian equilibrium. Let  $l$  be the favorite object. We argue that  $p_l^* > 0$ . Suppose that  $p_l^* = 0$ . Since  $p^* \in \Delta^L$ , there must be an object  $k \neq l$  with  $p_k^* > 0$ . Some agent  $i$  is consuming object  $k$ , meaning  $x_k^{i*} > 0$ . For this agent, substituting all such consumption for

an equivalent consumption of object  $l$  saves expenses and increases utility. Hence  $x^{i*} \notin \underline{d}^i(p^*)$ . This contradiction shows that  $p_l^* > 0$ . Notice that, in consequence,  $e^i(v^i, p^*) > 0$  for all  $i$ .

Our next step is to establish that  $\underline{m} = \min\{m^i(p^*) : 1 \leq i \leq I\} > 0$ . First, if  $\underline{m} = \min\{e^i(v^i, p^*) : 1 \leq i \leq I\}$  then we are done because  $e^i(v^i, p^*) > 0$ . Ruling out this case, there must exist  $i$  with  $m^i(p^*) < e^i(v^i, p^*)$ , which implies that  $\sum_i m^i(p^*) = \sum_i p^* \cdot \omega^i$ . Now, if

$$(2) \quad \underline{m} = \min\{e^i(u^i(\omega^i), p^*) : 1 \leq i \leq I\},$$

then there is  $h$  with  $\underline{m} = m^h(p^*) \leq e^i(u^i(\omega^i), p^*)$  for all  $i$ ; which implies by the definition of the income functions  $m^i$  that  $m^i(p^*) = e^i(u^i(\omega^i), p^*)$  for all  $i$ . But  $e^i(u^i(\omega^i), p^*) \leq p^* \cdot \omega^i$  for all  $i$  and

$$\sum_i e^i(u^i(\omega^i), p^*) = \sum_i m^i(p^*) = \sum_i p^* \cdot \omega^i,$$

which by  $e^i(u^i(\omega^i), p^*) \leq p^* \cdot \omega^i$  implies that  $e^i(u^i(\omega^i), p^*) = p^* \cdot \omega^i$  for all  $i$ . So  $m^i(p^*) = p^* \cdot \omega^i$  for all  $i$ . Now  $p_l^* > 0$  and  $\omega_l^i > 0$  gives that  $m^i(p^*) > 0$ .

Finally, if Equation (2) does not hold, then  $0 \leq \min\{e^i(u^i(\omega^i), p^*) : 1 \leq i \leq I\} < \underline{m}$ . So  $m^i(p^*) > 0$  for all  $i$ .

Now, by a standard argument,  $m^i(p^*) > 0$  for all  $i$  implies that the quasi-equilibrium  $(x^*, p^*)$  is a Walrasian equilibrium.

#### 9.4. Properties of $x^*$ .

9.4.1. *Pareto optimality.* Suppose that  $y^i \in \Delta_-^L$  and that  $u^i(y^i) \geq u^i(x^{i*})$ . Then we must have  $p^* \cdot y^i \geq m^i(p^*)$  because otherwise  $p^* \cdot y^i < m^i(p^*) \leq e^i(v^i, p^*)$  and if  $i$  is satiated, then  $p^* \cdot y^i < e^i(v^i, p^*)$  and  $u^i(y^i) \geq v^i$  contradicts the definition of  $e^i$ ; if  $i$  is not satiated, there would exist  $z^i$  with  $p^* \cdot z^i < m^i(p^*)$  and  $u^i(z^i) > u^i(y^i) \geq u^i(x^{i*})$ .

Obviously if  $y^i \in \Delta_-^L$  and  $u^i(y^i) > u^i(x^{i*})$  then  $p^* \cdot y^i > m^i(p^*)$ . So if  $y = (y^i)$  Pareto dominates  $x^*$  then  $\sum_i p^* \cdot y^i > \sum_i m^i(p^*)$ . But by Equation (1) this is impossible if  $y$  is an allocation.

9.4.2. *Individual rationality.* To show that  $x^*$  is individually rational it suffices to notice that  $m^i(p^*) \geq e^i(u^i(\omega^i), p^*)$  for all  $i$ .

9.4.3. *No justified envy.* Suppose that  $i$  envies  $j$  at  $x^*$ . This implies that  $i$  is not satiated, hence  $m^i(p^*) < e^i(v^i, p^*)$ . It also implies that  $m^i(p^*) < m^j(p^*)$  as  $m^i(p^*) < p^* \cdot x^j = m^j(p^*)$ . By Lemma 6, then,  $m^j(p^*) = e^j(u^j(\omega^j), p^*)$ .

We obtain that

$$p^* \cdot x^i = m^i(p^*) < m^j(p^*) = e^j(u^j(\omega^j), p^*),$$

and hence  $u^j(x^i) < u^j(\omega^j)$  by definition of expenditure function. So  $i$ 's envy is not justified.

## 10. PROOF OF THIRD STATEMENT IN THEOREM 1

As discussed, the first two statements of the theorem follow immediately from Theorem 3. We proceed to prove the last statement.

The allocation with the desired properties (individually rational, Pareto optimal and has no strong justified envy) is obtained as the limit of a sequence of equilibrium allocations whose existence is guaranteed by Theorem 2. To this end, consider an additional object  $e$ , and an economy with the following extended endowments and preferences:

$$\begin{aligned} \omega^{i\alpha} &= \alpha(\vec{0}, 1) + (1 - \alpha)(\omega^i, 0) \\ U^i(x_1^i, \dots, x_L^i, x_e^i) &= u^i(x_1^i, \dots, x_L^i) + \theta x_e^i \\ \theta &> \sup_{i,l,x \in \Delta_-} \frac{\partial u^i(x)}{\partial x_l} \end{aligned}$$

For any  $\alpha > 0$  the economy is under the hypotheses of Theorem 2. Take a sequence  $(\alpha_t, x_t)$  where  $\alpha_t \rightarrow 0$  and each  $x_t$  is a Walrassian equilibrium allocation in the  $\alpha_t$ -extended economy. Since  $(x_t)$  is bounded, we can assume it converges. Hence  $(\alpha_t, x_t) \rightarrow (0, x^*)$ . Obviously,  $x^*$  is an allocation that corresponds to the original economy, since  $x_e^{i*} = \omega_e^{i0} = 0$  for all  $i$ . We shall prove that  $x^*$  meets the conditions stated in the Theorem.

**10.1. Individual rationality.** Suppose that there is  $i$  with  $U^i(x^*) < U^i(\omega^{i0}) = U^i(\omega^i, 0)$ . By continuity of preferences  $U^i(x^*) > U^i(\omega^{i\alpha_t})$  for  $t$  high enough, contradicting the definition of  $x_t$ .

**10.2. No strong justified envy.** Suppose that  $i$  envies  $j$  at  $x^*$ . By continuity of preferences, for  $t$  high enough,  $i$  envies  $j$  at  $x_t$ . By construction,  $x_t$  is NJE in the  $\alpha_t$ -extended economy. For all those  $t$  it must be the case that  $U^j(x_t^i) < U^j(\omega^{j\alpha_t})$ . Hence, by continuity of preferences we obtain  $U^j(x^{i*}) \leq U^j(\omega^{j0})$ , yielding NSJE.

**10.3. Pareto-optimality.** Suppose that  $x^*$  is not Pareto-optimal. Let  $x'$  be an allocation with  $x'_e = 0$  that Pareto dominates  $x^*$ . For any  $\varepsilon \in (0, 1)$ , consider

$$x'_{\varepsilon t} = x_t + \varepsilon(x' - x^*),$$

and observe that  $x'_{\varepsilon t} \rightarrow x^* + \varepsilon(x' - x^*)$ .

By linearity of preferences, and the fact that  $x'$  Pareto dominates  $x^*$ , we have that  $U^i(x'_{\varepsilon t}) \geq U^i(x_t^i)$  for all  $i$  with at least one strict inequality. We have seen, by Theorem 2, that  $x_t$  is Pareto-optimal at the  $\alpha_t$ -extended economy. Consequently, for any  $t$  and any  $\varepsilon \in (0, 1)$ ,  $x'_{\varepsilon t}$  cannot be an allocation of the  $\alpha_t$ -extended economy.

Now, given that

$$\begin{aligned} \sum_i x'_{\varepsilon t} &= \sum_i x_t^i + \varepsilon(\sum_i x^{i'} - \sum_i x^{i*}) \\ &= \sum_i \omega^{i\alpha_t} + \varepsilon(\sum_i \omega^i - \sum_i \omega^i) = \sum_i \omega^{i\alpha_t}, \end{aligned}$$

the market clearing aspect of being an allocation is met. So for  $x'_{\varepsilon t}$  not to be an allocation it must be the case that, for every  $\varepsilon$  and  $t$ , there is at least one agent  $i$  such that  $x'_{\varepsilon t} \notin \Delta_-$ .

Observe that  $x'_{\varepsilon t} \notin \Delta_-$  means that we are in one of two cases

- (1)  $\sum_l x'_{\varepsilon, t, l} > 1$ , or
- (2)  $x'_{\varepsilon, t, l} < 0$  for some  $l$  (or both).

Moreover, note by definition of  $x'_{\varepsilon, t}$  that if we are in case (1) then we are in (1) for any  $\varepsilon' \geq \varepsilon$ , and if we are in case (2) then we are in (2) for any  $\varepsilon' \geq \varepsilon$ .

Let  $I_t^1$  be the set of agents  $i$  for whom  $x'_{\varepsilon, t}$  is in case (1) for all  $\varepsilon \in (0, 1)$ , and  $I_t^2$  be the set of agents  $i$  for whom  $x'_{\varepsilon, t}$  is in case (2) for all  $\varepsilon \in (0, 1)$ .

Given that for any  $\varepsilon \in (0, 1)$ , no matter how small, there exists  $i$  with  $x'_{\varepsilon, t} \notin \Delta_-$  for all  $\varepsilon' \geq \varepsilon$ , and that the set of agents is finite,  $I_t^1 \cup I_t^2 \neq \emptyset$  for all  $t$ .

Suppose first that  $I_t^1 \neq \emptyset$  for infinitely many  $t$ . Again, since the set of agents is finite, we can wlog assume that there exists a subsequence with the property that  $I_t^1 = I^* \neq \emptyset$  for all  $t$  large enough. Select an agent  $i^* \in I^*$ . Then

$$1 < \sum_l x'_{\varepsilon, t, l} = \sum_l x_{tl}^{i^*} + \varepsilon(\sum_l x_l^{i^*'} - \sum_l x_l^{i^*})$$

for all  $\varepsilon$  means that  $\sum_l x_{tl}^{i^*} = 1$ , as  $x_{tl}^{i^*} \in \Delta_-$ . Since this is true for all  $t$  large enough and  $x_t \rightarrow x^*$ ,  $\sum_l x_l^{i^*} = 1$ . Now, we must have  $\sum_l x_l^{i^*'} - \sum_l x_l^{i^*} > 0$ , which implies that  $\sum_l x_l^{i^*'} > 1$ , contradicting that  $x'$  is an allocation.

Suppose now that  $I_t^1 = \emptyset$  for all but finitely many  $t$ . Then  $I_t^2 \neq \emptyset$  for infinitely many  $t$ . Again, since the set of agents is finite, we can wlog assume that there exists a subsequence with the property that  $I_t^2 = I^* \neq \emptyset$  for all  $t$  large enough. Select an agent  $i^* \in I^*$ . Using the finiteness of the number of objects, there exists  $l$  with the property that for all  $t$  large enough,

$$(\forall \varepsilon \in (0, 1))(x_{tl}^{i^*} + \varepsilon(x_l^{i^{*'}} - x_l^{*i^*}) = x_{\varepsilon, t, l}^{i^{*'}} < 0).$$

Given that  $x_t^{i^*} \in \Delta_-$ , this can only be true for all  $\varepsilon$  if  $x_{t,l}^{i^*} = 0$ , and  $x_l^{i^{*'}} - x_l^{*i^*} < 0$ . Now  $x_t \rightarrow x^*$  means that  $x_l^{*i^*} = 0$ , so we have  $x_l^{i^{*'}} < 0$ , which contradicts that  $x'$  is an allocation.

## 11. PROOF OF THE OBSERVATIONS ON PAGE 14

First we provide an example to the effect that two students may switch their places in a given school's priority list, and that as a result the student who went up in priority may end with a worse outcome.

Consider an example with three schools:  $\{s, s', s''\}$ , and three children:  $\{a, b, c\}$ . Suppose that school priorities, and children preferences are as follows.

$s$	$s'$	$s''$	$a$	$b$	$c$
$b$	$b$	$c$	$s'$	$s$	$s$
$c$	$c$	$a$	$s''$	$s'$	$s''$
$a$	$a$	$b$	$s$	$s''$	$s'$

Then the student-optimal stable matching is  $\mu(s) = b$ ,  $\mu(s') = a$  and  $\mu(s'') = c$ . On the other hand, if  $a$  and  $b$  switch roles in school  $s$ 's priority structure, then the student-optimal stable matching is  $\mu(s) = c$ ,  $\mu(s') = b$  and  $\mu(s'') = a$ .

Next we turn to the claim that it is computationally hard to decide whether a given student and school are matched. Suppose given a matching market  $(S, C, \succeq)$  in which  $\succeq$  may have indifferences. Then we prove that

**Proposition 1.** *Given  $s \in S$  and  $c \in C$  it is NP hard to decide whether there exists some tie breaking of  $\succeq$  for which  $s$  and  $c$  are matched at a stable matching.*

The proof is an adaptation of existing results in the computer science literature on stable matching. Specifically, we adapt the results in [Manlove, Irving, Iwama, Miyazaki, and Morita \(2002\)](#).

**Lemma 7.** *Maximum cardinality stable and Pareto optimal matching is NP hard.*

*Proof.* The proof amounts to verifying that the construction in the proof of Lemma 1 in [Manlove, Irving, Iwama, Miyazaki, and Morita \(2002\)](#) obtains a Pareto optimal matching for the market under consideration. We shall use the same notation as in the paper of Manlove et. al.

First observe that the construction does not pin down every agents' ranking. We shall here choose the same ranking for every agent in each instance of square brackets for the description of agents' preferences.

We construct a matching  $M'$  from  $M$ . For each edge  $\{m_i, w_j\}$  in  $M$ , let  $(m_i, w_j) \in M'$  and  $(m'_i, y_j) \in M'$  if  $j = j_i$ , and  $(m'_i, w_j) \in M'$  and  $(m_i, y_j) \in M'$  if  $j = k_i$ . This means that there is no possible Pareto improvement possible that involves the men  $m_i$  or  $m'_i$  for any  $i$  that are connected in the maximal matching  $M$ . For the remaining men  $m_q, m'_q$  we have  $(m'_q, y_q) \in M'$  (meaning that  $m'_q$  cannot be made better of), and  $(m_q, z_i)$  for  $z_i \in Z$  chosen in the (common) order of  $Z$ . This means that to make  $m_q$  better off there are two possibilities. In the first, he would need to obtain the partner of some other agent who has a higher ranked partner in  $Z$ , but this would make the relevant man worse off. In the second, he would be matched to a woman in  $\{y_q, w_{j_q}, w_{k_q}\}$ , but  $y_q$  has been assigned to  $m'_q$ , who ranks her first, and  $w_{j_q}$  and  $w_{k_q}$  who must be matched already (or  $M$  would not be maximal). Finally, consider the men in  $X$ . They all have the same ranking of the women in  $W$ , so it is impossible to obtain a Pareto improvement that involves only men in  $X$ . For an improvement that involves both men in  $M \cup M'$  and in  $X$ , note that only the men that are matched to a woman in  $Z$  can be made better off with a partner in  $W$ . But the women in  $Z$  are unacceptable to men in  $X$ . So no Pareto improvement is possible that involves men in  $X$ .  $\square$

**Lemma 8.** *Let  $\mu$  be a stable and efficient matching in a market  $(S, C, \succeq)$  with ties. Then there is a (common) tie-breaking order of  $S$  for which  $\mu$  is stable and efficient in the resulting market with strict preferences.*

*Proof.* Let  $\mu$  be a student-optimal stable matching in a market  $(S, C, \succeq)$  with indifferences in school rankings. For any  $s, s' \in S$  with  $\mu(c) = s$ ,  $s \sim_c s'$  and  $c \succ_{s'} \mu(s')$ , let  $s P s'$ . We claim that  $P$  is acyclic. Suppose, towards a contradiction, that  $s^1 P s^2 P \dots P s^K = s^1$ . Then  $s^k \sim_{\mu(s^k)} s^{k+1}$ . The matching  $\mu'$  that coincides with  $\mu$  everywhere except that  $\mu'(s^{k+}) = \mu(s^k)$  is also stable and Pareto dominates  $\mu$ ; a contradiction.

Since  $P$  is acyclic it has an extension to a linear order  $>$  on  $S$ . If we use  $>$  as a common tie breaker for  $\succeq_c$ , then by definition,  $\mu$  will be stable in the resulting market with strict preferences. The reason is that  $\mu$  is clearly individually rational and if  $c \succ_s \mu(s)$  then stability in  $(S, C, \succeq)$  means that  $\mu(c) \succeq_c s$ ; then either  $\mu(c) \succ_c s$ , or  $\mu(c) \sim_c s$  so that  $\mu(c) P s$ , and thus  $\mu(c) > s$ . Either way,  $(c, s)$  is not a blocking pair in the market after tie-braking.  $\square$

**Lemma 9.** *It is NP hard to decide whether a given student and school are matched in a Pareto efficient stable matching.*

*Proof.* Following [Manlove, Irving, Iwama, Miyazaki, and Morita \(2002\)](#) we focus on complete (meaning no agents are single) stable matchings. The proof is an adaptation of the proof of their Theorem 6.

Let  $(S, C, \succeq)$  be any market and consider a student-school pair  $(s, c)$ ,  $s \notin S$  and  $c \notin C$ . Consider the market  $(S \cup \{s\}, C \cup \{c\}, \succeq')$  in which every agent's preferences in  $S \cup C$  over agents in  $S \cup C$  remains the same, where  $s$  ranks  $c$  first and  $c$  ranks  $s$  last among acceptable partners in  $\succeq$ . Suppose that  $c$  is ranked last by all  $s' \in S$  and that  $s$ , together with all other  $s'$  who were unacceptable in  $\succeq$  is ranked first (in arbitrary strict order) by  $c'$  for all  $c' \in S$ .

Now we claim that there is a complete stable and Pareto efficient matching  $\mu$  of  $(S, C, \succeq)$  if and only if  $\mu \cup \{(s, c)\}$  is a complete, stable and Pareto efficient matching of  $(S \cup \{s\}, C \cup \{c\}, \succeq')$ . First, it is obvious that  $\mu$  is complete and individually rational if and only if  $\mu \cup \{(s, c)\}$  has these properties.

Second, suppose that  $\mu$  is stable but  $\mu \cup \{(s, c)\}$  is not. Then there must exist a blocking pair involving either  $s$  or  $c$ , as  $\mu$  was stable. It cannot involve  $s$ , who gets their first choice, so it must be  $c$ , however  $c$  is ranked last by any  $s' \in S$  and since  $\mu$  was complete,  $\mu(s') \in C$ . A contradiction. Obviously, if  $\mu \cup \{(s, c)\}$  then so is  $\mu$  in  $(S, C, \succeq)$  as the ranking among agents in  $S$  and  $C$  does not change when going from  $\succeq'$  to  $\succeq$ .

Finally, if  $\mu$  is Pareto optimal then so is  $\mu \cup \{(s, c)\}$  because  $c$  gets their first option and it is impossible to change their allocation without making them worse off; so any Pareto improvement would have to have been feasible in  $(S, C, \succeq)$ . Conversely, it is clear that if  $\mu \cup \{(s, c)\}$  then so is  $\mu$ , as any putative Pareto improvement in the smaller market would have been feasible in the larger.  $\square$

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