CONSTRAINED PSEUDO-MARKET EQUILIBRIUM

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Abstract. We propose a market solution to the problem of resource allocation subject to various constraints, such as those imposed by considerations of diversity or geographical distribution. Constraints are "priced," and agents are charged to the extent that their purchases affect the value (at equilibrium prices) of the relevant constraints. The result is a constrained-efficient market equilibrium outcome. The outcome is fair whenever the constraints do not single out individual agents, which happens, for example with geographical distribution constraints. In economies with endowments, moreover, our equilibrium outcomes are constrained efficient and approximately individually rational.

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Date: First version: September 2019. This version: February 2020.

We thank Eric Budish, Fuhito Kojima, Andy McLennan, Hervé Moulin, and Tayfun Sönmez for comments. Echenique thanks the National Science Foundation for its support through the grants SES-1558757 and CNS-1518941, and the Simons Institute at UC Berkeley for its hospitality while part of the paper was written. Miralles acknowledges financial support from the Spanish Ministry of Economy and Competitiveness, through the Severo Ochoa Programme for Centers of Excellence in R&D (SEV-2015-0563). Zhang acknowledges financial support from National Natural Science Foundation of China (Grant No. 71903093).

Contents

1. Introduction

We propose a pseudo-market mechanism for resource allocation under *constraints* when monetary transfers are forbidden. The mechanism dates back to Hylland and Zeckhauser (1979) who consider the allocation of a number of indivisible objects to an equal number of unit-demand agents. We consider more general environments and allow for various constraints on allocations. For example, we can solve the allocation of school seats to students subject to diversity constraints; or hospital positions to doctors subject to geographical restrictions; or courses to students when students face some minimal competency requirements. We can also solve the well-known roommate matching model, as well as general models of coalition formation. Given a set of constraints, and under standard continuity and convexity assumptions, our mechanism achieves a (constrained) efficient outcome in equilibrium, which is a random assignment that describes the probability that each agent obtains each resource. Moreover, when constraints do not single out any particular agent, the equilibrium is efficient and fair; and when agents start out with property rights over initial endowments, we can achieve efficiency and approximate individual rationality.

We differ from the literature in that we do not take a constraint structure as the primitive in our model. The influential work by Budish, Che, Kojima, and Milgrom (2013) and the recent development by Akbarpour and Nikzad (forthcoming) are concerned with the implementability of random assignments. The constraint structures they characterize build foundations for many applications but also rule out many others. From our perspective, any set of constraints characterizes a finite set of feasible ex-post assignments, and the convex hull of feasible ex-post assignments defines the set of implementable random assignments. We take the convex hull as the primitive. In this sense we can address "any" constraints that pin down a well-defined set of feasible ex-post assignments.

The key idea in our proposal is to *price constraints*. We use linear inequalities to characterize the "upper-right" boundary of the convex hull of feasible assignments, and all these inequalities take the form of ceiling constraints. The unit demand and capacity constraints in Hylland and Zeckhauser's model are examples of such inequalities. We price these inequalities and obtain personalized prices, so an agent faces prices for resources that are composed from the prices of the inequalities that involve them. For example, if there is a ceiling constraint on how many units of an object that can go to a group of agents, the agents in the group will pay the price of the constraint when they buy probability shares of the object. If those agents are also involved in other constraints and different agents are involved in different constraints, then the final personalized prices they face can be different. But if two agents are always involved in the same constraints, they will face equal prices. So equal budgets ensure that they will not envy each other. For another example, if there is a floor constraint on how many units of an object that should go to a group of agents, the floor constraint will be translated into ceiling constraints on how many units of the object that can go to the other agents and how many units of the other objects that can go to the group. The equilibrium prices of these ceiling constraints will ensure that, in equilibrium, the floor constraint is satisfied.

Our idea is perhaps familiar from the role of shadow prices in optimization with constraints, but the familiarity is deceptive. Imagine using the dual variables (or Lagrange multipliers) associated with each constraint in order to decentralize an allocation that is constrained efficient. We run into two issues. One is that some constraints may impose a lower bound on consumed quantities, which would lead to negative prices. The other is that decentralizing a constrained efficient allocation would require transfers, as in the second welfare theorem.[1] With transfers there is little hope to achieve an outcome that is fair, or individually rational when there are endowments. Our approach avoids these issues. We simplify the problem by working with a subset of the constraints (the upper-right boundary), and ensure that they have positive prices. Individual rationality can be ensured when agents have endowments. And, as long as the constraints do not themselves induce unfairness by treating agents differently, it is possible to obtain a fair outcome in equilibrium.

We present a general result on the existence of a market equilibrium with desirable properties, and develop several applications. The first application is to the object allocation model under ceiling and floor constraints as described by Budish, Che, Kojima, and Milgrom (2013). Our result delivers a constrained version of the Hylland-Zeckhauser equilibrium, where (1) each agent gets a fixed exogenous budget and chooses an optimal probabilistic assignment given equilibrium prices, (2) markets clear, and (3) the resulting allocation satisfies the desired constraints. In particular, we derive the inequalities that we price in the hospital-doctor matching

[1]We discuss the second welfare theorem without transfers developed by Miralles and Pycia (2020) in the related literature section (Section 8). Note that the outcome of the second welfare theorem may induce envy, even in a completely standard economy with no constraints other than the total availability of goods.

with geographically distributional constraints and in the controlled school choice with type-specific quotas, and show that in equilibrium there is no envy among doctors and no envy among students of equal types.

The second application is to the allocation of objects that agents regard as “bads”. These objects represent duties or tasks that agents dislike, but where a minimum number of duties/tasks need to be fulfilled. The problem is different from the usual object allocation model in that agents’ bliss point is zero consumption. Borrowing the idea of describing labor supply as consumption of leisure, we endow agents with bads and let them buy the option of not consuming bads. Then the floor constraints on the supply of bads become ceiling constraints on the supply of such artificial “goods”. On the other hand, the unit demand constraints become floor constraints on the consumption of goods, and our method is used to deal with the floor constraints.

A third application is to the roommate problem, which is usually presented as an example in which stable matchings do not exist. We consider the problem as one of resource allocation with constraints, and show that a market for roommates has a Hylland-Zeckhauser equilibrium. Our result implies the existence of an outcome that is stable in the sense that no individual agent wishes to deviate from the partnerships that they buy into at market prices. We also discuss how to solve more general models of coalition formation, and the allocation of objects in bundles.

Our last application concerns economies with endowments. As the result of some pre-existing initial allocation, each agent is endowed with some objects. The initial allocation satisfies the desired constraints. Our main result implies the existence of an equilibrium where agents’ incomes are not exogenous, and are instead *partially* obtained from selling their endowments at equilibrium prices. Hylland and Zeckhauser (1979) have shown that an equilibrium with full endogenous incomes may not exist, but we show that it is possible to let part of incomes be endogenous. As a consequence, we achieve an equilibrium that is approximately individually rational. We apply the result to the time exchange problem in which agents are willing to serve the others in exchange for the others’ services. The price of a service reveals how much the agents collectively value it, and any agent who can provide the service is rewarded through income.

Related literature. Constrained resource allocation has received a lot of attention in recent years. The work by Kojima, Sun, and Yu (2018), Gul, Pesendorfer, and

Zhang (2019) and ours seems to be the first to look at constrained allocation by way of a market mechanism. The former two papers study the role of gross substitutes in a general model of discrete allocation. Despite a similar focus on markets and constraints, the results in our papers are very different; see Section 8 for more details. In studying constraints, we are motivated by the early work of Budish, Che, Kojima, and Milgrom (2013), Ehlers, Hafalir, Yenmez, and Yildirim (2014) and Kamada and Kojima (2015). Our results apply to the same kind of constraints they consider, but we differ substantially in methodology.

Aside from how we deal with constraints, our approach to generating income from endowments is closely related to, but distinct from, Mas-Colell (1992), Le (2017), and McLennan (2018). We provide a detailed comparison in Section 8. We also provide a detailed comparison with other work of the pseudo-market mechanism in that section.

2. The model

2.1. Notational conventions.

For vectors $x, y \in \mathbf{R}^n$, $x \leq y$ means that $x_i \leq y_i$ for all $i = 1, \ldots, n$; $x < y$ means that $x \leq y$ and $x \neq y$; and $x \ll y$ means that $x_i < y_i$ for all $i = 1, \ldots, n$. The set of all $x \in \mathbf{R}^n$ with $0 \leq x$ is denoted by $\mathbf{R}^n_+$, and the set of all $x \in \mathbf{R}^n$ with $0 \ll x$ is denoted by $\mathbf{R}^n_{++}$. Inner products are denoted as $x \cdot y = \sum_i x_i y_i$.

Let $X \subseteq \mathbf{R}^n$ be convex. A function $u : X \to \mathbf{R}$ is

- *quasi-concave* if for any $x, z \in X$ and $\lambda \in [0, 1]$,
$$\min\{u(z), u(x)\} \leq u(\lambda z + (1 - \lambda)x).$$
- *semi-strictly quasi-concave* if it is quasi-concave, and for any $x, z \in X$ and $\lambda \in (0, 1)$, $u(z) \neq u(x)$ implies that
$$\min\{u(z), u(x)\} < u(\lambda z + (1 - \lambda)x).$$
- *concave* if, for any $x, z \in X$ and $\lambda \in (0, 1)$,
$$\lambda u(z) + (1 - \lambda)u(x) \leq u(\lambda z + (1 - \lambda)x).$$
- *expected utility* if there exists a vector $v \in \mathbf{R}^n$ with $u(x) = v \cdot x$ for all $x \in X$.
- C^1 if it can be extended to a continuously differentiable function defined on an open set that contains X.
- *strictly increasing* if $x > x'$ implies that $u(x) > u(x')$.

Given a set $A \subseteq \mathbf{R}^n$, let $\text{co}(A)$ denote the *convex hull* of A in $\mathbf{R}^n$: the intersection of all convex sets that contain A.

A pair (a, b), with $a \in \mathbf{R}^n$ and $b \in \mathbf{R}$ defines a *linear inequality* $a \cdot x \leq b$. We say that a linear inequality (a, b) has *non-negative coefficients* if $a \geq 0$. Any linear inequality (a, b) defines a (closed) *half-space* $\{x \in \mathbf{R}^n : a \cdot x \leq b\}$.

A *polyhedron* in $\mathbf{R}^n$ is a set that is the intersection of a finite number of closed half-spaces. A *polytope* in $\mathbf{R}^n$ is a bounded polyhedron. Two special polytopes are the *simplex* in $\mathbf{R}^n$:

$$\Delta^n = \{x \in \mathbf{R}^L_+ : \sum_{l=1}^{L} x_l = 1\},$$

and the *subsimplex*

$$\Delta^n_- = \{x \in \mathbf{R}^L_+ : \sum_{l=1}^{L} x_l \leq 1\}.$$

When n is understood, we use the notation Δ and Δ_-.

2.2. **The economy.** An *economy* is a tuple $\Gamma = (I, O, (Z_i, u_i)_{i \in I}, (q_l)_{l \in O})$, where

- I is a finite set of *agents*;
- O is a finite set of *objects*, with $L = |O|$;
- $Z_i \subseteq \mathbf{R}^L_+$ is i's *consumption space*;
- $u_i : Z_i \to \mathbf{R}$ is i's *utility function*;
- $q_l \in \mathbf{R}_{++}$ is the amount of $l \in O$.

Let $L = |O|$ be the number of *object types*, and $N = |I|$ be the number of agents. Denote the set of objects by $O = \{1, 2, \ldots, L\}$.

An *assignment* in Γ is a vector

$$x = (x_{i,l})_{i \in I, l \in O} \text{ with } x_i \in Z_i,$$

where $x_{i,l}$ is the amount (or probability share) of object l received by agent i. Sometimes we interpret an assignment x as a matrix with a row for each agent and a column for each object. Let $\mathcal{A}$ denote the set of all assignments in Γ.

For now we restrict attention to $Z_i = \mathbf{R}^L_+$, but the consumption space will be restricted further as we introduce constraints.

2.3. **Constraints.** In many market design applications, objects are indivisible and randomization over deterministic assignments is used to ensure fairness. In contrast, we say that assignment x is *deterministic* if every $x_{i,l}$ is an integer. When an assignment is not deterministic we some times call it a random assignment, to emphasize that its entries may not be integer. It is, however, possible to apply our model to the assignment of infinitely divisible goods.

Exogenous constraints are often imposed on deterministic assignments. For example, the usual *unit-demand constraints* require that $\sum_{l\in O} x_{i,l} \leq 1$ for all $i \in I$, and the *supply constraints* require that $\sum_{i\in I} x_{i,l} \leq q_l$ for all $l \in O$. Floor constraints may be used to capture certain distributional objectives. For example, one may want a certain minimum number of doctors to be assigned to rural areas, or a lower bound on the number minority students that are assigned to a particular school; or that all students take at least two math courses. A deterministic assignment is feasible if it satisfies all exogenous constraints. An (random) assignment is *feasible* if it belongs to the convex hull of feasible deterministic assignments. The convex hull is a polytope since the number of feasible deterministic assignments is usually bounded, and therefore finite.

In this paper, we do not start from an explicit model of constraints. Instead, we introduce constraints implicitly through a primitive nonempty set $\mathcal{C} \subseteq \mathcal{A}$, and call all elements of $\mathcal{C}$ *feasible assignments*.[2] We require that $\mathcal{C}$ be a polytope. A *constrained allocation problem* is a pair $(\Gamma, \mathcal{C})$ in which Γ is an economy and $\mathcal{C}$ is the set of feasible assignments in Γ. Throughout the paper we discuss several special cases of constrained allocation problems, including the roommate problem, the Hylland-Zeckhauser model, and a model of "bads."

2.4. **Pre-processing of constraints.** Define the **lower contour set** of $\mathcal{C}$ to be

$$\text{lcs}(\mathcal{C}) = \{x \in \mathbf{R}_+^{NL} : \exists x' \in \mathcal{C} \text{ such that } x \leq x'\}.$$

Lemma 1. *There exists a finite set Ω of linear inequalities with non-negative coefficients such that*[3]

$$lcs(\mathcal{C}) = \bigcap_{(a,b)\in\Omega} \{x \in \mathbf{R}_+^{LN} : a \cdot x \leq b\}.$$

Proof. Consider

$$D = \{x' \in \mathbf{R}^{NL} : x' \leq x \text{ for some } x \in \mathcal{C}\}$$

and note that $\text{lcs}(\mathcal{C}) = D \cap \mathbf{R}_+^{NL}$. Write D as $\mathcal{C} - \mathbf{R}_+^{NL}$; thus, since $\mathcal{C}$ is a polytope, D is finitely generated. Then by Theorem 19.1 in Rockafellar (1970) D is polyhedral, and therefore the intersection of finitely many halfspaces. Let Ω be the set of linear

[2] We apply our results to a model with constraint structures as primitives in Section 4.1.

[3] Lemma 1 is used by Balbuzanov (2019) to define a generalization of the probabilistic serial mechanism that accommodates constraints.

inequalities (a, b) defining this collection of halfspaces, so for each $(a, b) \in \Omega$ we have the halfspace $\{x' \in \mathbf{R}^{NL} : a \cdot x' \leq b\}$. Since for each i and l there is $x' \in D$ with arbitrarily small $x'_{i,l}$, we must have $a \geq 0$. Hence Ω defines a finite collection of linear inequalities with non-negative coefficients.

To finish the proof, note that if $z \in D \setminus \text{lcs}(\mathcal{C})$ then $z \notin \mathbf{R}^{NL}_{+}$. □

For any $c = (a, b) \in \Omega$, define

$$\text{supp}(c) = \{(i, l) \in I \times O : a_{i,l} > 0\}.$$

Define $a_i = (a_{i,l})_{l \in O}$ to be the vector of coefficients relevant to i in (a, b).

Given that $\mathcal{C}$ is nonempty, there are two types of inequalities $(a, b) \in \Omega$: those with $b = 0$ and those with $b > 0$. If $b = 0$, then for any $x \in \mathcal{C}$ we must have $x_{i,l} = 0$ for all $(i, l) \in \text{supp}(c)$. We can, without loss of generality, assume that there is exactly one such inequality because if $(a, 0), (a', 0) \in \Omega$ then these can be substituted by $((\max\{a_{i,l}, a'_{i,l}\}), 0)$, and if there is no inequality with $b = 0$ in Ω then we can include the trivial inequality $(0, 0)$ in Ω. Let $(a^0, 0) \in \Omega$ be this unique inequality. When $(a^0, 0)$ is nontrivial, it forbids some agents from consuming certain objects. We say that l is a *forbidden object* for agent i when $a^0_{i,l} > 0$.

Among the remaining inequalities $\Omega \backslash \{(a^0, 0)\}$, we say (a, b) is an individual constraint for agent i if for all $j \neq i$ and $l \in O$, $a_{j,l} = 0$. In words, (a, b) only restricts i's consumption. Let Ω^i denote the set of all individual constraints for i. We use $(a^0, 0)$ and individual constraints to refine i's consumption space. Let $\mathcal{X}_i$ be the set of vectors $x_i \in Z_i$ such that $x_{i,l} = 0$ if l is a forbidden object for i and x_i satisfies all of i's individual constraints. That is,

$$\mathcal{X}_i = \{x_i \in \mathbf{R}^L_+ : a^0_i \cdot x_i \leq 0 \text{ and } a_i \cdot x_i \leq b \text{ for all } (a, b) \in \Omega^i\}.$$

Let $\Omega^* = \Omega \backslash \big(\{(a^0, 0)\} \bigcup \cup_{i \in I} \Omega^i\big)$ collect the remaining inequalities. The elements of Ω^* will be "priced."

2.5. **Normative properties.** Given a constrained allocation problem $(\Gamma, \mathcal{C})$, we discuss the efficiency and fairness that can be achieved subject to how allocations are constrained. A feasible allocation $x \in \mathcal{C}$ is *weakly $\mathcal{C}$-constrained Pareto efficient* if there is no feasible assignment $y \in \mathcal{C}$ such that $u_i(y_i) > u_i(x_i)$ for all i. Moreover, $x \in \mathcal{C}$ is *$\mathcal{C}$-constrained Pareto efficient* if there is no feasible assignment $y \in \mathcal{C}$ such that $u_i(y_i) \geq u_i(x_i)$ for all i with at least one strict inequality for one agent.

Fairness rules out envy among agents who are treated symmetrically by the primitive constraints. We say that two agents i and j are of *equal type* if $\mathcal{X}_i = \mathcal{X}_j$ and, for all $(a, b) \in \Omega^*$, $a_i = a_j$. We say an assignment x is *envy-free* if, for every two agents i and j, $u_i(x_i) \geq u_i(x_j)$ and $u_j(x_j) \geq u_j(x_i)$. We say x is *equal-type envy-free* if envy-freeness holds for every two agents of equal type.

2.6. **Equilibrium.** For each $c = (a, b) \in \Omega^*$, we introduce a price p_c. Given a price vector $p = (p_c)_{c \in \Omega^*} \in \mathbf{R}^{\Omega^*}$, the personalized price vector faced by any agent $i \in I$ is defined to be $p_i = (p_{i,l})_{l=1}^L$ such that

$$p_{i,l} = \sum_{(a,b) \in \Omega^*} a_{i,l} p_{(a,b)}.$$

A pair (x^*, p^*) is a *pseudo-market equilibrium* for $(\Gamma, \mathcal{C})$ if

(1) $x_i^* \in \arg\max_{x_i \in \mathcal{X}_i} \{u_i(x_i) : p_i^* \cdot x_i \leq 1\}$.
(2) $x^* \in \mathcal{C}$.
(3) For any $c = (a, b) \in \Omega^*$, $\sum_{(i,l)} a_{i,l} x_{i,l}^* < b$ implies that $p_c^* = 0$.

3. Main theorem

Our main result is the following theorem.

Theorem 1. *Suppose that agents' utility functions are continuous, quasi-concave and strictly increasing.*

- *There exists a pseudo-market equilibrium (x^*, p^*) in which x^* is weakly $\mathcal{C}$-constrained Pareto efficient.*
- *If agents' utility functions are semi-strictly quasi-concave, there exists a pseudo-market equilibrium (x^*, p^*) in which x^* is $\mathcal{C}$-constrained Pareto efficient.*
- *Every pseudo-market equilibrium assignment is equal-type envy-free.*

Theorem 1 is implied by Theorem 2 in Section 7. It generalizes ideas in Hylland and Zeckhauser (1979) to constrained allocation problems through the novel procedure of pricing constraints. We should emphasize that the first welfare theorem does not hold in our model: one can exhibit examples of Pareto inefficient pesudo-market equilibria, and even of Pareto-ranked equilibrium allocations. Crucial to Theorem 1 is the cheapest bundle property: A pseudo-market equilibrium (x, p) satisfies the *cheapest-bundle property* if, for each i, x_i minimizes expenditure $p_i \cdot z_i$ among all the $z_i \in \mathcal{X}_i$ for which $u_i(z_i) = u_i(x_i)$. The notion of a cheapest bundle, and its role in obtaining efficiency, was already established by Hylland and Zeckhauser (1979).

From our proof it is easy to see that any pseudo-market equilibrium satisfying the cheapest-bundle property is $\mathcal{C}$-constrained Pareto efficient.

4. Object allocation under constraints

4.1. **Constraint structure.** Our first application is the problem of allocating indivisible objects fairly and subject to constraints. Each object $l \in O$ is available in fixed integer supply, and each agent demands at most one copy of any object. We explicitly describe constraints of the form introduced by Budish, Che, Kojima, and Milgrom (2013). Many constraints in real-life problems are of such form.

A *constraint* is defined by a tuple $(S, \underline{q}_S, \overline{q}_S)$ where $S \subset I \times O$ is a subset of agent-object pairs, and $(\underline{q}_S, \overline{q}_S)$ is a pair of non-negative integers with $\underline{q}_S < \overline{q}_S$. The integers $\underline{q}_S$ and $\overline{q}_S$ are respectively called the *floor quota* and the *ceiling quota*. So we require that $\overline{q}_S \geq 1$. Let $q_S = (\underline{q}_S, \overline{q}_S)$. An assignment x satisfies $(S, \underline{q}_S, \overline{q}_S)$ if

$$\underline{q}_S \leq \sum_{(i,l)\in S} x_{i,l} \leq \overline{q}_S. \tag{1}$$

For any $i \in I$ and $l \in O$, a *singleton constraint* (S, q_S) is such that $S = \{(i,l)\}$ and $q_S = (0,1)$. This means that i can obtain at most one copy of l. For any $i \in I$, a *row constraint* (S, q_S) is such that $S = \{i\} \times O$ and $q_S = (0, q_i)$ where $q_i \in \mathbb{N}$. This means that i obtains at most q_i objects. Unit demand is an example of a row constraint. For any $l \in O$, a *column constraint* (S, q_S) is such that $S = I \times \{l\}$ and $q_S = (0, q_l)$. This means that at most q_l copies of l can be assigned.

A *constraint structure* $\mathcal{H}$ is a collection of constraints. The set of feasible assignments implied by $\mathcal{H}$ is defined to be

$$\mathcal{C} = \{x \in \mathbf{R}_+^{NL} : \underline{q}_S \leq \sum_{(i,l)\in S} x_{i,l} \leq \overline{q}_S \text{ for all } (S, q_S) \in \mathcal{H}\}.$$

We assume that $\mathcal{H}$ contains the singleton constraints for all agent-object pairs, the row constrains for all agents and the column constraints for all objects. Moreover, we assume that there exists $x \in \mathcal{C}$ with $\underline{q}_S < \sum_{(i,o)\in S} x_{i,o} < \overline{q}_S$ for all $S \in \mathcal{H}$. When $\mathcal{H}$ satisfies these assumptions, we say that it is *allocative*.

When $\mathcal{H}$ is arbitrary, $\mathcal{C}$ may not be equal to the convex hull of deterministic assignments satisfying the constraints in $\mathcal{H}$. In other words, the assignments in $\mathcal{C}$ may not be *implementable*. In that case, to use our method we first need to derive the convex hull, which is the (real) set of feasible assignments.

Budish, Che, Kojima, and Milgrom (2013) prove that a sufficient and necessary condition for $\mathcal{C}$ to be implementable is that $\mathcal{H}$ is a *bihierarchy*. Formally, a constraint structure $\mathcal{H}$ is a *hierarchy* if for every distinct S and S' in $\mathcal{H}$, either $S \subset S'$, or $S' \subset S$, or $S \cap S' = \emptyset$. $\mathcal{H}$ is a bihierarchy if there exist two hierarchies $\mathcal{H}_1$ and $\mathcal{H}_2$ such that $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$ and $\mathcal{H}_1 \cap \mathcal{H}_2 = \emptyset$. $\mathcal{H}_1$ is the set of sub-row, row, and sup-row constraints, while $\mathcal{H}_2$ is the set of sub-column, column and sup-column constraints.[4] When $\mathcal{H}$ is a bihierarchy, $\mathcal{C}$ is the set of feasible assignments that we take as a primitive in our model. Then we can apply our method directly to $\mathcal{C}$.

Below we restrict attention to the situation that $\mathcal{H}$ is a bihierarchy and discuss two cases. In the first case, all floor quotas are zero. Then we can directly price the constraints in $\mathcal{H}$. In the second case there are nontrivial floor constraints. To characterize $\text{lcs}(\mathcal{C})$, we will derive a new set of ceiling constraints implied by existing ceiling and floor constraints. We present two examples to show how the new ceiling constraints are derived. The general case is much like the two examples.

4.2. **No floor constraints.** Suppose that all floor quotas are zero. Then $\mathcal{C} = \text{lcs}(\mathcal{C})$ and we can directly price non-individual constraints in $\mathcal{H}$, the set of which is denoted by $\mathcal{H}^*$. Here individual constraints consist of singleton, sub-row, and row constraints.

Now two agents i and j are of *equal type* if $\mathcal{X}_i = \mathcal{X}_j$ and, for all $S \in \mathcal{H}^*$ and $l \in O$, $(i, l) \in S$ if and only if $(j, l) \in S$. We say $\mathcal{H}$ is *anonymous* if every two agents are of equal type. If $\mathcal{H}$ is anonymous, every constraint in $\mathcal{H}^*$ must be a column or sup-column constraint. Under anonymous constraints, every pseudo-market equilibrium is envy-free. An example of anonymous constraints is the Japanese medical residency match with regional caps studied by Kamada and Kojima (2015). Suppose that agents are doctors and objects are hospital positions. Each constraint takes the form

$$\big(I \times O', (0, \bar{q}_{O'})\big)$$

where $O' \subseteq O$ is the set of hospitals in a geographic region (a city or a prefecture). Here $\bar{q}_{O'}$ is the regional cap used to control the maximum number of doctors that the region O' can employ. A collection of such constraints is anonymous because each constraint does not distinguish among the identities of individual doctors.

4.3. **Floor constraints.** Floor constraints appear widely in applications. We discuss two examples. The first is the Japanese medical residency match mentioned

[4] A constraints (S, q_S) is a sub-row constraint if $S = \{i\} \times O'$ for some $i \in I$ and $O' \subset O$, and it is a sup-row constraint if $S = I' \times O$ for some $I' \subset I$. Sub-column and sup-column constraints are similarly defined.

above. By introducing regional caps to restrict the number of doctors assigned to urban hospitals, the Japanese government wants to increase the number of doctors assigned to rural hospitals. The government's ideal distribution of doctors can be described by a collection of constraints with both floor and ceiling quotas.

Specifically, the hospitals O are located in K disjoint regions. Accordingly, there is a partition of hospitals $O = R_1 \cup R_2 \cup \cdots \cup R_K$ such that every R_k is the set of hospitals in a region. So we simply refer to R_k as a region. For each region R_k, there is a constraint

$$\underline{q}_{R_k} \leq \sum_{l \in R_k} \sum_{i \in I} x_{i,l} \leq \overline{q}_{R_k}.$$

We assume that there are enough hospital positions because we can always add null hospitals. Below we derive the inequalities in Ω and show that they are anonymous. So every pseudo-market equilibrium is envy-free.

Let $\mathcal{R} = \{R_1, R_2, \ldots, R_K\}$ denote the set of regions. For each $\ell \in \{1, 2, \ldots, K\}$, let $\mathcal{R}_\ell$ be the set of sets each of which is the union of ℓ distinct regions. That is,

$$\mathcal{R}_\ell = \{R_{k_1} \cup R_{k_2} \cup \cdots \cup R_{k_\ell} : \{k_1, k_2, \ldots, k_\ell\} \subset \{1, 2, \ldots, K\}\}.$$

So $\mathcal{R}_1 = \mathcal{R}$.

Consider the following inequalities

$$(2) \qquad \begin{cases} 0 \leq \sum_{l \in O} x_{i,l} \leq 1 & \text{for all } i \in I, \\ 0 \leq \sum_{i \in I} x_{i,l} \leq q_l & \text{for all } l \in O, \\ 0 \leq \sum_{i \in I, l \in R} x_{i,l} \leq \overline{q}_R & \text{for all } \ell \in \{1, \ldots, K\} \text{ and } R \in \mathcal{R}_\ell, \end{cases}$$

where $\overline{q}_R$ is (re)defined according to the following procedure:

- For every $R \in \mathcal{R}_1$, redefine the ceiling quota to be

$$\overline{q}_R = \min \Big\{ \overline{q}_R, N - \sum_{R' \in \mathcal{R} \setminus \{R\}} \underline{q}_{R'} \Big\}.$$

  Note that $\overline{q}_R \geq \underline{q}_R$ because $N \geq \sum_{R' \in \mathcal{R}} \underline{q}_{R'}$, and $\overline{q}_R$ is weakly smaller than the original ceiling quota.

- For every $R = R_{k_1} \cup R_{k_2} \in \mathcal{R}_2$, define the ceiling quota to be

$$\overline{q}_R = \min \Big\{ \overline{q}_{R_{k_1}} + \overline{q}_{R_{k_2}}, N - \sum_{R' \in \mathcal{R} \setminus \{R_{k_1}, R_{k_2}\}} \underline{q}_{R'} \Big\}.$$

- In general, for every $R = R_{k_1} \cup R_{k_2} \cup \cdots \cup R_{k_\ell} \in \mathcal{R}_\ell$, define the ceiling quota to be

$$\overline{q}_R = \min\big\{\overline{q}_{R\setminus\{R_{k_x}\}} + \overline{q}_{R_{k_x}} \text{ for every } x \in \{1,2,\ldots,\ell\}, N - \sum_{R'\in\mathcal{R}\setminus\{R_{k_1},\ldots,R_{k_\ell}\}} \underline{q}_{R'}\big\}.$$

We prove that lcs($\mathcal{C}$) is characterized by the inequalities in (2).

Proposition 1. *lcs*$(\mathcal{C}) = \{x \in \mathbf{R}_+^{NL} : x$ *satifsies the inequalities in (2)*$\}$.

Proof. We denote by A the set characterized by the inequalities in (2). It is easy to see that $A = \text{lcs}(A)$. By the procedure to define $\overline{q}_R$, all elements of $\mathcal{C}$ satisfy (2). So $\mathcal{C} \subset A$ and thus lcs$(\mathcal{C}) \subset A$. To prove that $A \subset \text{lcs}(\mathcal{C})$, we first prove a claim.

Claim. *For every* $\ell \in \{2,\ldots,K\}$, *every* $R = R_{k_1} \cup R_{k_2} \cup \cdots \cup R_{k_\ell} \in \mathcal{R}_\ell$, *and every* $x \in \{1,\ldots,\ell\}$,

$$\overline{q}_R \geq \underline{q}_{R_{k_x}} + \overline{q}_{R\setminus\{R_{k_x}\}}.$$

Proof of Claim. Base case $\ell = 2$: For every $R = R_{k_1} \cup R_{k_2} \in \mathcal{R}_2$, if $\overline{q}_R = \overline{q}_{R_{k_1}} + \overline{q}_{R_{k_2}}$, then the claim holds obviously. Otherwise, $\overline{q}_R = N - \sum_{R'\in\mathcal{R}\setminus\{R_{k_1},R_{k_2}\}} \underline{q}_{R'}$. By definition, $N - \sum_{R'\in\mathcal{R}\setminus\{R_{k_1}\}} \underline{q}_{R'} \geq \overline{q}_{R_{k_1}}$. So $\overline{q}_R = N - \sum_{R'\in\mathcal{R}\setminus\{R_{k_1},R_{k_2}\}} \underline{q}_{R'} \geq \overline{q}_{R_{k_1}} + \sum_{R'\in\mathcal{R}\setminus\{R_{k_1}\}} \underline{q}_{R'} - \sum_{R'\in\mathcal{R}\setminus\{R_{k_1},R_{k_2}\}} \underline{q}_{R'} = \overline{q}_{R_{k_1}} + \underline{q}_{R_{k_2}}$. Similarly, we prove that $\overline{q}_R \geq \underline{q}_{R_{k_1}} + \overline{q}_{R_{k_2}}$.

Induction step: Suppose the claim is true for $1,2,\ldots,\ell$. Then we prove that it is also true for $\ell+1$. For any $R = R_{k_1} \cup R_{k_2} \cup \cdots \cup R_{k_{\ell+1}} \in \mathcal{R}_{\ell+1}$, if $\overline{q}_R = \overline{q}_{R\setminus\{R_{k_x}\}} + \overline{q}_{R_{k_x}}$ for some $x \in \{1,2,\ldots,\ell+1\}$, then it is obvious that $\overline{q}_R \geq \overline{q}_{R\setminus\{R_{k_x}\}} + \underline{q}_{R_{k_x}}$. By the induction assumption, for every $y \neq x$, $\overline{q}_{R\setminus\{R_{k_x}\}} \geq \underline{q}_{R_{k_y}} + \overline{q}_{R\setminus\{R_{k_x},R_{k_y}\}}$. So $\overline{q}_R \geq \underline{q}_{R_{k_y}} + \overline{q}_{R\setminus\{R_{k_x},R_{k_y}\}} + \overline{q}_{R_{k_x}} \geq \underline{q}_{R_{k_y}} + \overline{q}_{R\setminus\{R_{k_y}\}}$. The claim is proved.

Otherwise, $\overline{q}_R = N - \sum_{R'\in\mathcal{R}\setminus\{R_{k_1},\ldots,R_{k_{\ell+1}}\}} \underline{q}_{R'}$. By definition, for every x, $N - \sum_{R'\in\mathcal{R}\setminus\{R_{k_1},\ldots,R_{k_{\ell+1}}\}\cup\{R_{k_x}\}} \underline{q}_{R'} \geq \overline{q}_{R\setminus\{R_{k_x}\}}$. So $\overline{q}_R \geq \overline{q}_{R\setminus\{R_{k_x}\}} + \underline{q}_{R_{k_x}}$.

So by induction, we prove Claim.

Define $A' = \{x \in A : \nexists x' \in A \text{ such that } x < x'\}$. It is clear that $A = \text{lcs}(A')$. We prove that $A' \subset \mathcal{C}$. Suppose there exists $x \in A'$ such that $x \notin \mathcal{C}$. Because x satisfies all original ceiling constraints that define $\mathcal{C}$, x must violate the floor constraint of some R_k. That is, $\sum_{i\in I, l\in R_k} x_{i,l} < \underline{q}_{R_k}$. Then there must exist some doctor i such that $\sum_{l\in O} x_{i,l} < 1$, since otherwise $\sum_{i\in I,l\in O\setminus\{R_k\}} x_{i,l} = N - \sum_{i\in I,l\in R_k} x_{i,l} > N - \underline{q}_{R_k} \geq \overline{q}_{O\setminus\{R_k\}}$, which contradicts the assumption that $x \in A$. Because $\underline{q}_{R_k} \leq \sum_{l\in R_k} q_l$, there must exist $l \in R_k$ such that $\sum_{i\in I} x_{i,l} < q_l$. Now consider a new assignment x' such that $x'_{i,l} = x_{i,l} + \epsilon$ where $0 < \epsilon < \min\{1 - \sum_{l\in O} x_{i,l}, q_l - \sum_{i\in I} x_{i,l}, \underline{q}_{R_k} - \sum_{i\in I, l\in R_k} x_{i,l}\}$, and x' coincides with x in the other cells. So $x < x'$. Below we prove that $x' \in A$, which contradicts the assumption that $x \in A'$.

Suppose towards a contradiction that $x' \notin A$. Let $\ell > 1$ be the smallest index such that there exists $R \in \mathcal{R}_\ell$ with $\sum_{i\in I, l\in R} x'_{i,l} > \overline{q}_R$. It is clear that $R_k \subset R$. By Claim, $\overline{q}_R \geq \underline{q}_{R_k} + \overline{q}_{R\setminus R_k}$. So

$$\sum_{i\in I, l\in R} x'_{i,l} > \underline{q}_{R_k} + \overline{q}_{R\setminus R_k}.$$

Because ϵ is chosen such that $\sum_{i\in I, l\in R_k} x'_{i,l} < \underline{q}_{R_k}$. So

$$\sum_{i\in I, l\in R\setminus R_k} x'_{i,l} > \overline{q}_{R\setminus R_k}.$$

But it means that $\sum_{i\in I, l\in R\setminus R_k} x_{i,l} > \overline{q}_{R\setminus R_k}$, which contradicts $x \in A$. So $x' \in A$. □

The second application we discuss is controlled school choice. When implementing school choice, a consideration for many school districts is demographic diversity. We present a model in which the students I are simply classified into minorities I^m and majorities I^M. Let the number of minorities be N^m and the number of majorities be N^M. Each school l has a pair of quotas $(\overline{q}_l^m, \underline{q}_l^m)$ for minorities, and a pair of quotas $(\overline{q}_l^M, \underline{q}_l^M)$ for majorities. So besides the capacity constraint, each school l has the constraints

$$\underline{q}_l^m \leq \sum_{i\in I^m} x_{i,l} \leq \overline{q}_l^m,$$
$$\underline{q}_l^M \leq \sum_{i\in I^m} x_{i,l} \leq \overline{q}_l^M.$$

Of course, we assume that $\underline{q}_l^m + \underline{q}_l^M \leq q_l$.

The inequalities to characterize $\text{lcs}(\mathcal{C})$ can be derived similarly as above. The only difference is that we need to take into account of the interaction between the quotas for the two student types within each school. After that we can deal with the assignments for two types separately.

Formally, consider the following inequalities

$$\begin{cases} 0 \leq \sum_{l \in O} x_{i,l} \leq 1 & \text{for all } i \in I, \\ 0 \leq \sum_{i \in I} x_{i,l} \leq q_l & \text{for all } l \in O, \\ 0 \leq \sum_{i \in I^m, l \in O'} x_{i,l} \leq \overline{q}^m_{O'} & \text{for all nonempty } O' \subset O, \\ 0 \leq \sum_{i \in I^M, l \in O'} x_{i,l} \leq \overline{q}^M_{O'} & \text{for all nonempty } O' \subset O, \end{cases} \tag{3}$$

where $\overline{q}^m_{O'}$ and $\overline{q}^M_{O'}$ are (re)defined as follows:

- For every $l \in O$, redefine the ceiling quotas to be
$$\overline{q}^m_l = \min\{\overline{q}^m_l, q_l - \underline{q}^M_l, N^m - \sum_{l' \in O \setminus \{l\}} \underline{q}^m_{l'}\},$$
$$\overline{q}^M_l = \min\{\overline{q}^M_l, q_l - \underline{q}^m_l, N^M - \sum_{l' \in O \setminus \{l\}} \underline{q}^M_{l'}\}.$$
- For every non-singleton $O' \subset O$, define the ceiling quotas to be
$$\overline{q}^m_{O'} = \min\{\overline{q}^m_{O' \setminus \{l\}} + \overline{q}^m_l \text{ for every } l \in O', N^m - \sum_{l' \in O \setminus O'} \underline{q}^m_{l'}\},$$
$$\overline{q}^M_{O'} = \min\{\overline{q}^M_{O' \setminus \{l\}} + \overline{q}^M_l \text{ for every } l \in O', N^M - \sum_{l' \in O \setminus O'} \underline{q}^M_{l'}\}.$$

Proposition 2. *lcs*$(\mathcal{C}) = \{x \in \mathbf{R}^{NL}_+ :$ *x satifsies the inequalities in (3)*$\}$.

Proof. It can be proved in a very similar way as in Proposition 1. □

Note that besides the unit demand constraints, the other inequalities in (3) do not distinguish among the identities of the students of each type. So every pseudo-market equilibrium is envy-free among the students of each type. When students are classified into more than two types, the inequalities to characterize lcs$(\mathcal{C})$ can be derived similarly as above.

5. A market for roommates, and other problems

Our method can be used to solve problems in which constraints varied, and the set of feasible assignments cannot be easily characterized from an ex-ante view. In this section we first present our solution to the roommate problem. There are models of market design that use stability, or core-stability, as the criterion for design but where no stable outcomes that satisfy the constraint exist (Ehlers, Hafalir, Yenmez, and Yildirim, 2014; Kamada and Kojima, 2015). One of the oldest example is the roomate problem: it is well known that there are in general no stable outcomes in the roomate problem. In contrast, we shall obtain as a simple corollary of our theorem that there are competitive equilibria in the market for roommates.

We interpret O as a copy of I, and think of $l \in O$ as a different name for agent $i \in I$. So $N = L$. If x is an assignment, interpret $x_{i,j} = 1$ as agents i and j forming a partnership, or becoming roommates. When i is alone without a roommate, we have $x_{i,i} = 1$. In consequence, we restrict attention to assignments x where $x_{i,j} = x_{j,i}$, meaning that the matrix $(x_{i,j})_{i\in I, j\in I}$ is *symmetric*.

We say that an assignment x is a *matching* if (1) $x_{i,j} \in \{0,1\}$ for all $(i,j) \in I \times I$, (2) x is symmetric, (3) x satisfies the unit demand constraints with equality ($\sum_j x_{i,j} = 1$) and (4) x satisfies the allocation constraints with equality ($\sum_i x_{i,j} = 1$). Define $\mathcal{C}$ to be the convex hull of all matchings.

Note that $\mathcal{C}$ is not equal to the set of symmetric assignments that satisfy the unit demand and allocation constraints, dropping the integrality constraint $x_{i,j} \in \{0,1\}$. Katz (1970) proves that the latter set is the convex hull of all matrices of the form $(1/2)(P + P')$ where P is a permutation matrix with no even cycles greater than 2. Edmonds (1965) provides a characterization of $\mathcal{C}$, which is used in the proof of Proposition 3 below.

It is instructive to work out the set of inequalities Ω for this case. To this end, let $\mathcal{F}$ be the set of subsets $F \subseteq I \times I$ such that (1) for all i, $(i,i) \notin \mathcal{F}$ and (2) for every $(i,j) \in F$, $(j,i) \notin F$. For each $F \in \mathcal{F}$, let G_F be the graph with vertex set I and edge set $\{\{i,j\} : (i,j) \in F \text{ or } (j,i) \in F\}$. Denote the cardinality of the maximum independent edge set of G_F by k_F. For every $i \in I$, let $\mathcal{J}_i$ be the set of subsets $J \subset (\{i\} \times I) \cup (I \times \{i\})$ such that $(i,i) \in J$ and for every $j \neq i$, either $(i,j) \in J$ or $(j,i) \in J$ but not both. Then $\text{lcs}(\mathcal{C})$ is characterized by the following inequalities.

Proposition 3.

$$lcs(\mathcal{C}) = \left(\bigcap_{\emptyset \neq F \in \mathcal{F}} \{x \in \mathbf{R}_+^{I\times I} : \sum_{(i,j)\in F} x_{i,j} \leq k_F\} \right) \bigcap \left(\bigcap_{i \in I, J \in \mathcal{J}_i} \{x \in \mathbf{R}_+^{I\times I} : \sum_{(i',j')\in J} x_{i',j'} \leq 1\} \right).$$

Proof. Let D denote the set on the right-hand side of the proposition. We first prove that $D \subset \text{lcs}(\mathcal{C})$. For every $x \in D$, consider the matrix x' obtained by letting $x'_{i,j} = \max\{x_{i,j}, x_{j,i}\}$ for all $(i,j) \in I \times I$. Then x' is symmetric and $x \leq x'$. We prove that $x' \in D$. For any $\emptyset \neq F \in \mathcal{F}$, suppose to the contrary that $\sum_{(i,j)\in F} x'_{i,j} > k_F$. Then we define $F' \subset I \times I$ such that for every $(i,j) \in F$, if $x_{i,j} \geq x_{j,i}$, let $(i,j) \in F'$, and otherwise let $(j,i) \in F'$. So $G_{F'}$ and G_F have the same (undirected) edge set, and thus $k_F = k_{F'}$. However, $\sum_{(i,j)\in F'} x_{i,j} = \sum_{(i,j)\in F} x'_{i,j} > k_{F'}$, which contradicts that

$x \in D$. Similarly we can prove that for every i and every $J \in \mathcal{J}_i$, $\sum_{(i',j')\in J} x'_{i',j'} \leq 1$. Thus, $x' \in D$.

Now define another matrix y by 1) for every $(i,j) \in I \times I$ with $i \neq j$, set $y_{i,j} = x'_{i,j}$, and 2) for every $i \in I$, $y_{i,i} = 1 - \sum_{j\neq i} x'_{i,j}$. It is clear that y is symmetric and that $x' \leq y$ (as $\{i\} \times I \in \mathcal{J}_i$). For any $F \in \mathcal{F}$, $(i,i) \notin F$; hence $\sum_{(i,j)\in F} y_{i,j} = \sum_{(i,j)\in F} x'_{i,j} \leq k_F$. Since x' is symmetric and $x' \in D$, for every i and every $J \in \mathcal{J}_i$, $\sum_{(i',j')\in J} y_{i',j'} = 1$. So $y \in D$ and it is a bistochastic matrix.

Now we prove that $y \in \mathcal{C}$. Edmonds (1965) proves that a symmetric bistochastic matrix z belongs to $\mathcal{C}$ if and only if for every $r \in \mathbb{N}$ and every $I' \subset I$ with $|I'| = 2r+1$, $\sum_{(i,j)\in F} z_{i,j} \leq r$, where $F \subset I' \times I'$ such that there does not exist $(i,i) \in F$ and for every $(i,j) \in I' \times I'$ with $i \neq j$, either $(i,j) \in F$ or $(j,i) \in F$ but not both. For any such F, $k_F = r$ because I' is odd and we can form r pairs among the $2r$ elements of I' that can be paired. Since $F \in \mathcal{F}$, then, y satisfies Edmonds' inequalities and thus $y \in \mathcal{C}$. Since $x \leq x' \leq y$, $x \in \text{lcs}(\mathcal{C})$.

To prove $\text{lcs}(\mathcal{C}) \subset D$, consider any $x \in \mathcal{C}$. Then x is the convex combination of deterministic matchings x^k. For each $\emptyset \neq F \in \mathcal{F}$ and each i, there is at most one j with $x^k_{i,j} = 1$. By the definition of independent edge set, then $\sum_{(i,j)\in F} x^k_{i,j} \leq k_F$. So $\sum_{(i,j)\in F} x_{i,j} \leq k_F$. It is clear that x satisfies the other inequalities related to every $\mathcal{J}_i$. So $x \in D$. Then it means that $\text{lcs}(\mathcal{C}) \subset D$. □

A pseudo-market equilibrium implies a random matching x^* (a probability distribution over matchings) that is Pareto efficient. Of course, x^* needs not be stable in the game theoretic sense, but it corresponds to individual agents' optimizing behavior, as long as these agents take prices as given. Price taking behavior is a plausible assumption in a large centrally-run market for partnerships, like for example a market for roommates in college dormitories. A pseudo-market could be set up by the college, and equilibrium prices could in principle be enforced and be made transparent.

Example 1 (A market for roommates). Let $I = \{1,2,3\}$ and $L = 3$. For agent i, consuming object l is the same as having agent l as her roomate. Suppose that the agents' utilities are

	1	2	3
1	0	1	2
2	2	0	1
3	1	2	0

With these preferences, there are no stable matchings. However, there is a HZ equilibrium. In the equilibrium, the price of the following constraint is two:

$$x_{2,1} + x_{1,3} + x_{3,2} \leq 1,$$

the price of the following constraint is one:

$$x_{1,2} + x_{2,3} + x_{3,1} \leq 1,$$

and the price of every other constraint is zero. Then, agent 1's personalized price vector is $(0, 1, 2)$, 2's personalized price vector is $(2, 0, 1)$, and 3's personalized price vector is $(1, 2, 0)$. All of them choose the consumption $(1/3, 1/3, 1/3)$, and this is the symmetric equilibrium assignment.

5.1. **Coalition formation.** The application to roommates can be adapted to a general coalition-formation problem. Given a set of agents I, let O be the set of all *coalitions* from I; that is, $O = 2^I \setminus \{\emptyset\}$. A deterministic assignment is a partition of agents into coalitions, and can be represented by a matrix $x \in \{0, 1\}^{NL}$ such that $x_{i,l} = 1$ if and only if i joints the coalition $l \in O$. Unit demand constraints will imply that agents are members of a single coalition. We may then let $\mathcal{C}$ be the convex hull of the set of deterministic assignments. Then there exists a pseudo-market equilibrium, and the equilibrium assignment is a probability distribution over coalitions.

5.2. **Combinatorial allocation.** Our methods can also be used to solve other general combinatorial assignment/matching problems. Here we discuss allocation problems in which agents demand a bundle of objects, as in course allocation. There will be obvious capacity constraints, but course allocation may exhibit additional, and more problematic, constraints. For example, if a student i regards two courses l and l' as complements so that she has to take both of them or neither, then we have the constraint $x_{i,l} = x_{i,l'}$. If a student i regards two courses l and l' as substitutes, so that she has to take at most one of them, then we have the constraint $x_{i,l} \cdot x_{i,l'} = 0$.

The set of feasible (random) assignments in course allocation problems cannot be easily characterized. In particular, an assignment that seems ex-ante feasible may not be actually feasible. For example, suppose there are three agents $1, 2, 3$ and three objects a, b, c. Each object has one copy. The set of bundles is $O = \{ab, ac, bc\}$. The following random assignment is ex-ante feasible because it satisfies unit demand constraints of agents and allocation constraints of objects. But it is not feasible because bundles are not independent objects. When a bundle is assigned, the other two bundles become not available.

i	ab	ac	bc
1	1/2	0	0
2	0	1/2	0
3	0	0	1/2

We can still apply our results by starting from a collection of deterministic assigments. Let A be the set of objects, each of which has a number of copies. Let $O \subset 2^A$ be the set of bundles under consideration. A deterministic assignment is represented by a matrix $x \in \{0,1\}^{NL}$ such that $x_{i,l} = 1$ if and only if i obtains the bundle $l \in O$. Let $\mathcal{C}$ be the convex hull of the set of deterministic assignments. Then our theorem applies.

6. A market for "bads"

So far we have assumed that objects are "goods," in the sense that agents' utility functions are monotone increasing. In some applications, however, objects represent duties, or tasks, that agents dislike. Yet another application is to waste disposal, or pollution. A certain minimum amount of such "bads" have to be allocated; the question is to whom, and in which quantities?

The presence of bads give rise to floor constraints, but we cannot use our previous methods directly as all agents will choose zero consumption from their consumption space. We can, however, borrow an idea from the standard model of a labor market: labor supply is often described as consumption of leisure. So we endow every agent with a copy of every "bad," and allow them to buy the options of not consuming a bad. Such options become "goods," and our previous methods apply.

Specifically, for every $l \in O$, q_l denotes the minimum number of copies of l that have to be assigned. Every agent can be assigned at most one object (unit demand). If $\sum_{l \in O} q_l = N$, then every agent must obtain an object so that the problem becomes the one studied by Hylland and Zeckhauser (1979). Assume then that $\sum_{l \in O} q_l < N$. For every $x \in \Delta_-$ and every $i \in I$, $u_i(x)$ is strictly decreasing in x: if $x' > x$, then $u_i(x') < u_i(x)$.

We consider a dual problem $(I, \tilde{O}, \tilde{\Delta}_-, (\tilde{u}_i)_{i \in I}, (q_{\tilde{l}})_{\tilde{l} \in \tilde{O}})$ in which

- the set of objects is $\tilde{O} = \{\tilde{l}\}_{l \in O}$ where every $\tilde{l}$ is an artificial object dual to $l \in O$, and its supply is $q_{\tilde{l}} = N - q_l$. When an agent i consumes an amount

a of $\tilde{l}$, it is understood that i consumes $1-a$ of l. Because at least q_l of l need to be assigned, the number of copies of $\tilde{l}$ is $N - q_l$.

- The consumption space for every agent is $\tilde{\Delta}_- = \{x \in \mathbf{R}^L_+ : x_{\tilde{l}} \in [0,1]$ for every $l \in \tilde{O}, \sum_{\tilde{l}\in\tilde{O}} x_{\tilde{l}} \in [L-1, L]\}$. So the amount of objects in O that i will consume is $L - \sum_{\tilde{l}\in\tilde{O}} x_{\tilde{l}} \in [0,1]$.
- Every agent i has the utility function $\tilde{u}_i$ such that for every $x \in \tilde{\Delta}_-$, $\tilde{u}_i(x) = u_i(\mathbf{1} - x)$. When u_i is (semi-strictly) quasi-concave and strictly decreasing, $\tilde{u}_i$ is (semi-strictly) quasi-concave and strictly increasing.

In the dual problem, agents can consume multiple artificial objects. We use floor constraints on individual consumption, and can derive the inequalities to characterize lcs($\mathcal{C}$) as in Section 4.3. Then Theorem 1 applies to give a desirable outcome. We omit the details.

7. Endowment and α-slack equilibrium

We turn to a version of our model in which objects are initially owned by agents as endowments. Endowments are important in market design when the purpose is to re-assign resources. Often, one want to improve on an existing allocation. It is then important to be able to respect agents' property rights.[5] Moreover, there are models (such as time banks, briefly discussed in 7.5), in which the agents themselves provide the goods that are to be allocated.

7.1. **The economy and equilibrium.** Now an *economy* is a tuple $\Gamma = (I, (Z_i, u_i, \omega_i)_{i\in I})$, where

- I is a finite set of *agents*;
- $Z_i \subseteq \mathbf{R}^L_+$ is i's *consumption space*;
- $u_i : Z_i \to \mathbf{R}$ is i's *utility function*;
- $\omega_i \in Z_i$ is i's *endowment*.

The *aggregate endowment* is denoted by $\bar{\omega} = \sum_{i\in I} \omega_i$. For every $l \in O$, $\bar{\omega}_l$ is the amount of l in the economy.

A *constrained allocation problem with endowments* is a pair $(\Gamma, \mathcal{C})$ in which Γ is an economy and $\mathcal{C}$ is a set feasible assignments such that

(1) $\mathcal{C}$ is a polytope;
(2) $\omega = (\omega_i)_{i\in I} \in \mathcal{C}$; that is, ω is feasible.

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[5] Indeed if we want buy-in from agents in the market, we will have to ensure that they are not hurt in the re-allocation.

A feasible assignment $x \in \mathcal{C}$ is *acceptable* to agent i if $u_i(x_i) \geq u_i(\omega_i)$; x is *individually rational* (IR) if it is acceptable to all agents. We also define a notion of approximate individual rationality: for any $\varepsilon > 0$, x is ε-*individually rational* (ε-IR) if $u_i(x_i) \geq u_i(\omega_i) - \varepsilon$ for all $i \in I$.

Let $\mathcal{X}_i$ and Ω^* be defined as before. We say two agents i and j are of *equal type* if $\omega_i = \omega_j$, $\mathcal{X}_i = \mathcal{X}_j$, and for all $(a, b) \in \Omega^*$, $a_i = a_j$.

In a Walrasian equilibrium in the textbook exchange economy, agents' budgets are from selling their endowments at equilibrium prices. However, when agents have satiated preferences as in our model, a Walrasian equilibrium may not exist. See Example 3 in Section 7.3.

Our method to solve the nonexistence problem is to introduce an arbitrarily small exogenous budget. Given any price vector p, let p_i be the personalized price vector faced by any agent i as defined in Section 2.6. Then for any $\alpha \in [0, 1]$, we let i's budget be

$$\alpha + (1 - \alpha)p_i \cdot \omega_i.$$

So the budget is the convex combination of the exogenous budget of one used in the main model and the market value of i's endowment. When $\alpha = 1$, agents have the equal budget of one, and when $\alpha = 0$, agents' budgets are the market values of their endowments.

For any $\alpha \in [0, 1]$, we say (x^*, p^*) is an α**-slack equilibrium** if

(1) $x_i^* \in \arg\max_{x_i \in \mathcal{X}_i} \{u_i(x_i) : p_i^* \cdot x_i \leq \alpha + (1 - \alpha)p_i^* \cdot \omega_i\}$;
(2) $x^* \in \mathcal{C}$;
(3) For any $c = (a, b) \in \Omega^*$, $\sum_{(i,l)} a_{i,l} x_{i,l}^* < b$ implies that $p_c^* = 0$.

7.2. **Results.** We assume that for each $c \in \Omega^*$, $\sum_{(i,l) \in supp(c)} \omega_{i,l} > 0$. A sufficient condition for this assumption is that every agent owns a positive amount of every object. Then we obtain an extension of Theorem 1.

Theorem 2. *Suppose that agents' utility functions are continuous, quasi-concave and strictly increasing. For any* $\alpha \in (0, 1]$*:*

- *There exists an* α*-slack equilibrium* (x^*, p^*)*, and* x^* *is weakly* $\mathcal{C}$*-constrained Pareto efficient.*
- *If agents' utility functions are semi-strictly quasi-concave, there exists an* α*-slack equilibrium assignment* x^* *that is* $\mathcal{C}$*-constrained Pareto efficient.*
- *Every* α*-slack equilibrium assignment is equal-type envy-free.*

Theorem 2 ensures that we can choose $\alpha \in (0,1]$ arbitrarily, but since prices are endogenous it is not clear that the magnitude of α has any actual meaning. Our next result shows that it does. In fact, by choosing α arbitrarily small we ensure that agents' budgets approximate the market values of their endowments. In consequence, the α-slack equilibrium obtained is approximately individually rational.

Theorem 3. *Suppose that agents' utility functions are continuous, semi-strictly quasi-concave and strictly increasing. For any $\varepsilon > 0$, there is $\alpha \in (0,1]$ and an α-slack equilibrium (x^*, p^*) such that x^* is $\mathcal{C}$-constrained Pareto efficient and*

$$\max\{u_i(y) : y \in \mathcal{X}_i \text{ and } p_i^* \cdot y \leq p_i^* \cdot \omega_i\} - u_i(x_i^*) < \varepsilon.$$

In particular, x^* is ε-individually rational.

Below is an example of pricing constraints in the presence of endowments.

Example 2. Consider an economy with 2 agents and 2 objects. In addition to the unit demand and allocation constraints, suppose that we want that $x_{i,l} \geq \delta \in [0, 1/2]$ for all $i \in I$ and $l \in O$. In words, we have a minimum consumption requirement. We want all agents to consume at least δ units of each object.

Hence $\mathcal{C}$ consists of all assignments $x = (x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}) \in \mathbf{R}^4_+$ with $x_{i,1} + x_{i,2} \leq 1$ and $x_{1,l} + x_{2,l} \leq 1$ for all i and l, such that $x_{i,l} \geq \delta$.

Then we can let Ω^* contain the inequalities

$$c_{i,l} = ((0, \ldots, 1, \ldots, 0), 1 - \delta),$$

with $(0, \ldots, 1, \ldots, 0)$ having an entry of 1 in the position for i, l. There are four such constraints. Complete the description of Ω^* by adding the constraints $c_1 = ((1,0,1,0),1)$ and $c_2 = ((0,1,0,1),1)$. Clearly, $\mathcal{X}_i = \Delta_-$.

It should be clear that $\text{lcs}(\mathcal{C}) = \{x \in \mathbf{R}^4_+ : a \cdot x \leq b \text{ for all } (a,b) \in \Omega^*\}$.

Suppose that agents' endowments are $\omega_i = (1/2, 1/2)$, $i = 1, 2$, and that utility is linear with $u_1(1,0) = 10$, $u_1(0,1) = 1$, while $u_2(1,0) = 1$ and $u_2(0,1) = 10$. So the agents would like to trade away from the equal endowments to the allocation $(1,0,0,1)$. There is then a $(1-2\delta)$-slack equilibrium with $p_{c_{1,1}} = p_{c_{2,2}} = 1$ $p_{c_{1,2}} = p_{c_{2,1}} = 1/10$, $p_{c_1} = p_{c_2} = 0$. In this case the allocation is $(1-\delta, \delta, \delta, 1-\delta)$.

7.3. **The Hylland and Zeckhauser example.** A major application of Theorem 2 is to the object allocation model under the allocative and unit demand constraints. That is, $\mathcal{X}_i = \Delta_-$ for every i and $x \in \mathcal{C}$ if and only if $\sum_i x_i = \bar{\omega}$. There are exactly

L inequalities in Ω^*, one for each object l, expressing that $\sum_i x_{i,l} \leq \bar{\omega}_l$. All agents face equal personalized prices, so we write p_l for the price of l.

We present an example due to Hylland and Zeckhauser (1979) showing that a Walrasian equilibrium (a 0-slack equilibrium) may not exist, and show how the symmetric Pareto efficient assignment in the example can be sustained as an α-slack equilibrium with any $\alpha \in (0, 1]$.

Example 3. Given is an economy with three agents $1, 2, 3$ and two objects A, B. Object A has one copy and B has two copies. Agents have the following von-Neumann Morgenstern utilities:

i	$u_{i,A}$	$u_{i,B}$
1	100	1
2	100	1
3	1	100

Endowments are $\omega_i = (1/3, 2/3)$ for $i = 1, 2, 3$.

Claim 1. *There is no Walrasian equilibrium in Example 3.*

Proof. Suppose (towards a contradiction) that (x, p) is a Walrasian equilibrium. Suppose first that $p_B > 0$ and normalize it to one. Then all agents have the same positive budget. If $p_A = 0$, then 1 and 2 would each buy one copy of A, which is a contradiction. So p_A must be positive. The preferences of agents imply that 1 and 2 must each obtain a half of A. Therefore, $1/3p_A + 2/3 \geq 1/2p_A$, and we obtain $p_A \leq 4$. However, if $p_A < 4$, 1 and 2 would spend all of their budgets on A, and each obtain more than a half of A, which is a contradiction. So it must be that $1/3p_A + 2/3 = 1/2p_A$ and $p_A = 4$. But this means that at most 3 demands B and B must have excess supply, which is a contradiction.

Now suppose $p_B = 0$ and $p_A > 0$. Then 3 must obtain one copy of B. Since p_A is positive, 1 and 2 must each obtain a half of A. However, their budget $1/3p_A$ cannot afford such a consumption. $\square$

Consider the assignment x defined by:

i	$x_{i,A}$	$x_{i,B}$
1	1/2	1/2
2	1/2	1/2
3	0	1

Claim 2. *For any $\alpha \in (0,1]$, the price vector $p = (\frac{6\alpha}{1+2\alpha}, 0)$ and the assignment x constitute an α-slack equilibrium in Example 3.*

Proof. For any $\alpha \in (0,1]$ and $i = 1,2,3$,

$$\alpha + (1-\alpha) p \cdot \omega_i = \alpha + (1-\alpha)\frac{2\alpha}{1+2\alpha} = \frac{3\alpha}{1+2\alpha} = p \cdot x_i.$$

With such budgets, agents 1 and 2 can only afford a 1/2 share of A and a 1/2 share of B, which is the best consumption for them. Agent 3 chooses a copy of B for free. □

Note that in the above α-slack equilibrium, the endogenous value of agents' endowments is $2\alpha/(1+2\alpha)$. So the value of the exogenous part of the budget relative to the value of the endogenous part is

$$\frac{\alpha}{(1-\alpha)\dfrac{2\alpha}{1+2\alpha}} \to \frac{1}{2}$$

as $\alpha \to 0$. So when α shrinks to zero, the value of the exogenous income is not negligible. In the same spirit, the following proposition shows that the average endogenous budget will always be below the exogenous budget of one. This means that the economy needs outside "money."

Proposition 4. *If (x,p) is an α-slack equilibrium, then*

$$\frac{1}{N}\sum_{i=1}^{N} p \cdot \omega_i \leq 1$$

Proof. Note that $p \cdot (x_i - \omega_i) \leq \alpha(1 - p \cdot \omega_i)$. Sum over i to obtain:

$$0 = p \cdot \left(\sum_i x_i - \bar{\omega}\right) \leq \alpha(N - p \cdot \bar{\omega}).$$

□

7.4. **A market-based fairness property.** In the object allocation model of Section 7.3, it is possible to use our result to develop a kind of fairness property in the presence of endowments. Fairness, in the sense of absence of envy, is generally incompatible with individual rationality. Imagine an economy with two objects, where both agents prefer object 1 over object 2, and all the endowment of object 1 belongs to agent 1. Then, in any allocation, there will either be envy, or agent 1's individual rationality will be violated. So fairness has to be amended to account for the presence of endowment.[6]

In the object allocation model with allocative and unit demand constraints, in any α-slack equilibrium, if agent i envies agent j then it must be that j's endowment is worth more than i's at equilibrium prices. In a sense, this means that agents collectively value j's endowment more than i's. Our next result formalizes this idea.

Proposition 5. *Suppose that agents' utility functions are concave and C^1. Let (x,p) be an α-slack equilibrium. Denote by $S = \{i : u_i(x_i) = \max\{u_i(z_i) : z_i \in \Delta_-\}\}$ the set of satiated agents, and by $U = I \setminus S$ the set of others. Suppose that $\sum_{i\in U} x_i \gg 0$.*

If i envies j in x $(u_i(x_j) > u_i(x_i))$, then $p\cdot\omega_j > p\cdot\omega_i$, and there exists welfare weights $\theta \in \mathbf{R}^U_{++}$ such that if

$$v(t) = \sup\{\sum_{i\in U}\theta_i u_i(\tilde{x}_i) : (\tilde{x}_i) \in \Delta^U_- \text{ and } \sum_{i\in U}\tilde{x}_i \leq \bar{\omega} + t(\omega_i - \omega_j) - \sum_{i\in S} x_i\},$$

then $(x_i)_{i\in U}$ solves the problem for $v(0)$, and $v(t) < v(0)$ for all t small enough.

The meaning of Proposition 5 is that if an agent i envies agent j then j's endowment is more valuable than i's in two senses. First, it is more valuable at equilibrium prices. Second, the higher price valuation translates into a statement about how much agents value the endowments. In particular, j's endowment is more valuable than i's to a coalition of players U (a coalition that includes i!). It is more valuable to U in the sense that there are welfare weights for the members of U such that a change in agents' endowment towards having more of i's endowment and less of j's leads to a worse weighted utilitarian outcome. The result requires that $\sum_{i\in U} x_i \gg 0$ simply to ensure that when we subtract ω_j we do not force some agent to consume negative quantities of some object.[7]

7.5. **A market for time exchange.** In organizations such as *time banks*, members exchange time and skills without using monetary transfers.[8] A time exchange problem can be described as an object allocation model with endowments. Formally, O is the set of service types. For each agent i and service l, $\omega_{i,l}$ is the amount of l that i can provide. We could require that $\sum_{l\in O}\omega_{i,l} \leq 1$ for all i and every agent's demand

[6]The paper by Echenique, Miralles, and Zhang (2019) deals exclusively with this problem, but proposes a very different solution.

[7]The $\sum_{i\in U} x_i \gg 0$ hypothesis in Proposition 5 is stronger than what we need. It suffices that if $\omega_{j,l} > 0$ then $\sum_{i\in U} x_{i,l} > 0$.

[8]See Andersson, Cseh, Ehlers, and Erlanson (2019) for more description of real-life time banks.

is no greater than one. Here "one" could mean one day, one week, or one month. Services can be regarded as divisible because time is divisible. But of course, in real life time is often measured in integers such as hours, days, or weeks. Theorem 2 implies that we can find a market equilibrium to the problem. The value of an agent's endowment at equilibrium prices shows how much his endowment is valued by all agents. When the value is higher, the agent is rewarded by a better assignment.

8. Related Literature

Constrained resource allocation has received a lot of attention in recent years. Budish, Che, Kojima, and Milgrom (2013) identify the bihierarchy structure of constraint blocks in the assignment matrix as the sufficient and necessary condition for implementation. Akbarpour and Nikzad (forthcoming) extend this result by relaxing some constraints and considering approximate implementation. We circumvent the implementation issue by taking the set of implementable assignments as the primitive. Budish et al. allow for floor constraints in implementation but rule out them in their applications. In their extension of the pseudo-market mechanism, they consider column constraints, row constraints and sub-row constraints. By incorporating all row and sub-row constraints into agents' consumption spaces, they prove the existence of equilibria much like Hylland and Zeckhauser's. Their extension is a special case of ours. We can deal with more general constrains on both rows and columns, and allow for floor constraints. When there are no floor constraints, we directly price ceiling constraints, and when there are floor constraints, we translate floor constraints into a different set of ceiling constraints.

Ehlers, Hafalir, Yenmez, and Yildirim (2014) focus on the problem of controlled school choice (which was introduced by Abdulkadiroğlu and Sönmez (2003)), whereby school children have to be assigned seats at different schools to satisfy some diversity objective.[9] Kamada and Kojima (2015) are mainly (but not exclusively) motivated by the problem of allocating doctors to hospitals to satisfy geographic quotas. The objective of the quotas is to avoid an excessive concentration of doctors in urban areas.[10] Both papers proceed by adapting the notion of stability to capture the presence of constraints, and to add structure to the constraints being considered.

[9]Controlled school choice is also investigated by, among others, Ehlers (2010), Hafalir, Yenmez, and Yildirim (2013), Kominers and Sönmez (2013), Westkamp (2013), Echenique and Yenmez (2015), Aygun and Bó (2017), and Nguyen and Vohra (2017).

[10]See Kamada and Kojima (2017) for an overview.

To address more general constraints, Kamada and Kojima (2019) relax stability and focus on feasible, individually rational, and fair assignments. They demonstrate that the class of general upper-bound constraints on individual schools are the most permissive constraints under which a student-optimal fair matching exists. That class rules out floor constraints. Our paper can deal with the same kinds of constraints in the above papers, but we follow a different methodological tradition. Instead of a two-sided game-theoretic matching model, we consider object allocation and propose a competitive equilibrium solution. The above papers also investigate the role of incentives in their mechanisms. We expect our pseudo-market mechanism to be incentive compatible in large markets, but we choose to focus on existence, efficiency and fairness.[11]

The recent work of Balbuzanov (2019) considers a version of the probabilistic serial mechanism for object allocation subject to constraints. Like us, he works on a one-sided object allocation model, but the focus on probabilistic serial makes the analysis clearly distinct from ours. We borrow from this paper the idea, expressed in Lemma 1, allowing us to focus on non-negative linear inequalities.

The use of markets over lottery shares to solve centralized allocation problems was first proposed by Hylland and Zeckhauser (1979). They assume no constraints other than unit demands. They impose a fixed income for each agent, independent of prices. They also emphasize that equilibrium may not be efficient, and introduce the "cheapest bundle" property that we employ as well in our version of the first welfare theorem. Many other papers have followed Hylland and Zeckhauser in analyzing competitive equilibria as solutions in market design; see for instance, Budish (2011), Ashlagi and Shi (2015), Hafalir and Miralles (2015), He, Miralles, Pycia, and Yan (2018). Miralles and Pycia (2020) establish the second welfare theorem for the market with satiated preferences and token money: every Pareto efficient assignment may be supported in a Walrasian equilibrium with properly chosen budgets. But none of these papers consider constrained allocation problems.

Hylland and Zeckhauser make the point that an equilibrium may not exist in a model with endowments. Like us, Mas-Colell (1992), Le (2017) and McLennan (2018) also propose to avoid the non-existence issue by means of a hybrid income between the exogenous budget and the endogenous Walrasian income. A version of the hybrid model was first introduced by Mas-Colell (1992), who presents an

[11] He, Miralles, Pycia, and Yan (2018) prove the asymptotic strategy-proofness of their pseudo-market mechanism.

existence result with income that is the sum of a fixed income and a price-dependent income. His result requires the first component to be determined endogenously as part of the fixed point argument in the equilibrium existence result. Aside from the presence of constraints, our result differs from his by allowing us to obtain approximate individual rationality with a small exogenous α. In Le's (2017) notion of equilibrium, two identical objects may have different prices. As a consequence, there may be envy among identical agents, and it may be necessary for some agents to purchase a more expensive copy of a object when a cheaper one is available.[12] Envy among equals is problematic for normative reasons, and it is hard to implement such equilibria in a decentralized fashion.[13]

McLennan (2018) presents an existence result for equilibrium with "slack" in a general model that allows for production and encompasses our model as a special case. But his notion of equilibrium with slack differs from ours in important ways. Agents in his (and our) model may be satiated, and his notion of slack controls the distribution of transfers from satiated agents who spend less than their income to unsatiated agents. In contrast, our α parameter controls the role of endowments, allowing for α to specify the weight of equal incomes vs. (unequal) endowments. In fact, it is possible to construct an example to illustrate the difference between the two notions of equilibrium. In the example no agents are satiated, so the slack in McLennan's notion has no role to play, and his equilibrium allocations are independent of α; in contrast, our equilibrium allocations range from equal division to the autartical consumption of endowments, as α ranges from placing all weight on the exogenous income, to placing all weight on initial endowments.[14]

[12]In Example 3, a Le's equilibrium is as follows. Let $p = (100, 1, \frac{101}{2})$ be a price vector in which the latter two elements are the prices of the two copies of B. Then all agents have an income of 101/2. The unique optimal bundle for agents 1 and 2 is $x_i = (1/2, 1/2, 0)$. Agent 3 is willing to spend all his income on buying the more expensive copy of B, so $x_3 = (0, 0, 1)$.

Consider a variation of the example in which endowments become $\omega_1 = (1/3, 1/2, 1/6)$, $\omega_2 = (1/3, 1/6, 1/2)$, and $\omega_3 = (1/3, 1/3, 1/3)$. Then $p = (100, 1, \frac{101}{2})$ is still an equilibrium price, with $x_1 = (\frac{5}{12}, \frac{7}{12}, 0)$, $x_2 = (\frac{7}{12}, \frac{5}{12}, 0)$, and $x_3 = (0, 0, 1)$ being the equilibrium assignment. Observe that agent 1 envies 2, despite they have the same utility and the same endowment: $1/3$ of A and $2/3$ of B.

[13]One could interpret different prices for different copies of the same object as a novel endogenous transfer scheme, but we are unaware of a normative defense of this idea.

[14]We are grateful to Andy McLennan for this example, which can be found in his paper.

Kojima, Sun, and Yu (2018) and Gul, Pesendorfer, and Zhang (2019) consider market equilibrium in economies with gross substitutes utilities and constraints. Kojima et. al characterize the constraints that preserve the gross substitutes property of firms' demands in a transferable utility model (Kelso and Crawford's (1982) job matching model). Gross substitutes ensure equilibrium existence, and the authors show that the constraint structures have to take the form of "interval constraints." Gul et. al prove the existence of equilibrium in economies with a finite number of indivisible objects, and limited transfers or no transfers. They show that with limited transfers or no transfers, equilibrium requires random allocations and can be approached by the equilibrium with full transfers. They also show that equilibrium allocations satisfying certain constraints can be constructed by building these constraints into utility functions or incorporating them into a production technology. Different from them, we price constraints and can accommodate more general preferences and constraint structures.

Related to our applications, Manjunath (2016) proposes a competitive equilibrium notion for a two-sided fractional matching market. The double-indexed price system in his notion resembles our personalized price system, but he needs to deal with both sides' preferences. As a consequence, his equilibrium exists when there are transfers, but only approximately exists when transfers are forbidden. Andersson, Cseh, Ehlers, and Erlanson (2019) propose a time exchange model in which each agent provides a distinct service and has dichotomous preferences. They propose a priority mechanism to maximize the amount of exchanges among agents. Differently, in our model an agent can provide multiple services and different agents can provide the same service. Agents can express richer preferences and the prices in our market solution reveal on which service agents have more demand. Bogomolnaia, Moulin, Sandomirskiy, and Yanovskaia (2017, 2019) study the competitive equilibrium allocation of a mixed manna that contains "goods" and "bads". They prove that an equilibrium always exists. Our model is different than theirs in that agents have unit-demand constraints. So their existence result does not hold in our paper.

9. Proof of Theorem 1 and Theorem 2

We first prove the theorem by assuming that all utility functions are semi-strictly quasi-concave. We then explain the differences when utility functions are only quasi-concave. We also explain how the proof works for Theorem 1.

With an abuse of notation, we write $\sum_{l\in O} p_{i,l}x_{i,l}$ as $p_i \cdot x_i$. For each $c \in \Omega^*$, we have assumed that $\sum_{(i,l)\in supp(c)} \omega_{i,l} > 0$. It implies that $\sum_{(i,l)\in supp(c)} a^c_{i,l}\omega_{i,l} > 0$ (we write $c = (a^c, b^c)$).

We define a price ceiling

$$\bar{p} = \frac{2N}{\min_{c\in\Omega^*} \sum_{(i,l)\in supp(c)} a^c_{i,l}\omega_{i,l}},$$

and a price space $\mathcal{P} = [0, \bar{p}]^{|\Omega^*|}$.

Given $\alpha \in (0, 1]$, for every $p \in \mathcal{P}$, define

$$\begin{aligned}
v_i &= \max\{u_i(x_i) : x_i \in \mathcal{X}_i\},\\
B_i(p, \alpha) &= \{x_i \in \mathcal{X}_i : p_i \cdot x_i \leq \alpha + (1-\alpha)p_i \cdot \omega_i\},\\
d_i(p) &= \operatorname{argmax}\{u_i(x_i) : x_i \in B_i(p, \alpha)\},\\
\underline{d}_i(p) &= \operatorname{argmin}\{p \cdot x_i : x_i \in d_i(p)\},\\
V_i(p) &= \max\{u_i(x_i) : x_i \in B_i(p, \alpha)\}.
\end{aligned}$$

Lemma 2. *If $V_i(p) < v_i$ then $d_i(p) = \underline{d}_i(p)$.*

Proof. Let $x_i \in d_i(p)$. We shall prove that $p_i \cdot x_i = \alpha + (1-\alpha)p_i \cdot \omega_i$, which means we are done because it implies that all bundles in $d_i(p)$ cost the same at prices p. Let $z_i \in \mathcal{X}_i$ be such that $u_i(z_i) = v_i > u_i(x_i)$. For any $\varepsilon \in (0, 1)$, since $\mathcal{X}_i$ is convex, $\varepsilon z_i + (1-\varepsilon)x_i \in \mathcal{X}_i$. By the semi-strict quasi-concavity of u_i, $u_i(\varepsilon z_i + (1-\varepsilon)x_i) > u_i(x_i)$. This means that, for any $\varepsilon \in (0, 1)$,

$$\varepsilon p_i \cdot z_i + (1-\varepsilon)p_i \cdot x_i > \alpha + (1-\alpha)p_i \cdot \omega_i.$$

But this is only possible, for arbitrarily small ε, if $p_i \cdot x_i \geq \alpha + (1-\alpha)p_i \cdot \omega_i$. Since $x_i \in B_i(p, \alpha)$, we have $p_i \cdot x_i = \alpha + (1-\alpha)p_i \cdot \omega_i$. □

Lemma 3. *If $V_i(p) = v_i$, then*

$$\underline{d}_i(p) = \arg\min\{p_i \cdot x_i : u_i(x_i) = v_i \text{ and } x_i \in \mathcal{X}_i\}.$$

Proof. Let $x_i \in \underline{d}_i(p)$. Then for any $z_i \in \mathcal{X}_i$ with $p_i \cdot z_i < p_i \cdot x_i$, we have $z_i \in B_i(p, \alpha)$. So $u_i(z_i) < v_i$ by definition of $\underline{d}_i$. Therefore, if $z_i \in \operatorname{argmin}\{p_i \cdot x_i : u_i(x_i) = v_i \text{ and } x_i \in \mathcal{X}_i\}$, then

$$p_i \cdot z_i = p_i \cdot x_i,$$

and therefore

$$\underline{d}_i(p) \supseteq \arg\min\{p_i \cdot x_i : u_i(x_i) = v_i \text{ and } x_i \in \mathcal{X}_i\}.$$

The converse set inclusion follows similarly because if x_i is not in the right-hand set, there would exist $z_i \in \mathcal{X}_i$ with $p_i \cdot z_i < p_i \cdot x_i$ and $u_i(z_i) = v_i$, which is not possible as such z_i would be in $B_i(p, \alpha)$. □

Lemma 4. *d_i is upper hemicontinuous.*

Proof. Let $(x^n, p^n) \to (x, p)$, with $x^n \in d_i(p^n)$. Suppose that there is $x' \in B_i(p, \alpha)$ with $u_i(x') > u_i(x)$. If $p_i \cdot x' < \alpha + (1 - \alpha) p_i \cdot \omega_i$, then this strict inequality will be true for p^n with n large enough; a contradiction, as u_i is continuous. If $p_i \cdot x' = \alpha + (1 - \alpha) p_i \cdot \omega_i$, then $\alpha > 0$ implies that $p_i \cdot x' > 0$. Then there is $\lambda \in (0, 1)$ large enough that $u_i(\lambda x') > u_i(x)$, $p_i \cdot (\lambda x') < p_i \cdot x'$, and $\lambda x' \in \mathcal{X}_i$. The argument for the case of a strict inequality then applies. □

Remark 1. Lemma 4 uses crucially that $\alpha > 0$.

Lemma 5. *$\underline{d}_i(p)$ is upper hemi-continuous.*

Proof. To prove upper hemi-continuity, we shall prove that $\underline{d}_i$ has a closed graph. Let $(x_i^n, p^n) \to (x_i, p)$ with $x_i^n \in \underline{d}_i(p^n)$ for all n.

First, consider the case where $V_i(p) < v_i$. By the maximum theorem, V_i is continuous, so $V_i(p^n) < v_i$ for all large enough n. Then Lemma 2 implies that $x_i \in \underline{d}_i(p)$ as d_i is upper hemi-continuous.

Second, consider the case where $V_i(p) = v_i$. We know that $x_i \in d_i(p)$ as d_i is upper hemi-continuous. Suppose (towards a contradiction) that $x_i \notin \underline{d}_i(p)$. Then there is $y_i \in d_i(p)$ with

$$p_i \cdot y_i < p_i \cdot x_i \leq \alpha + (1 - \alpha) p_i \cdot \omega_i.$$

Then for all n large enough,

$$p_i^n \cdot y_i < \alpha + (1 - \alpha) p_i \cdot \omega_i.$$

Since $y_i \in d_i(p)$ and $V_i(p) = v_i$, $u_i(y) = v_i$. This means that $V_i(p^n) = v_i$ for all n large enough, as $y_i \in B_i(p^n, \alpha)$. Then, by Lemma 3, $x_i^n \in \operatorname{argmin}\{p_i^n \cdot x_i : u_i(x_i) = v_i \text{ and } x_i \in \mathcal{X}_i\}$ for all n large enough. But the correspondence

$$p \mapsto \operatorname{argmin}\{p_i \cdot x_i : u_i(x_i) = v_i \text{ and } x \in \mathcal{X}_i\}.$$

is upper hemicontinous, by the maximum theorem. So

$$x_i \in \operatorname{argmin}\{p_i \cdot x_i : u_i(x_i) = v_i \text{ and } x \in \mathcal{X}_i\},$$

which is a contradiction. □

It is easy to see that $d_i(p)$ is nonempty, compact- and convex-valued. So $\underline{d}_i(p)$ is also nonempty, compact- and convex-valued. For every $c \in \Omega^*$, define the aggregate demand on c by

$$D_c(p) = \sum_{(i,l)\in supp(c)} a^c_{i,l}\underline{d}_{i,l}(p) = \cup\{a \cdot x : x \in \times_i \underline{d}_i(p)\}.$$

Define the aggregate demand correspondence by

$$D(p) = (D_c(p))_{c\in\Omega^*},$$

and the excess demand correspondence by

$$z(p) = D(p) - \{\mathbf{b}\},$$

where $\mathbf{b} = (b^c)_{c\in\Omega^*}$.

Consider the correspondence $\varphi : \mathcal{P} \to \mathcal{P}$ defined by

$$\varphi_c(p) = \{\min\{\max\{0, z_c + p_c\}, \bar{p}\} : z \in z(p)\} \text{ for all } c \in \Omega^*.$$

$D(p)$, and therefore $z(p)$, are upper hemicontinuous, convex-valued, and compact-valued. Thus, φ is upper hemi-continuous, convex-valued and compact-valued. By Kakutani's fixed point theorem, there exists $p^* \in \mathcal{P}$ with $p^* \in \varphi(p^*)$.

Note that there exists $z^* \in z(p^*)$ such that

$$p^*_c = \min\{\max\{0, z^*_c + p^*_c\}, \bar{p}\} \text{ for all } c \in \Omega^*. \tag{4}$$

Choose $x^* \in \mathbf{R}^{NL}_+$ such that $x^*_i \in \underline{d}_i(p^*)$ for all i and $a \cdot x^* - b = z^*_{(a,b)}$ for all $(a, b) \in \Omega^*$. We shall prove that (x^*, p^*) is an α-slack Walrasian equilibrium.

Lemma 6. *$p^* \cdot z^* \geq 0$.*

Proof. If $p^* \cdot z^* < 0$, then there is some $c \in \Omega^*$ with $p^*_c > 0$ and $z^*_c < 0$. By Equation 4, then, $p^*_c = p^*_c + z^*_c$, which is not possible as $z^*_c < 0$. □

Lemma 7. *$p^*_c < \bar{p}$ for all $c \in \Omega^*$.*

Proof. Suppose towards a contradiction that there exists $c^* \in \Omega^*$ for which $p^*_{c^*} = \bar{p}$. Then $p^*_c > 0$. Now, $x^*_i \in B_i(p^*, \alpha)$ means that

$$p^*_i \cdot x^*_i \leq \alpha + (1 - \alpha)p^*_i \cdot \omega_i,$$

which is equivalent to

$$p^*_i \cdot (x^*_i - \omega_i) \leq \alpha(1 - p^*_i \cdot \omega_i).$$

Summing over i, we obtain that

$$\sum_{i\in I} p^*_i \cdot (x^*_i - \omega_i) \le \alpha(N - \sum_{i\in I} p^*_i \cdot \omega_i),$$

which is equivalent to

$$\sum_{i\in I}\sum_{l\in O} p^*_{i,l}(x^*_{i,l} - \omega_{i,l}) \le \alpha(N - \sum_{i\in I}\sum_{l\in O} p^*_{i,l}\omega_{i,l}) \tag{5}$$

Note that

$$\begin{aligned}\sum_{i\in I}\sum_{l\in O} p^*_{i,l}\omega_{i,l} &= \sum_{i\in I}\sum_{l\in O}\Big(\sum_{c\in\Omega^*:(i,l)\in supp(c)} a^c_{i,l}p^*_c\Big)\omega_{i,l} \\ &= \sum_{c\in\Omega^*} p^*_c\Big(\sum_{(i,l)\in supp(c)} a^c_{i,l}\omega_{i,l}\Big) \qquad (6)\end{aligned}$$

Now, by definition of $\bar{p}$, we have that

$$\sum_{c\in\Omega^*} p^*_c\Big(\sum_{(i,l)\in supp(c)} a^c_{i,l}\omega_{i,l}\Big) \ge \bar{p}\Big(\sum_{(i,l)\in supp(c^*)} a^{c^*}_{i,l}\omega_{i,l}\Big) \ge \bar{p}\min_{c\in\Omega^*}\Big(\sum_{(i,l)\in supp(c)} a^c_{i,l}\omega_{i,l}\Big) = 2N.$$

Thus, using this inequality and equations (5) and (6), we obtain that

$$\sum_{i\in I}\sum_{l\in O} p^*_{i,l}(x^*_{i,l} - \omega_{i,l}) \le \alpha(N - \sum_{i\in I}\sum_{l\in O} p^*_{i,l}\omega_{i,l}) < 0. \tag{7}$$

On the other hand,

$$\begin{aligned} p^*\cdot z^* &= \sum_{c\in\Omega^*} p^*_c\Big(\sum_{(i,l)\in supp(c)} a^c_{i,l}x^*_{i,l} - b^c\Big). \\ &\le \sum_{c\in\Omega^*} p^*_c\Big(\sum_{(i,l)\in supp(c)} a^c_{i,l}x^*_{i,l} - \sum_{(i,l)\in supp(c)} a^c_{i,l}\omega_{i,l}\Big) \qquad (8) \\ &= \sum_{i\in I}\sum_{l\in O}\sum_{\{c\in\Omega^* \text{ st } (i,l)\in supp(c)\}} p^*_c a^c_{i,l}(x^*_{i,l} - \omega_{i,l}) \\ &= \sum_{i\in I}\sum_{l\in O} p^*_{i,l}(x^*_{i,l} - \omega_{i,l}) \qquad (9) \\ &< 0, \qquad (10)\end{aligned}$$

where (8) follows because for each $c \in \Omega^*$, $\sum_{(i,l)\in supp(c)} a^c_{i,l}\omega_{i,l} \le b^c$, (9) follows as

$$p^*_{i,l} = \sum_{\{c\in\Omega^* \text{ st } (i,l)\in supp(c)\}} p^*_c a^c_{i,l},$$

and (10) follows from (7).

Finally, (10) is absurd as it contradicts Lemma 6. □

Proof of Theorem 2. We claim that (x^*, p^*) is an α-slack Walrasian equilibrium. If $p_c^* > 0$, since $p_c^* < \bar{p}$, then $p_c^* = z_c^* + p_c^*$, which implies $z_c^* = 0$. If $p_c^* = 0$, then $z_c^* + p_c^* \leq 0$, which implies $z_c^* \leq 0$. Recall that $z_c^* = a^c \cdot x^* - b^c$. So this implies that x^* satisfies all inequalities in Ω^*. By definition of $\mathcal{X}_i$, x^* satisfies then all inequalities in Ω. Hence,

$$x^* \in \text{lcs}(\mathcal{C}).$$

Moreover, if $z_c^* < 0$, it must be that $p_c^* = 0$, as $p_c^* > 0$ implies $z_c^* = 0$.

It remains to show that $x^* \in \mathcal{C}$. Suppose to the contrary that $x^* \notin \mathcal{C}$. Since $x^* \in \text{lcs}(\mathcal{C})$, there exists $x' \in \mathcal{C}$ such that $x^* \leq x'$. Then $x^* \neq x'$, so there is $(i^*, l^*) \in I \times O$ with $x^*_{i^*,l^*} < x'_{i^*,l^*}$. By definition of $\mathcal{C}$, $x'_{i^*} \in Z_i$.

Consider y_{i*} defined as $y_{i^*,l} = x^*_{i^*,l}$ for all $l \neq l^*$, and $y_{i^*,l^*} = x'_{i^*,l^*}$. Since $x' \in \mathcal{C}$ and $x'_{i^*,l^*} > 0$, l^* cannot be a forbidden object for i^*. Hence, $x'_{i^*} \in \mathcal{X}_{i^*}$ and therefore $y_{i^*} \in \mathcal{X}_i$.

Moreover, for any $c = (a, b) \in \Omega^*$, if $(i^*, l^*) \in \text{supp}(c)$ then $a \cdot x^* < a \cdot x' \leq b$ and therefore $z_c^* < 0$ (c must not be binding at x^*). Hence $p_c^* = 0$. In consequence,

$$\begin{aligned}\sum_{l \in O} p^*_{i^*,l} y_{i^*,l} &= \sum_{l \neq l^*} p^*_{i^*,l} y_{i^*,l} + \Big(\sum_{(a,b) \in \Omega^*} \underbrace{p^*_{(a,b)} a_{i^*,l^*}}_{=0} \Big) y_{i^*,l^*} \\ &= \sum_{l \neq l^*} p^*_{i^*,l} x^*_{i^*,l} \\ &\leq \alpha + (1-\alpha) p_i^* \cdot \omega_i.\end{aligned}$$

Thus $y_{i^*} \in B_{i^*}(p^*, \alpha)$ and $x^*_{i^*} < y_{i^*}$, contradicting the strict monotonicity of u_{i^*} and that $x_i^* \in d_i(p^*)$.

We next prove that x^* is $\mathcal{C}$-constrained Pareto efficient. Suppose towards a contradiction that x is an feasible allocation that Pareto dominates x^*. Given that $x \in \mathcal{C}$, $x_i \in \mathcal{X}_i$. Then, for all $i \in I$, $u_i(x_i) \geq u_i(x_i^*)$, so by definition of $\underline{d}_i$ we have that

$$p_i^* \cdot x_i \geq p_i^* \cdot x_i^*.$$

And for some $j \in I$, $u_j(x_j) > u_j(x_j^*)$, so by utility maximization,

$$p_j^* \cdot x_j > p_j^* \cdot x_j^*.$$

Thus,

$$\sum_{i \in I} p_i^* \cdot x_i > \sum_{i \in I} p_i^* \cdot x_i^*.$$

This is equivalent to

$$\sum_{c\in\Omega^*} p_c^*\Big(\sum_{(i,l)\in supp(c)} a_{i,l}^c x_{i,l}\Big) > \sum_{c\in\Omega^*} p_c^*\Big(\sum_{(i,l)\in supp(c)} a_{i,l}^c x_{i,l}^*\Big).$$

So there must exist $c \in \Omega^*$ such that $p_c^* > 0$ and

$$\sum_{(i,l)\in supp(c)} a_{i,l}^c x_{i,l} > \sum_{(i,l)\in supp(c)} a_{i,l}^c x_{i,l}^*.$$

However, $p_c^* > 0$ implies that $z_c^* = 0$ (c is binding at x^*), and thus x violates c and is not feasible, which is a contradiction.

Equal-type envy-freeness follows the fact that agents of equal type have equal consumption space and equal budgets, and face equal personalized prices. □

Remark 2. The proof uses semi-strict quasi-concavity only in the proof of upper hemicontinuity of $\underline{d}_i$. To prove existence of an equilibrium without imposing the cheapest-bundle property, observe that continuity and quasiconcavity of u_i is enough to ensure that d_i is upper hemicontinuous, and convex- and compact-valued. If z is defined from d_i in place of $\underline{d}_i$, the proof can be written same as above. To prove that every α-slack equilibrium assignment x^* is weakly $\mathcal{C}$-constrained Pareto efficient, suppose towards a contradiction that there exists a feasible assignment x such that for all $i \in I$, $u_i(x_i) > u_i(x_i^*)$. By utility maximization, for all $i \in I$,

$$p_i^* \cdot x_i > p_i^* \cdot x_i^*.$$

Thus,

$$\sum_{i\in I} p_i^* \cdot x_i > \sum_{i\in I} p_i^* \cdot x_i^*.$$

So we obtain a contradiction as before.

Remark 3 (Proof of Theorem 1). The above proof can be easily adapted to prove Theorem 1. We first change the price space to be $\mathcal{P} = [0, \bar{p}]^{|\Omega^*|}$, where

$$\bar{p} = \frac{N}{b_{min}} + 1, \text{ and } b_{min} = \min\{b : (a,b) \in \Omega^*\}.$$

By letting $\alpha = 1$, Lemma 2 to Lemma 6 do not change.

Lemma 7 becomes easier to prove. Suppose $p_c^* = \bar{p}$ for some $c \in \Omega^*$. Then $z_c^* + p_c^* \geq \bar{p}$ implies that $z_c^* \geq 0$. So c must be binding, and for every $(i,l) \in supp(c)$, $p_{i,l}^* \geq a_{i,l} p_c^*$. However, this is impossible because

$$\textstyle\sum_{(i,l)\in supp(c)} a_{i,l} x_{i,l}^* \leq \sum_{(i,l)\in supp(c)} a_{i,l} \frac{1}{p_{i,l}^*} \leq \sum_{(i,l)\in supp(c)} \frac{1}{p_c^*} \leq \frac{N}{p_c^*} < b_{min}.$$

Then we can prove as above that (x^*, p^*) is a pseudo-market equilibrium, and x^* is (weakly) $\mathcal{C}$-constrained Pareto efficient.

10. Proof of Theorem 3

Let d_H denote the Hausdorff distance between two sets in $\mathbf{R}^L$. So,

$$d_H(A, B) = \max\{\sup\{\inf\{\|x-y\| : y \in B\} : x \in A\}, \sup\{\inf\{\|x-y\| : x \in A\} : y \in B\}\}.$$

Let $B_i(p, \alpha)$ denote the budget set of agent i given a price vector p and slack $\alpha \in [0, 1]$. Let $\bar{B}_i(p, \alpha) = \{x_i \in \mathcal{X}_i : p_i \cdot x_i \leq \alpha + (1-\alpha)p_i \cdot \omega_i\}$ denote the budget line. Note that $B_i(p, \alpha) = \{x_i \in \mathcal{X}_i : \exists y \in \bar{B}_i(p, \alpha) \text{ s.t. } x \leq y\}$.

Lemma 8. *For any $\delta > 0$, there is $\alpha > 0$ such that if p is an α-slack equilibrium price vector found in Theorem 2, then for any i, either $p_i \cdot \omega_i < 1$ or $d_H(\bar{B}_i(p, \alpha), \bar{B}_i(p, 0)) < \delta$.*

Proof. Consider the price $\bar{p}$ defined in the proof of Theorem 2. If p is a price obtained in Theorem 2, then $p \in [0, \bar{p}]^{|\Omega^*|}$. Note that $\bar{p}$ is independent of α.

Let $K = \sup\{\|x\| : x \in \mathcal{X}_i, 1 \leq i \leq N\}$. Now choose $\alpha \in (0, 1)$ such that

$$\sup\{\left|1 - \frac{\alpha + (1-\alpha)p_i \cdot \omega_i}{p_i \cdot \omega_i}\right| K : p \in [0, \bar{p}]^{|\Omega^*|} \text{ and } p_i \cdot \omega_i \geq 1\} < \delta$$

Observe that when $p_i \cdot \omega_i \geq 1$, $B_i(p, \alpha) \subseteq B_i(p, 0)$. So for any $x \in B_i(p, \alpha)$, $\inf\{\|x - y\| : y \in \bar{B}_i(p, 0)\} = \|x - x\| = 0$. Hence,

$$\sup\{\inf\{\|x - y\| : y \in \bar{B}_i(p, 0)\}, x \in \bar{B}_i(p, \alpha)\} = 0.$$

On the other hand, if we let $x \in \bar{B}_i(p, 0)$, then $\gamma x \in \bar{B}_i(p, \alpha)$, where

$$\gamma = \frac{\alpha + (1-\alpha)p_i \cdot \omega_i}{p_i \cdot \omega_i}.$$

Since $\gamma \leq 1$, $\gamma \in \mathcal{X}_i$.

Note that

$$\|x - \gamma x\| = |1 - \gamma| \, \|x\| < \delta.$$

Thus $\inf\{\|x - y\| : y \in \bar{B}_i(p, \alpha)\} < \delta$, and therefore

$$\sup\{\inf\{\|x - y\| : y \in \bar{B}_i(p, \alpha)\}, x \in \bar{B}_i(p, 0)\} < \delta.$$

Thus $d_H(B_i(p, 0), B_i(p, \alpha)) < \delta$. ∎

To prove the theorem, let $\delta > 0$ be such that, for any $p \in [0, \bar{p}]^{\Omega^*}$, if $d_H(B_i(p,0), B_i(p,\alpha)) < \delta$ then

$$|\max\{u_i(x) : x \in B_i(p,\alpha)\} - \max\{u_i(x) : x \in B_i(p,0)\}| < \varepsilon.$$

For such δ, let α be as in Lemma 8.

For any i, if $p_i \cdot \omega_i < 1$ then $B_i(p,0) \subseteq B_i(p,\alpha)$, so

$$\max\{u_i(y) : y \in \Delta_- \text{ and } p_i \cdot y \leq p_i \cdot \omega_i\} - u_i(x) < 0 < \varepsilon.$$

If, on the contrary, $p_i \cdot \omega_i \geq 1$, then Lemma 8 implies that $d_H(B_i(p,0), B_i(p,\alpha)) < \delta$, and the result follows from the definition of δ.

11. Proof of Proposition 5

Our first observation establishes the relation between envy and the value of endowments at equilibrium prices.

Lemma 9. *Let (x,p) be a Walrasian equilibrium with slack $\alpha \in (0,1]$. If i envies j, then $p \cdot (x_j - x_i) > 0$ and $p \cdot (\omega_j - \omega_i) > 0$.*

Proof. Let i envy j, so $u_i(x_j) > u_i(x_i)$. Then utility maximization implies that

$$\alpha + (1-\alpha)p \cdot \omega_j \geq p \cdot x_j > \alpha + (1-\alpha)p \cdot \omega_i \geq p \cdot x_i,$$

where the strict inequality follows because $x_j \in \Delta_-$. So $p \cdot (x_j - x_i) > 0$ and $p \cdot (\omega_j - \omega_i) > 0$. □

Now consider a α-slack Walrasian equilibrium (x,p). Agent i's maximization problem is:

$$\max_{x \in \mathbf{R}^L_+} u_i(x) + \lambda_i(I_i - p \cdot x) + \gamma_i(1 - \mathbf{1} \cdot x)$$

Where $I_i = \alpha + (1-\alpha)p \cdot \omega_i$, λ_i is a multiplier for the budget constraint, and γ_i for the $\sum_l x_{i,l} \leq 1$ constraint.

Utility functions are C^1. The first-order conditions for the maximization problems are then:

$$\partial_l u_i(x_i) - \lambda_i p_l - g_i \begin{cases} = 0 & \text{if } x_{i,l} > 0 \\ \leq 0 & \text{if } x_{i,l} = 0, \end{cases}$$

where $\partial_l u_i(x_i)$ denotes the partial derivative of u_i with respect to $x_{i,l}$.

Observe that if $p \cdot x_i < \alpha + (1-\alpha)p \cdot \omega_i$, then the budget constraint is not binding and $\lambda_i = 0$. As a consequence, $u_i(x_i) = \max\{u_i(z_i) : z_i \in \Delta_-\}$. Let $S = \{i \in [N] : p \cdot x_i < \alpha + (1-\alpha)p \cdot \omega_i\}$ be the set of *satiated* consumers. Let

$U = \{i \in [N] : p \cdot x_i = \alpha + (1-\alpha)p \cdot \omega_i\}$ be the set of *unsatiated*, and observe that we can let $\lambda_i > 0$ for all $i \in U$. Consider the two stage social program:

Stage 1:

$$\max_{\tilde{y} \in (\Delta_-)^S} \sum_{i \in S} u_i(\tilde{y}_i)$$

Stage 2:

$$\max_{\tilde{y} \in (\Delta_-)^U} \sum_{i \in U} \tfrac{1}{\lambda_i} u_i(\tilde{y}_i)$$
$$\sum_{i \in U} \tilde{y}_i \leq \bar{\omega} - \sum_{i \in S} x_i$$

Note that $(x_i)_{i \in S}$ solves Stage 1, while satisfying $\sum_{i \in S} x_i \leq \bar{w}$, and that given $(x_i)_{i \in S}$, $(x_i)_{i \in U}$ solves Stage 2. That this is so follows from the fact that $(x_i)_{i \in U}$ solves the first-order conditions for the Stage 2 problem with Lagrange multiplier p for the constraint that $\sum_{i \in U} \tilde{y}_i \leq \bar{\omega} - \sum_{i \notin S} x_i$.

Now use the assumption that $\sum_{i \in U} x_i \gg 0$. This means that there exists $\bar{t} > 0$ such that if $t \in (0, \bar{t}]$ then the set of $\tilde{y} \in (\Delta_-)^U$ such that $\sum_{i \in U} \tilde{y}_i \leq \bar{\omega} + t(\omega_i - \omega_j) - \sum_{i \notin S} x_i$ is nonempty (and, for constraint qualification, contains an element that satisfies all constraints with slack).

Consider the problem

$$\max_{\tilde{y} \in (\Delta_-^U)} \sum_{i \in U} \tfrac{1}{\lambda_i} u_i(\tilde{y}_i)$$
$$\sum_{i \in U} \tilde{y}_i \leq \bar{\omega} + t(\omega_i - \omega_j) - \sum_{i \in S} x_i$$

Note that for each $t \in (0, \bar{t}]$ there exists $(\nu(t), \gamma(t), \alpha(t))$ such that

$$v(t) = \sup\{\sum_{i \in U} \frac{1}{\lambda_i} u_i \cdot \tilde{y}_i + \nu(t) \cdot (\bar{\omega} - \sum_{i \in S} \tilde{y}_i + t(\omega_i - \omega_j)) - \sum_{i \in U} \tilde{y}_i) + \sum_{i \in U} \gamma_i(t)(1 - \sum_{l \in O} \tilde{y}_{i,l}) + \sum_{i \in U} \alpha_i(t) \tilde{y}_{i,l}.\}$$

Here $\nu(t)$ is the Lagrange multiplier for the constraint that $\sum_{i \in U} \tilde{y}_i \leq \bar{\omega} - \sum_{i \in S} x_i + t(\omega_i - \omega_j)$, while $\gamma(t)$ and $\alpha(t)$ are the Lagrange multipliers for the constraint that $(\tilde{y}_i) \in (\Delta_-)^N$. Choose a selection $(\nu(t), \gamma(t), \alpha(t))$ such that $\nu(0) = p$.

Let $\tilde{\omega} = \bar{\omega} - \sum_{i \in S} x_i$. The saddle point inequalities imply that

$$\begin{aligned}
(t'-t)\nu(t)\cdot(\omega_i - \omega_j) = & \sum_{i\in U} \frac{1}{\lambda_i} u_i(x_i(t')) + \nu(t)\cdot(\tilde{\omega} + t'(\omega_i - \omega_j) - \sum_{i\in U} x_i(t')) \\
& + \sum_{i\in U} \gamma_i(t)(1 - \sum_{l\in O} x_{i,l}(t')) + \sum_{i\in U} \alpha_i(t) x_{i,l}(t') \\
& - \left(\sum_{i\in U} \frac{1}{\lambda_i} u_i(x_i(t')) + \nu(t)\cdot(\tilde{\omega} + t(\omega_i - \omega_j) - \sum_{i\in U} x_i(t')) \right. \\
& \left. + \sum_{i\in U} \gamma_i(t)(1 - \sum_{l\in O} x_{i,l}(t')) + \sum_{i\in U} \alpha_i(t) x_{i,l}(t') \right) \\
& \geq v(t') - v(t)
\end{aligned}$$

Now recall that $\nu(0) = p$. Then Lemma 9, together with the above inequality, imply that

$$0 > p\cdot(\omega_i - \omega_j)t' \geq v(t') - v(0)$$

for all $t' > 0$ with $t' \leq \bar{t}$.

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