The Edgeworth Conjecture with Small Coalitions and Approximate Equilibria in Large Economies

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- Scope of the “competitive hypothesis,” or validity of price-taking assumption.
- New algorithmic “testing” question.

Price-taking behavior

Francis Ysidro Edgeworth 1884

"... the reason why the complex play of competition tends to a simple uniform result – what is arbitrary and indeterminate in contract between individuals becoming extinct in the jostle of competition – is to be sought in a principle which pervades all mathematics, the principle of limit, or law of great numbers as it might perhaps be called."

Competitive hypothesis

- Core convergence theorem (Aumann; Debreu-Scarf): in a large economy, where no agent is "unique," bargaining power dissipates and the outcome of bargaining approximates a Walrasian equilibrium
- Competitive prices emerge as terms of trade in bargaining.

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- Competitive prices emerge as terms of trade in bargaining.
- Requires coailitions of *arbitrary* size.

Our results – I

Coalitions of size

$$\mathcal{O}\left(\frac{h^2\ell}{\varepsilon^2}\right)$$

suffice, where:

- h is the heterogeneity of the economy
- ℓ is the number of goods
- $\varepsilon > 0$ approximation factor.
- We use the Debreu-Scarf replica model.

Our results – II

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Given an economy $\mathcal{E}$ and an allocation x , are there prices p such that (x, p) is a Walrasian equilibrium?

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We provide a poly time algorithm that (under certain sufficient conditions) decides the question.

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We provide a poly time algorithm that (under certain sufficient conditions) decides the question.

Hardness of Walrasian eq.

Context: existing hardness results for Walrasian equilibria: Chen, Dai, Du, and Teng (2009); Codenotti, Saberi, Varadarajan, and Ye (2006); Vazirani and Yannakakis (2010); Garg, Mehta, Vazirani, and Yazdanbod (2017)

Our contribution: finding prices is easy even when finding a W-Eq. is hard. Specifically:

- Leontief utilities
- Piecewise-linear concave utilities

Economies

An *exchange economy* comprises of

- a set of consumers $[h] := \{1, 2, \ldots, h\}$,
- a set of goods, $[\ell] := \{1, 2, \ldots, \ell\}$.

Each consumer i described by

- A utility function $u_i : \mathbb{R}^\ell_+ \mapsto \mathbb{R}$
- An *endowment vector* $\omega_i \in \mathbb{R}^\ell_+$.

An exchange economy $\mathcal{E}$ is a tuple $((u_i, \omega_i))_{i=1}^h$.

Assumptions on u_i

- u_is are continuous and monotone increasing.
- utilities are continuously differentiable
- and α-strongly concave, with $\alpha > 0$: $u : \mathbb{R}^\ell \mapsto \mathbb{R}$, is said to be α-strongly concave within a set $\mathcal{R} \subset \mathbb{R}^\ell$ if

$$u(y) \leq u(x) + \nabla u(x)^T (y - x) - \frac{\alpha}{2} \|y - x\|^2.$$

$\nabla u(x)$ is the gradient of the function u at point x

Allocations

An *allocation* in $\mathcal{E}$ is

$$\overline{x} = (\overline{x}_i)_{i=1}^{h} \in \mathbb{R}_+^{h\ell} \quad \text{st} \quad \sum_{i=1}^{h} \overline{x}_i = \sum_{i=1}^{h} \omega_i.$$

Utility normalization

Utilities are normalized so that $u_i(x_i) \in [0, 1)$ for all consumers $i \in [h]$ and all allocations $(x_i)_i \in \mathbb{R}_+^{h\ell}$.

- An *allocation* in $\mathcal{E}$ is $\overline{x} = (\overline{x}_i)_{i=1}^{h} \in \mathbb{R}_{+}^{h\ell}$, s.t $\sum_{i=1}^{h} \overline{x}_i = \sum_{i=1}^{h} \omega_i$.

The Core

- An *allocation* in $\mathcal{E}$ is $\overline{x} = (\overline{x}_i)_{i=1}^h \in \mathbb{R}_+^{h\ell}$, s.t $\sum_{i=1}^h \overline{x}_i = \sum_{i=1}^h \omega_i$.
- A nonempty subset $S \subseteq [h]$ is a *coalition*.
- $(y_i)_{i \in S}$ is an *S-allocation* if $\sum_{i \in S} y_i = \sum_{i \in S} \omega_i$.

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- $(y_i)_{i \in S}$ is an *S-allocation* if $\sum_{i \in S} y_i = \sum_{i \in S} \omega_i$.
- A coalition S *blocks* the allocation $\overline{x} = (\overline{x}_i)_{i=1}^h$ in $\mathcal{E}$ if $\exists$ an S-allocation $(y_i)_{i \in S}$ s.t $u_i(y_i) > u(\overline{x}_i)$ for all $i \in S$.
- The *core* of $\mathcal{E}$ is the set of all allocations that are not blocked by any coalition.

The κ-core

The *κ-core* of $\mathcal{E}$, for $\kappa \in \mathbb{Z}_+$, is the set of allocations that are not blocked by any coalition of cardinality at most κ.

Note:

- Core: *all* 2^h coalitions
- κ-core: small coalitions
- κ-core: few ($\binom{h}{\kappa}$) coalitions

Equilibrium and approximate equilibrium

A *Walrasian equilibrium* is a pair $(p, \overline{x}) \in \mathbb{R}^{\ell}_{+} \times \mathbb{R}^{h\ell}_{+}$ s.t

1. $p \in \mathbb{R}^{\ell}_{+}$ is a *price vector*
2. $p^T \overline{x}_i = p^T \omega_i$ and, for all bundles $y \in \mathbb{R}^{\ell}_{+}$ with the property that $u_i(y) > u_i(\overline{x}_i)$, we have $p^T y_i > p^T \omega_i$.
3. $\sum_{i=1}^{h} \overline{x}_i = \sum_{i=1}^{h} \omega_i$ (**supply equals the demand**).

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3. $\sum_{i=1}^{h} \overline{x}_i = \sum_{i=1}^{h} \omega_i$ (**supply equals the demand**). i.e $\overline{x} = (\overline{x}_i)_{i \in [h]} \in \mathbb{R}_+^{h\ell}$ is an allocation

Approximate Walrasian equilibrium

A ε-*Walrasian equilibrium* is a pair $(p, \overline{x}) \in \mathbb{R}_+^\ell \times \mathbb{R}_+^{h\ell}$ in which $p \in \Delta$ and

(i) $|p^T \overline{x}_i - p^T \omega_i| \leq \varepsilon$ and

(ii) for any bundle $y \in \mathbb{R}_+^\ell$, with the property that $u_i(y) > u_i(\overline{x}_i)$, we have $p^T y > p^T \omega_i - \varepsilon / h$.

iii) $\overline{x}$ is an allocation (**supply equals the demand**).

Replica economies

Let $\mathcal{E} = ((u_i, \omega_i))_{i \in [h]}$ be an exchange economy.
The *n-th replica* of $\mathcal{E}$, for $n \geq 1$, is the exchange economy $\mathcal{E}^n = ((u_{i,t}, \omega_{i,t}))_{i \in [n], t \in [h]}$, with nh consumers. In $\mathcal{E}^n$ the consumers are indexed by (i, t), with index $i \in [n]$ and type $t \in [h]$, and they satisfy:

$$u_{i,t} = u_t \text{ and } \omega_{i,t} = \omega_t.$$

Equal treatment property

An allocation in $\mathcal{E}^n$ has the *equal treatment property* if all consumers of the same type are allocated identical bundles.

Equal treatment of equals

Let $\mathcal{E} = ((u_i, \omega_i))_{i \in [h]}$ be an exchange economy.

Lemma (Equal treatment property)

Suppose each u_i is strictly monotonic, continuous, and strictly concave. Then, every h-core allocation of $\mathcal{E}^n$ satisfies the equal treatment property.

Core convergence: Debreu-Scarf (1963)

Let $\mathcal{E} = ((u_i, \omega_i))_{i\in[h]}$ be an exchange economy.

Theorem (Debreu-Scarf Core Convergence Theorem)

Suppose u_i is st. monotonic, cont., and strictly quasiconcave.
If the allocation $\overline{x} \in \mathbb{R}_+^{h\ell}$ is in the core of $\mathcal{E}^n$ for all $n \geq 1$,
$\Longrightarrow \exists\ p \in \mathbb{R}_+^{\ell}$ *s.t $(p, \overline{x})$ is a Walrasian equilibrium.*

Main result

Let $\mathcal{E} = ((u_i, \omega_i))_{i \in [h]}$ be an exchange economy with h consumers and ℓ goods.

Theorem

Let $\varepsilon > 0$. Suppose u_i is st. monotonic, C^1, and α-strongly concave. If the allocation $\overline{x}$ is in the κ-core of $\mathcal{E}^n$, for

$$n \geq \kappa \geq \frac{16}{\alpha}\left(\frac{\lambda \ell h}{\varepsilon} + \frac{h^2}{\varepsilon^2}\right).$$

Then $\exists\ p \in \Delta$ s.t $(p, \overline{x})$ is an ε-Walrasian equilibrium).
Here, λ is the Lipschitz constant of the utilities.

Testing

Assume black-box access to utilities and their gradients.

Testing

Let $\mathcal{E} = ((u_i, \omega_i))_{i \in [h]}$ be an exchange economy.

Theorem (Testing Algorithm)

Suppose that each u_i is monotonic, C^1, and strongly concave. Then, there exists a polynomial-time algorithm that, given an allocation $\overline{y}$ in $\mathcal{E}$, decides whether $\overline{y}$ is an ε-Walrasian allocation.

Testing

Remark

Analogous results are possible without strong concavity: Leontief and PLC utilities, for instance.

Ideas in the proof.

Approximate Caratheodory

Theorem

Let $x \in cvh(\{x_1, \ldots, x_K\}) \subseteq \mathbf{R}^n$, $\varepsilon > 0$ *and* p *an integer with* $2 \leq p < \infty$. *Let* $\gamma = \max\{\|x_k\|_p : 1 \leq k \leq K\}$. *Then there is a vector* x' *that is a convex combination of at most*

$$\frac{4p\gamma^2}{\varepsilon}$$

of the vectors $x_1, \ldots, x_K$ *such that* $\|x - x'\|_p < \varepsilon$.

See Barman (2015).

Upper contour sets

Let $\overline{y} = (\overline{y}_i)_{i \in [h]}$ be an allocation.
Let

$$V_i := \left\{ y \in \mathbb{R}^{\ell}_{+} \mid u_i(y) \geq u_i(\overline{y}_i) \right\}$$

be the *upper contour set* of i at $\bar{y}$.
Obs: V_i is closed and convex.

Upper contour sets

Inducing i to buy $\bar{y}_i$ amounts to

- supporting V_i at $\bar{y}_i$ with some prices p_i.
- ensuring that i has the right income

Equilibrium: $p_i = p$ for all i.

Upper contour sets

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- supporting V_i at $\bar{y}_i$ with some prices p_i.
- ensuring that i has the right income

Equilibrium: $p_i = p$ for all i.

The second welfare thm. relies on separating $\sum_i V_i$. from $\sum_i \omega_i$ $\Longrightarrow$ obtain p. Use *transfers* to ensure that income is right.

The Debreu-Scarf relies on separating $\cup_i V_i$. Problem is: $\cup_i V_i$ may not be convex.

Upper contour sets

Let $\eta \in (0,1)$.
Let $V_i^\eta := \left\{y \in \mathbb{R}_+^\ell \mid u_i(y) \geq u_i(\overline{y}_i) + \eta\right\}$ of i at $\bar{y}$.
Let $Q_i^\eta := \left\{z \in \mathbb{R}^\ell \mid u_i(z + \omega_i) \geq u_i(\overline{y}_i) + \eta\right\}$.
By definition, $z \in Q_i^\eta$ iff $(z + \omega_i) \in V_i^\eta$.

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Let $Q_i^\eta := \left\{z \in \mathbb{R}^\ell \mid u_i(z + \omega_i) \geq u_i(\overline{y}_i) + \eta\right\}$.
By definition, $z \in Q_i^\eta$ iff $(z + \omega_i) \in V_i^\eta$.
We also consider $\widehat{Q}_i^\eta$, a bounded subset of Q_i^η; specifically,

$$\widehat{Q}_i^\eta := Q_i^\eta \ \cap \ \left\{z \in \mathbb{R}^\ell \ : \ \|z\| \leq \sqrt{\frac{2(\lambda\ell\delta + 1)}{\alpha}}\right\},$$

Core lemma

Lemma

$$(-\delta)\mathbf{1} \in cvh\left(\bigcup_{i=1}^{h} Q_i^{\eta}\right) \quad \textit{iff} \quad (-\delta)\mathbf{1} \in cvh\left(\bigcup_{i=1}^{h} \widehat{Q}_i^{\eta}\right).$$

Lemma

If $\overline{x} = (\overline{x}_i)_{i\in[h]}$ is in the κ-core of $\mathcal{E}^n$, then

$$(-\delta)\,\mathbf{1} \notin cvh\left(\bigcup_{i=1}^{h} P_i^{\eta}\right).$$

Crucial characterization

Lemma

$$(-\delta)\mathbf{1} \in cvh\left(\bigcup_{i=1}^{h} Q_i^{\eta}\right) \quad \textit{iff} \quad (-\delta)\mathbf{1} \in cvh\left(\bigcup_{i=1}^{h} \widehat{Q}_i^{\eta}\right).$$

Lemma

An allocation $\overline{y}$ is an ε-Walrasian allocation of $\mathcal{E}$ iff

$$(-\delta)\,\mathbf{1} \notin cvh\left(\bigcup_{i=1}^{h} \widehat{Q}_i\right).$$

Piece-wise linear concave: PLC

$$u_i(x) = \min_k \left\{ \sum_j U^k_{i,j} x_j \; + \; T^k_i \right\}$$

Piece-wise linear concave: PLC

$$\Lambda := \max_{i \in [h], x \in \mathbb{R}^{\ell}_{+}} \left\{ \|x - \omega_i\| \;:\; u_i(x) \leq u_i\left(\sum_i \omega_i\right) \right\} \tag{1}$$

Piece-wise linear concave: PLC

$$\widetilde{Q}_i := Q_i \cap \left\{ z \in \mathbb{R}^\ell \mid \; \|z\| \le \Lambda \right\} \tag{2}$$

For each consumer i, the subset $\widetilde{Q}_i$ is compact, convex, and has a nonempty interior.

Lemma

Let $\overline{y}$ be an allocation in an exchange economy $\mathcal{E}$ with PLC utilities. Suppose that the sets Q_i and $\widetilde{Q}_i$, for $i \in [h]$, are as defined above. Then, with parameter $\delta > 0$, we have

$$(-\delta)\mathbf{1} \in cvh\left(\bigcup_{i=1}^{h} Q_i\right) \quad \textit{iff} \quad (-\delta)\mathbf{1} \in cvh\left(\bigcup_{i=1}^{h} \widetilde{Q}_i\right).$$

Piece-wise linear concave

Lemma

An allocation $\overline{y}$ is an ε-Walrasian allocation in a PLC economy $\mathcal{E}$ iff

$$(-\delta)\,\mathbf{1} \notin cvh\left(\bigcup_{i=1}^{h} \widetilde{Q}_i\right).$$

Piece-wise linear concave

Lemma

An allocation $\overline{y}$ is an ε-Walrasian allocation in a PLC economy $\mathcal{E}$ iff

$$(-\delta)\,\mathbf{1} \notin cvh\left(\bigcup_{i=1}^{h} \widetilde{Q}_i\right).$$

Theorem

There exists a polynomial-time algorithm that—given an allocation $\overline{y} = (\overline{y}_i)_{i\in[n]}$ in an exchange economy $\mathcal{E} = ((u_i, \omega_i))_{i\in[n]}$ with PLC utilities—determines whether $\overline{y}$ is an ε-Walrasian allocation, or not.

Literature I

Core convergence:

- Debreu and Scarf (1963),Aumann (1964), Anderson (1978).
- Surveys: Hildenbrand (1974) and Anderson (1992).
- Schmeidler (1972); Grodal (1972); Vind (1972).
- Mas-Colell (1979)

Complexity of core/equilibrium:

- Chen, Dai, Du, and Teng (2009); Codenotti, Saberi, Varadarajan, and Ye (2006); Vazirani and Yannakakis (2010); Garg, Mehta, Vazirani, and Yazdanbod (2017)

- Wooldridge (2012)

Conclusion

- We provide a core convergence result for the κ-core: the set of allocations that cannot be blocked by small coalitions.
- We introduce a new "testing" problem: when is an allocation a (approx.) Walrasian equilibrium allocation.
- The ideas behind our core convergence result furnish us with an algorithm that decides the testing question.

Anderson, R. M. (1978): "An elementary core equivalence theorem," *Econometrica*, pp. 1483–1487.

——— (1992): "The core in perfectly competitive economies," *Handbook of game theory with economic applications*, 1, 413–457.

Aumann, R. J. (1964): "Markets with a continuum of traders," *Econometrica*, pp. 39–50.

Barman, S. (2015): "Approximating nash equilibria and dense bipartite subgraphs via an approximate version of Caratheodory's theorem," in *Proceedings of the forty-seventh annual ACM symposium on Theory of computing*, pp. 361–369. ACM.

Chen, X., D. Dai, Y. Du, and S.-H. Teng (2009): "Settling the Complexity of Arrow-Debreu Equilibria in Markets with Additively Separable Utilities," in *FOCS '09: Proceedings of the 2009 50th Annual IEEE Symposium on Foundations of Computer Science*, pp. 273–282, Washington, DC, USA. IEEE Computer Society.

Codenotti, B., A. Saberi, K. Varadarajan, and Y. Ye (2006): "Leontief economies encode nonzero sum two-player games," in *Proceedings of the seventeenth annual ACM-SIAM symposium on Discrete algorithm*, pp. 659–667. Society for Industrial and Applied Mathematics.

Debreu, G., and H. Scarf (1963): "A limit theorem on the core of an economy," *International Economic Review*, 4(3), 235–246.

References III

Garg, J., R. Mehta, V. V. Vazirani, and S. Yazdanbod (2017): "Settling the complexity of Leontief and PLC exchange markets under exact and approximate equilibria," in *Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing*, pp. 890–901. ACM.

Grodal, B. (1972): "A second remark on the core of an atomless economy," *Econometrica*, pp. 581–583.

Hildenbrand, W. (1974): *Core and Equilibria of a Large Economy*. Princeton university press.

Mas-Colell, A. (1979): "A refinement of the core equivalence theorem," *Economics Letters*, 3(4), 307–310.

Schmeidler, D. (1972): "A remark on the core of an atomless economy," *Econometrica*, 40(3), 579.

References IV

VAZIRANI, V. V., AND M. YANNAKAKIS (2010): "Market Equilibrium under Separable, Piecewise-Linear, Concave Utilities," in *Proc. Innovations in Computer Science*, pp. 156–165, Beijing, China.

VIND, K. (1972): "A third remark on the core of an atomless economy," *Econometrica*, 40(3), 585.

WOOLDRIDGE, G. C. E. E. M. J. (2012): *Computational aspects of cooperative game theory*, vol. 5 of *Synthesis digital library of engineering and computer science.; Synthesis lectures on artificial intelligence and machine learning, 16*. Morgan and Claypool Publishers.