Fairness and efficiency for probabilistic allocations with endowments

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Antonio and Jun:
Discrete allocation
For example

- Jobs to workers
- Courses to students
- Organs to patients
- Schools to children
- Offices to professors.
Desiderata

- Efficiency
- Fairness
- Property rights.
Pareto optimality.

An assignment is efficient if there is no alternative (feasible) assignment that makes everyone better off and at least one agent strictly better off.
Alice *envies* Bob at an assignment if she would like to have what Bob got.

An assignment is *fair* if no agent envies another agent.
Fairness requires randomization.

If Alice and Bob want the same office $\Rightarrow$ flip a coin.
When there is a conflict between efficiency and fairness, policy makers (and society?) often prioritize fairness.

Hence fairness is a priority in market design.

So we’ll work with random assignments.
Pseudomarkets

Can we be fair and efficient?

Yes: use pseudomarkets
Assign workers to jobs.

- $L$ jobs.

- A lottery: $x^i = (x_1^i, x_2^i, \ldots, x_L^i)$

- $x_l^i$ = probability that $i$ is assigned job $l$. 

Pseudomarkets: Hylland and Zeckhauser 1979
Assign workers to jobs.

- $L$ jobs.
- A lottery: $x^i = (x_1^i, x_2^i, \ldots, x_L^i)$
- $x_l^i$ = probability that $i$ is assigned job $l$.
- utility function $u^i(x^i)$
- for ex. $u^i(x^i)$ can be an exp. utility.
A lottery $x^i$ satisfies

$$\sum_{i} x^i \leq 1$$

A lottery is an element of

$$\Delta_- = \{ x \in \mathbb{R}^L_+ : \sum_{j=1}^{L} x_j \leq 1 \}$$

$u^i : \Delta_- \rightarrow \mathbb{R}$ (cont. & mon.)
Agents: $I = \{1, \ldots, N\}$.

Objects: $S = \{s_1, \ldots, s_L\}$.

$u^i : \Delta_- \rightarrow \mathbb{R}$ (cont. & mon.)
An allocation is $x = (x^i)_{i=1}^N$, with $x^i \in \Delta_L$, s.t

$$\sum_{i \in I} x^i_s = 1$$
i envies j at x if \( u^i(x^j) > u^i(x^i) \)

An allocation \( x \) is *fair* if no agent envies another agent at \( x \).
An allocation $x$ is \textit{fair} if no agent envies another agent at $x$. 

$$x^i = (1/L, \ldots, 1/L) \implies \text{no envy}$$
An allocation $x$ is *Pareto optimal* (PO) if there is no allocation $y$ s.t.

$$u^i(y^i) \geq u^i(x^i) \text{ for all } i \text{ and } u^j(y^j) > u^j(x^j)$$

for some $j$. 


Hylland and Zeckhauser (1979)
An *HZ-equilibrium* is a pair \((x, p)\), with \(x \in \Delta_N\) and \(p = (p_s)_{s \in S} \geq 0\) s.t.

1. \(\sum_{i=1}^N x^i = (1, \ldots, 1)\)
2. \(x^i\) solves

\[
\max \{u^i(z^i) : z^i \in \Delta_- \text{ and } p \cdot z^i \leq 1\}
\]

Condition (1): supply = demand.
Condition (2): \(x^i\) is \(i\)'s demand at prices \(p\) and income = 1.

Observe:

- Income is independent of prices
- Not a “closed” model (Monopoly money).
Suppose that each $u^i$ is linear (expected utility).

**Theorem (Hylland and Zeckhauser (1979))**

*There is a HZ equilibrium allocation. It is envy-free and PO.*
This paper:

Fair assignment with endowments.
Why endowments?

- Endowments are relevant for *any* problem where we don’t start from scratch.
- Existing allocation matters. Want agents to buy into market design, hence respect property rights.
Why endowments?

- Endowments are relevant for *any* problem where we don’t start from scratch.
- Existing allocation matters. Want agents to buy into market design, hence respect property rights.
- School choice:
  - Property rights are captured by priorities.
  - As property rights, priorities are equivocal; not transparent.
  - Endowments are explicit property rights.
  - For ex., guarantee a:
    1. chance at a good school;
    2. neighborhood school;
    3. slot for a sibling.
This paper:
- Assignment with endowments
- Make agents *unequal*
- Conflict between no-envy and property rights.
No envy: fairness for equals
Agents have unequal endowments
No envy may violate property rights.
This paper:

▶ We propose a notion of fairness for unequally endowed agents
▶ Prove it can be achieved with efficiency and individual rationality.
▶ Can be obtained as a market outcome.
▶ And respecting general constraint structures.
Related Literature

- Justified envy w/endowments: Yilmaz (2010)

More references in the paper...
Each $i$ has an *endowment* $\omega^i \in \Delta$.

$\omega^i$ is an initial lottery.

Suppose that $\sum_i \omega^i = (1, \ldots, 1)$.

For example, suppose schools are allocated via a lottery. Admission probabilities reflect: neighborhood school (walk-zone priority), sibling priority, or test scores.
Agents: \( I = \{1, \ldots, N\} \).

Objects: \( S = \{s_1, \ldots, s_L\} \). Suppose \( N = L \).

For each \( i \in I \),

\[ u^i : \Delta_- \rightarrow \mathbb{R} \]

\[ \omega^i \in \Delta \]

\[ \sum_i \omega^i = (1, \ldots, 1) \]
A *Walrasian equilibrium* is a pair \((x, p)\) with \(x \in \Delta^N\), \(p \geq 0\) s.t.

1. \(\sum_{i=1}^{N} x^i = \sum_{i=1}^{N} \omega^i\); and
2. \(x^i\) solves

\[
\text{Max} \{ u^i(z^i) : z^i \in \Delta_- \text{ and } p \cdot z^i \leq p \cdot \omega^i \}
\]
Proposition (Hylland and Zeckhauser (1979))

*There are economies in which all agents’ utility functions are expected utility, that posses no Walrasian equilibria.*
Budget set

$p$

$\omega^i$
Budget set

\[ \omega_i \]

\((1, 1)\)

simplex

\(p\)

\(\omega^i\)
no Walras’ Law
non-responsive demand

\[ \omega^i \]
HZ Example

3 agents; exp. utility

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Endowments: $\omega^i = (1/3, 2/3)$. 
3 agents; exp. utility

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Endowments: $\omega^i = (1/3, 2/3)$.

Obvious allocation:

$x^1 = x^2 = (1/2, 1/2)$

$x^3 = (0, 1)$
HZ Example

Obvious allocation

\( \omega^i \)

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Fairness & Efficiency
HZ Example

\[ \omega^i \]

1/2 2/3

1/3

Echenique-Miralles-Zhang  Fairness & Efficiency
HZ Example

Echenique-Miralles-Zhang Fairness & Efficiency
HZ Example

\[ \omega^i \]

\[ \frac{1}{2} \] \quad \[ \frac{1}{3} \] \quad \[ \frac{2}{3} \]
Moreover, . . .

- the first welfare theorem fails.
- There are Pareto ranked Walrasian equilibria.
Our results
Let $x$ be an allocation.

- $x$ is **weak Pareto optimal** (wPO) if $\nexists$ an allocation $y$ s.t. $u^i(y^i) > u^i(x^i)$ for all $i$.

- $\varepsilon$-**weak Pareto optimal** ($\varepsilon$-PO), for $\varepsilon > 0$, if $\nexists$ an allocation $y$ s.t. $u^i(y^i) > u^i(x^i) + \varepsilon$ for all $i$. 
Property rights

Let $x$ be an allocation.

- $x$ is **acceptable** to $i$ if $u^i(x^i) \geq u^i(\omega^i)$.
- $x$ is **individually rational** (IR) if it is acceptable to all agents.
Justified envy

\[ i \text{ envies } j \text{ at } x \text{ if } u^i(x^j) > u^i(x^i). \]

Such envy will be tolerated (i.e. not be justified) only if \( j \)'s endowment is “good enough.”
i envies j at x if $u^i(x^j) > u^i(x^i)$.

Such envy will be tolerated (i.e. not be justified) only if $j$ regards $x^i$ as \textit{unacceptable}. 
Justified envy

\[ i \text{ envies } j \text{ at } x \text{ if } u^i(x^j) > u^i(x^i). \]

Such envy will be tolerated (i.e. not be justified) only if
\[ u^j(\omega^j) > u^i(x^i) \]
i has *justified envy* towards j at allocation x if

\[ u^i(x^j) > u^i(x^i) \text{ and } u^j(x^i) \geq u^j(\omega^j). \]
Let \( x \) be an allocation.

\( x \) has *no justified envy* (NJE) if no agent has justified envy towards any other agent at \( x \).
Observe: NJE and IR imply *equal treatment of equals*. 
Let $x$ be an allocation.

$x$ has *no justified envy* (NJE) if no agent has justified envy towards any other agent at $x$. 
\begin{itemize}
  \item $i$ has \textit{strong justified envy} (SJE) towards $j$ at $x$ if $u^i(x^j) > u^i(x^i)$ and $u^j(x^i) > u^j(\omega^j)$.
  \item For $\varepsilon > 0$, $i$ has \textit{$\varepsilon$-justified envy} ($\varepsilon$-JE) towards $j$ at $x$ if $u^i(x^j) > u^i(x^i)$ and $u^j(x^i) > u^j(\omega^j) - \varepsilon$.
\end{itemize}
Justified envy

no $\varepsilon$-justified envy $\implies$ no justified envy $\implies$ no strong just. envy
Theorem

Suppose utility functions are concave.

1. ∃ an allocation that is $\varepsilon$-IR, $\varepsilon$-PO and has no $\varepsilon$-justified envy;
2. ∃ an allocation that is IR, wPO and has no strong justified envy.
3. Moreover, if utility functions are expected utility ∃ an allocation that is IR, PO and has no strong justified envy.
Theorem

Suppose utility functions are quasi-concave, and that

♠

Then there exists continuous functions $m^i : \Delta \to \mathbb{R}_+$ and $(x, p) = ((x^i)_{i=1}^I, p) \in (\Delta^I_-) \times \Delta$, such that

1. $\sum_i x^i = \sum_i \omega^i$ (x is an allocation; or, "supply equals demand").

2. x is Pareto optimal, individually rational and has no justified envy.

3. $x^i \in \text{argmax}\{u^i(z^i) : z^i \in \Delta_- \text{ and } p \cdot z^i \leq m^i(p)\}$
- Given as primitive a set $A^C$ of allocations.
- The *feasible* allocations.
- Assume $A^C$ is convex and compact.

For example:
- Distributional constraints.
- Geographical constraints.
- etc.
i has an *justified envy* towards j at an allocation $x \in \mathcal{A}^c$ if

$$u^i(x^j) > u^i(x^i), \quad u^j(x^i) \geq u^j(\omega^j) \text{ and } x_{i\leftrightarrow j} \in \mathcal{A}^c.$$
Constraints

\( i, j \in I \) are of equal type if

\[
\text{for all } x \in A^c, \ x_i \leftrightarrow j \in A^c.
\]
Theorem

Suppose agents’ utility functions are concave and that $\omega \in \mathcal{A}^C$.

1. For any $\varepsilon > 0$, there exists an allocation that is $\varepsilon$-IR, $\varepsilon$-PO and has no equal-type $\varepsilon$-justified envy;

2. There exists an allocation that is IR, wPO, and has no strong equal-type justified envy.
Ideas
Theorem

Suppose utility functions are concave.

1. ∃ an allocation that is $\varepsilon$-IR, $\varepsilon$-PO and has no $\varepsilon$-justified envy;

2. ∃ an allocation that is IR, wPO and has no strong justified envy.

3. Moreover, if utility functions are expected utility ∃ an allocation that is IR, PO and has no strong justified envy.
Consider problem

$$\text{Max} \sum_{i} \lambda_i u^i(x_i)$$

s.t. $x$ is an allocation.

Obtain a NJE allocation from this problem

by choosing right welfare weights, $(\lambda_i) \in \Delta^N$.

(Actual proof uses an approximation to this problem, hence the $\varepsilon$).
KKM Lemma

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Fairness & Efficiency
Theorem

Suppose utility functions are quasi-concave, and that

\[ \therefore \]

Then there exists continuous functions \( m^i : \Delta \to \mathbb{R}_+ \) and \( (x, p) = ((x^i)_{i=1}^I, p) \in (\Delta_-^I) \times \Delta \), such that

1. \( \sum_i x^i = \sum_i \omega^i \) (\( x \) is an allocation; or, “supply equals demand”).
2. \( x \) is Pareto optimal, individually rational and has no justified envy.
3. \( x^i \in \argmax\{u^i(z^i) : z^i \in \Delta_- \text{ and } p \cdot z^i \leq m^i(p)\} \)

\[ \therefore \exists l \text{ s.t. for any } i \in I \text{ and } x^i \in \Delta_-^l, \text{ decreasing consumption of any object } k \neq l \text{ in favor of } l \text{ leads to an increase in } u^i; \text{ and } \omega_i^l > 0. \]
\[ e^i(v, p) = \inf \{ p \cdot x : u^i(x) \geq v \}, \]
for \( p \in \Delta^L \) and \( v \in \mathbb{R} \).

Let \( v^i = \sup u^i(\Delta^L) \) be the utility of agent \( i \) when she is satiated.
For any scalar $m \geq 0$ and $p \in \Delta^L$, let

$$\mu^i(m, p) = \text{median}(\{e^i(u^i(\omega^i), p), m, e^i(v^i, p)\}).$$

Consider the function

$$\varphi(m, p) = \sum_i \mu^i(m, p) - \sum_i p \cdot \omega^i.$$

Observe that

- $e^i(u^i(\omega^i), p) \leq e^i(v^i, p)$.
- $\mu^i$ is continuous and $m \mapsto \mu^i(m, p)$ weakly monotone increasing.
- $\varphi$ is continuous and $m \mapsto \varphi(m, p)$ weakly monotone increasing.
- $\varphi(m, p) \leq 0$ for $m \geq 0$ small enough as $e^i(u^i(\omega^i), p) \leq p \cdot \omega^i$. 

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Fairness & Efficiency
in the case that \( \sum_i e^i(v^i, p) < \sum_i p \cdot \omega^i \), we let \( m^i(p) = e^i(v^i, p) \).

in the case that \( \sum_i e^i(v^i, p) \geq \sum_i p \cdot \omega^i \), we have that \( \varphi(m, p) \leq 0 \) for \( m \geq 0 \) small enough, and \( \varphi(m, p) \geq 0 \) for \( m \geq 0 \) large enough. So \( \exists m^* \geq 0 \) with \( \varphi(m^*, p) = 0 \). Now let \( m^i(p) = \mu^i(m^*, p) \).
Suppose that $i$ envies $j$ at $x^*$. This implies that $i$ is not satiated, hence $m^i(p^*) < e^i(v^i, p^*)$.

It also implies that $m^i(p^*) < m^j(p^*)$ as $m^i(p^*) < p^* \cdot x^j = m^j(p^*)$.

On can then show that, by defn. of $m^j$, $m^j(p^*) = e^j(u^j(\omega^j), p^*)$. 
We obtain that
\[ p^* \cdot x^i = m^i(p^*) < m^j(p^*) = e^j(u^j(\omega^j), p^*), \]
and hence \( u^j(x^i) < u^j(\omega^j) \) by definition of expenditure function. So \( i \)'s envy is not justified.