Savage in the Market

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Math. Econ. Conference – Wisconsin
September 27, 2014

- Model / Utility
- Data / Behavior

This paper:

- SEU
- Market behavior

Utility and behavior

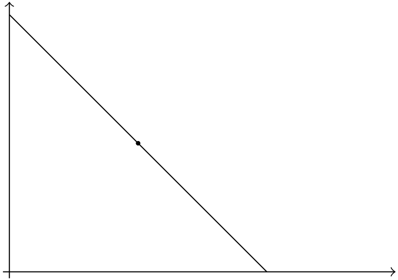
Model:

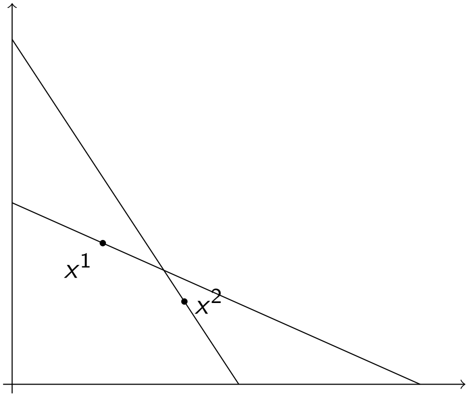
$$\max_{x \in \mathbf{R}^S_+} \quad U(x)$$

$$p \cdot x \leq I$$

Utility and behavior

Market behavior:




x^1
x^2

Utility and behavior

- Q: When is observable behavior consistent with utility max.?
- A: When SARP is satisfied.

This paper: *Subjective Expected Utility (SEU)*

$$\max_{x \in \mathbf{R}^S_+} \quad U(x)$$

$$p \cdot x \leq I$$

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$$\max_{x \in \mathbf{R}_+^S} \quad U(x)$$

$$p \cdot x \leq I$$

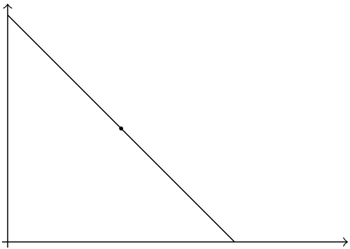
Where

$$U(x) = \sum_{s \in S} \mu_s u(x_s)$$

- $u : \mathbf{R}_+ \to \mathbf{R}$ st. inc. and concave;
- $\mu \in \Delta(S)$ a subjective prior.

This paper.

Market behavior:



- State-contingent consumption (monetary acts);
- complete markets;

This paper.

- Q: When is observable behavior consistent with SEU?
- A: When SARSEU is satisfied.

Warmup

Warmup

The 2×2 case.

- 2 states
- 2 observations

What is the meaning of this:

$$\max \mu_1 u(x_1) + \mu_2 u(x_2)$$
$$p_1 x_1 + p_2 x_2 \leq I$$

model for market behavior ?

Unobservables:

- Utility $u : \mathbf{R}_+ \to \mathbf{R}$
- Prior (μ_1, μ_2)

Observable:

- choices at different budgets

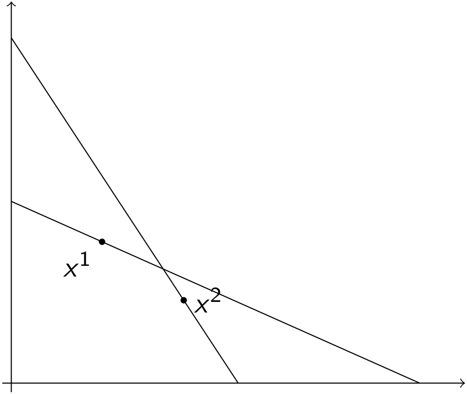
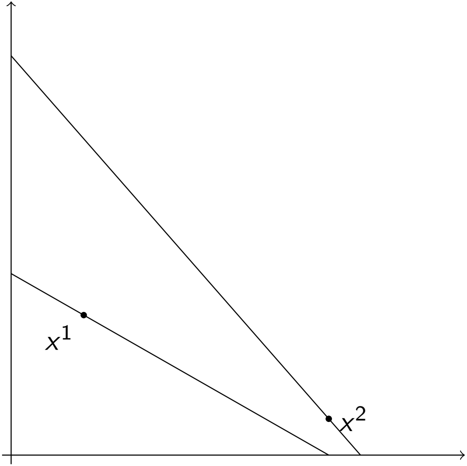
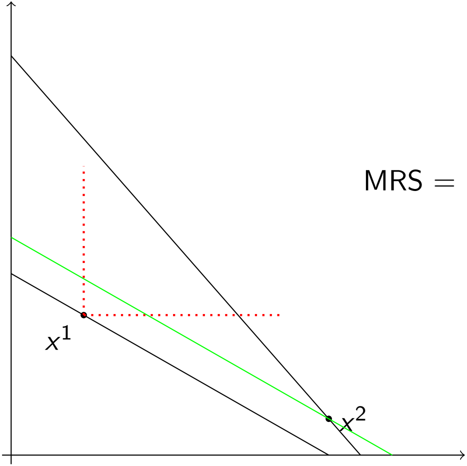
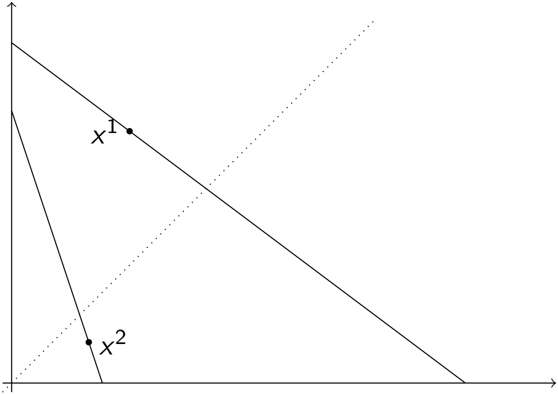


Figure : A violation of WARP.


x^1
x^2



$$\text{MRS} = \frac{\mu_{s_1} u'(x_{s_1})}{\mu_{s_2} u'(x_{s_2})}$$


x^1
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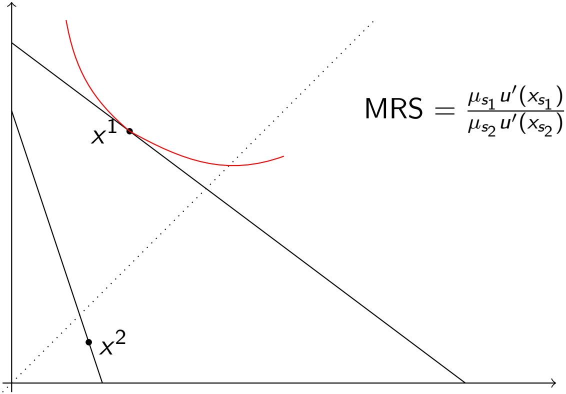
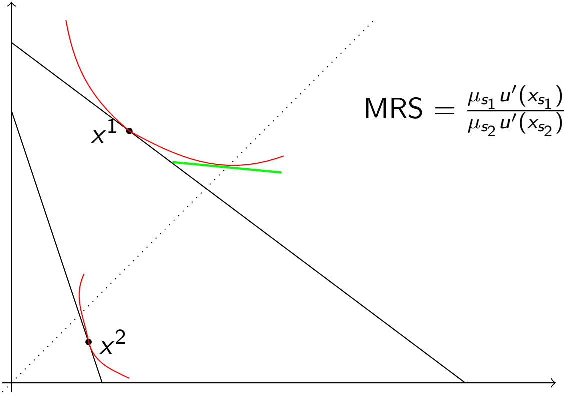
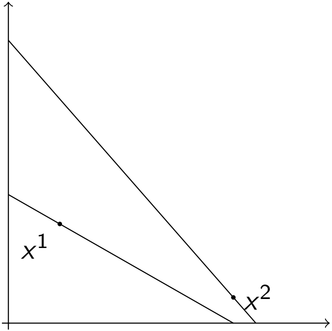


$$\text{MRS} = \frac{\mu_{s_1} u'(x_{s_1})}{\mu_{s_2} u'(x_{s_2})}$$

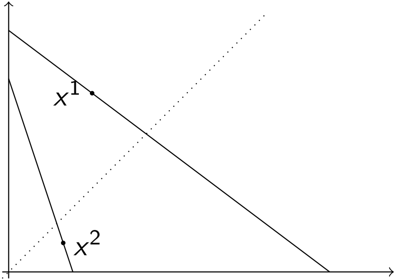


$$\text{MRS} = \frac{\mu_{s_1} u'(x_{s_1})}{\mu_{s_2} u'(x_{s_2})}$$

Axiom 1
Not:



Axiom 2
Not:



END of Warmup

Now: K observations and S states.

Main theorem:
A dataset is SEU rationalizable iff it satisfies the Strong Axiom of Revealed Subjective Expected Utility (SARSEU).

Plug

Echenique, Imai, Saito (2014)

- Discounting: $\sum \delta^t u(x_t)$
- Quasi-hyperbolic discounting $u(x_0) + \beta \sum \delta^t u(x_t)$.
- Empirical application to Andreoni-Sprenger's data.

Model

- Finite set S of states.
- Monetary acts: $x \in \mathbf{R}^S_+$.
- Price vectors: $p \in \mathbf{R}^S_{++}$

Notation: S is also the number of states.

Data

A *dataset* is a collection $(x^k, p^k)_{k=1}^K$ s.t.

- x^k is a monetary act;
- p^k is a price vector.

Notation

Let

- $\Delta^S_{++} = \{\mu \in \mathbf{R}^S_{++} | \sum_{s=1}^S \mu_s = 1\}$
- $\mathcal{C} = \{u : \mathbf{R}_+ \to \mathbf{R} | u \text{ is st. increasing and concave}\}$
- $B(p, I) = \{y \in \mathbf{R}^S_+ | p \cdot y \leq I\}$

Model

SEU

$$\max_{x \in \mathbf{R}^S_+} \quad \sum_{s \in S} \mu_s u(x_s)$$
$$\text{s.t} \sum_{s \in S} p_s x_s \leq I$$

SEU rational

$(x^k, p^k)_{k=1}^K$ is *subjective exp. utility rational (SEU rational)* if

- $\exists \mu \in \Delta_{++}^S$;
- and $u \in \mathcal{C}$ s.t.

$$\sum_{s \in S} \mu_s u(y_s) \leq \sum_{s \in S} \mu_s u(x_s^k),$$

for all $y \in B(p^k, p^k \cdot x^k)$ and all k.

Previous work:

- Varian
- Green & Srivastava
- Kubler, Selden & Wei

All assume observable μ.

Derive SARSEU; $K = 1$ and μ is *known*.

Derivation of SARSEU.

- $K = 1$
- μ *objective and known*
- u differentiable.

Derive SARSEU; $K = 1$ and μ is *known*.

$$\max_{x \in \mathbf{R}_+^S} \sum_{s \in S} \mu_s u(x_s)$$
$$\sum_{s \in S} p_s x_s \leq I$$

FOC:

$$\mu_s u'(x_s) = \lambda p_s$$
$$u'(x_s) = \lambda (p_s / \mu_s) = \lambda \rho_s$$

Here ρ is observable.

Derive SARSEU; $K = 1$ and μ is *known*.

$$u'(x_s) = \lambda(p_s/\mu_s) = \lambda\rho_s$$

So,

$$\frac{u'(x_s)}{u'(x_{s'})} = \frac{\cancel{\lambda}\rho_s}{\cancel{\lambda}\rho_{s'}} = \frac{\rho_s}{\rho_{s'}}$$

Derive SARSEU; $K = 1$ and μ is *known*.

$$u'(x_s) = \lambda(p_s/\mu_s) = \lambda\rho_s$$

So,

$$\frac{u'(x_s)}{u'(x_{s'})} = \frac{\cancel{\lambda}\rho_s}{\cancel{\lambda}\rho_{s'}} = \frac{\rho_s}{\rho_{s'}}$$

Axiom (Downward sloping demand):

$$x_s > x_{s'} \Rightarrow \frac{\rho_s}{\rho_{s'}} \leq 1$$

Derive SARSEU - general K and subjective μ

$$\max_{x \in \mathbf{R}_+^S} \sum_{s \in S} \mu_s u(x_s)$$
$$\sum_{s \in S} p_s x_s \leq I$$

FOC:

$$\mu_s u'(x_s) = \lambda p_s.$$

Derive SARSEU - general K and subjective μ

$$\max_{x \in \mathbf{R}^S_+} \sum_{s \in S} \mu_s u(x_s)$$
$$\sum_{s \in S} p_s x_s \leq I$$

FOC:

$$\mu_s u'(x_s) = \lambda p_s.$$

Hence,

$$\frac{u'(x^k_s)}{u'(x^{k'}_{s'})} = \frac{\mu_{s'}}{\mu_s} \frac{\lambda^k}{\lambda^{k'}} \frac{p^k_s}{p^{k'}_{s'}}.$$

$$\frac{u'(x_s^k)}{u'(x_{s'}^{k'})} = \frac{\mu_{s'}}{\mu_s}\frac{\lambda^k}{\lambda^{k'}}\frac{p_s^k}{p_{s'}^{k'}}.$$

Idea: Choose $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})$ so that unobservable μ_s and λ^k cancel out.

Example

Choose:

$$x_{s_1}^{k_1} > x_{s_2}^{k_2}, \quad x_{s_2}^{k_3} > x_{s_3}^{k_1}, \quad \text{and } x_{s_3}^{k_2} > x_{s_1}^{k_3}.$$

Then:

$$\frac{u'(x_{s_1}^{k_1})}{u'(x_{s_2}^{k_2})} \cdot \frac{u'(x_{s_2}^{k_3})}{u'(x_{s_3}^{k_1})} \cdot \frac{u'(x_{s_3}^{k_2})}{u'(x_{s_1}^{k_3})} = \left(\frac{\mu_{s_2}}{\mu_{s_1}} \frac{\lambda^{k_1}}{\lambda^{k_2}} \frac{p_{s_1}^{k_1}}{p_{s_2}^{k_2}}\right) \cdot \left(\frac{\mu_{s_3}}{\mu_{s_2}} \frac{\lambda^{k_3}}{\lambda^{k_1}} \frac{p_{s_2}^{k_3}}{p_{s_3}^{k_1}}\right) \cdot \left(\frac{\mu_{s_1}}{\mu_{s_3}} \frac{\lambda^{k_2}}{\lambda^{k_3}} \frac{p_{s_3}^{k_2}}{p_{s_1}^{k_3}}\right)$$

Example

Choose:

$$x_{s_1}^{k_1} > x_{s_2}^{k_2}, \quad x_{s_2}^{k_3} > x_{s_3}^{k_1}, \quad \text{and } x_{s_3}^{k_2} > x_{s_1}^{k_3}.$$

Then:

$$\frac{u'(x_{s_1}^{k_1})}{u'(x_{s_2}^{k_2})} \cdot \frac{u'(x_{s_2}^{k_3})}{u'(x_{s_3}^{k_1})} \cdot \frac{u'(x_{s_3}^{k_2})}{u'(x_{s_1}^{k_3})} = \left(\frac{\mu_{s_2}}{\cancel{\mu_{s_1}}} \frac{\cancel{\lambda^{k_1}}}{\lambda^{k_2}} \frac{p_{s_1}^{k_1}}{p_{s_2}^{k_2}} \right) \cdot \left(\frac{\mu_{s_3}}{\mu_{s_2}} \frac{\lambda^{k_3}}{\cancel{\lambda^{k_1}}} \frac{p_{s_2}^{k_3}}{p_{s_3}^{k_1}} \right)$$

$$\cdot \left(\frac{\cancel{\mu_{s_1}}}{\mu_{s_3}} \frac{\lambda^{k_2}}{\lambda^{k_3}} \frac{p_{s_3}^{k_2}}{p_{s_1}^{k_3}} \right)$$

$$= \frac{p_{s_1}^{k_1}}{p_{s_2}^{k_2}} \frac{p_{s_2}^{k_3}}{p_{s_3}^{k_1}} \frac{p_{s_3}^{k_2}}{p_{s_1}^{k_3}}$$

So by concavity of u,

$$\frac{p_{s_1}^{k_1}}{p_{s_2}^{k_2}} \frac{p_{s_2}^{k_3}}{p_{s_3}^{k_1}} \frac{p_{s_3}^{k_2}}{p_{s_1}^{k_3}} \leq 1$$

SARSEU

(Strong Axiom of Revealed Subjective Utility (SARSEU))

For any $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^{n}$ *s.t.*

1. $x_{s_i}^{k_i} > x_{s_i'}^{k_i'}$
2. *s appears as s_i (on the left of the pair) the same number of times it appears as s_i' (on the right);*
3. *k appears as k_i (on the left of the pair) the same number of times it appears as k_i' (on the right):*

$$\prod_{i=1}^{n} \frac{p_{s_i}^{k_i}}{p_{s_i'}^{k_i'}} \leq 1.$$

Main result

Theorem

A dataset is SEU rational if and only if it satisfies SARSEU.

The 2×2 case again

The 2×2 case again

Data:

$$\frac{u'(x_{s_1}^{k_1})}{u'(x_{s_2}^{k_1})} \frac{u'(x_{s_2}^{k_2})}{u'(x_{s_1}^{k_2})} = \frac{p_{s_1}^{k_1}}{p_{s_2}^{k_1}} \frac{p_{s_2}^{k_2}}{p_{s_1}^{k_2}}$$

Two cases:

The 2×2 case again

Data:

$$\frac{u'(x_{s_1}^{k_1})}{u'(x_{s_2}^{k_1})}\frac{u'(x_{s_2}^{k_2})}{u'(x_{s_1}^{k_2})} = \frac{p_{s_1}^{k_1}}{p_{s_2}^{k_1}}\frac{p_{s_2}^{k_2}}{p_{s_1}^{k_2}}$$

Two cases:

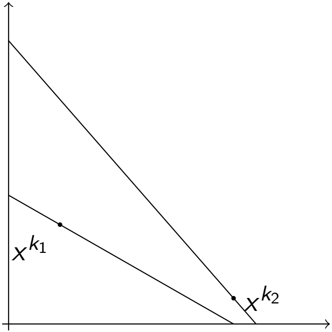
$$x_{s_1}^{k_1} > x_{s_2}^{k_1} \text{ and } x_{s_2}^{k_2} > x_{s_1}^{k_2} \Rightarrow \frac{p_{s_1}^{k_1}}{p_{s_2}^{k_1}}\frac{p_{s_2}^{k_2}}{p_{s_1}^{k_2}} \leq 1$$

$$x_{s_1}^{k_1} > x_{s_1}^{k_2} \text{ and } x_{s_2}^{k_2} > x_{s_2}^{k_1} \Rightarrow \frac{p_{s_1}^{k_1}}{p_{s_2}^{k_1}}\frac{p_{s_2}^{k_2}}{p_{s_1}^{k_2}} \leq 1$$

The 2×2 case again

$$\frac{u'(x_{s_1}^{k_1})}{u'(x_{s_2}^{k_1})}\frac{u'(x_{s_2}^{k_2})}{u'(x_{s_1}^{k_2})} = \frac{p_{s_1}^{k_1}}{p_{s_2}^{k_1}}\frac{p_{s_2}^{k_2}}{p_{s_1}^{k_2}}$$

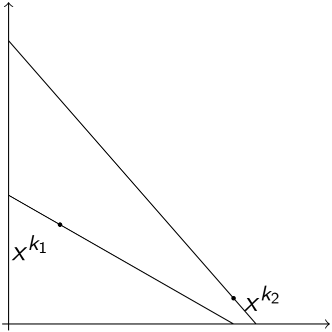
$$x_{s_1}^{k_1} > x_{s_1}^{k_2} \text{ and } x_{s_2}^{k_2} > x_{s_2}^{k_1} \Rightarrow \frac{p_{s_1}^{k_1}}{p_{s_2}^{k_1}}\frac{p_{s_2}^{k_2}}{p_{s_1}^{k_2}} \leq 1$$



The 2×2 case again

$$\frac{u'_{s_1}(x_{s_1}^{k_1})}{u'_{s_2}(x_{s_2}^{k_1})}\frac{u'_{s_2}(x_{s_2}^{k_2})}{u'_{s_1}(x_{s_1}^{k_2})} = \frac{p_{s_1}^{k_1}}{p_{s_2}^{k_1}}\frac{p_{s_2}^{k_2}}{p_{s_1}^{k_2}}$$

$$x_{s_1}^{k_1} > x_{s_1}^{k_2} \text{ and } x_{s_2}^{k_2} > x_{s_2}^{k_1} \Rightarrow \frac{p_{s_1}^{k_1}}{p_{s_2}^{k_1}}\frac{p_{s_2}^{k_2}}{p_{s_1}^{k_2}} \leq 1$$



(Strong Axiom of Revealed Subjective Utility (SARSEU))

For any $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^n$ *s.t.*

1. $x_{s_i}^{k_i} > x_{s_i'}^{k_i'}$
2. *s appears as* s_i *(on the left of the pair) the same number of times it appears as* s_i' *(on the right);*
3. *k appears as* k_i *(on the left of the pair) the same number of times it appears as* k_i' *(on the right):*

$$\prod_{i=1}^{n} \frac{p_{s_i}^{k_i}}{p_{s_i'}^{k_i'}} \leq 1.$$

(Strong Axiom of Revealed State-dependent Utility)

For any $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^n$ *s.t.*

1. $x_{s_i}^{k_i} > x_{s_i'}^{k_i'}$
2. $s_i = s_i'$.
3. *k appears as k_i (on the left of the pair) the same number of times it appears as k_i' (on the right):*

$$\prod_{i=1}^{n} \frac{p_{s_i}^{k_i}}{p_{s_i'}^{k_i'}} \leq 1.$$

Equivalently . . .

(Strong Axiom of Revealed State-dependent Utility)

For any cycle:

$$x_{s_1}^{k_1} > x_{s_1}^{k_2}$$
$$x_{s_2}^{k_2} > x_{s_2}^{k_3}$$
$$\vdots$$
$$x_{s_n}^{k_n} > x_{s_n}^{k_1},$$

it holds that:

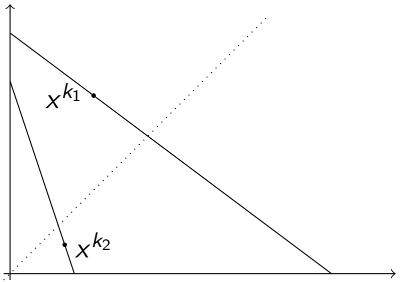
$$\prod_{i=1}^{n} \frac{p_{s_i}^{k_i}}{p_{s_i}^{k_{i+1}}} \leq 1$$

(using addition mod n).

The 2×2 case again

$$\frac{u'(x_{s_1}^{k_1})}{u'(x_{s_2}^{k_1})}\frac{u'(x_{s_2}^{k_2})}{u'(x_{s_1}^{k_2})} = \frac{p_{s_1}^{k_1}}{p_{s_2}^{k_1}}\frac{p_{s_2}^{k_2}}{p_{s_1}^{k_2}}$$

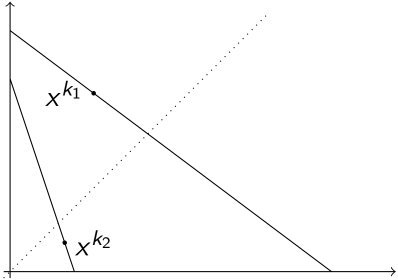
$$x_{s_1}^{k_1} > x_{s_2}^{k_1} \text{ and } x_{s_2}^{k_2} > x_{s_1}^{k_2} \Rightarrow \frac{p_{s_1}^{k_1}}{p_{s_2}^{k_1}}\frac{p_{s_2}^{k_2}}{p_{s_1}^{k_2}} \leq 1$$



The 2×2 case again

$$\frac{u'_{k_1}(x_{s_1}^{k_1})}{u'_{k_1}(x_{s_2}^{k_1})} \frac{u'_{k_2}(x_{s_2}^{k_2})}{u'_{k_2}(x_{s_1}^{k_2})} = \frac{p_{s_1}^{k_1}}{p_{s_2}^{k_1}} \frac{p_{s_2}^{k_2}}{p_{s_1}^{k_2}}$$

$$x_{s_1}^{k_1} > x_{s_2}^{k_1} \text{ and } x_{s_2}^{k_2} > x_{s_1}^{k_2} \Rightarrow \frac{p_{s_1}^{k_1}}{p_{s_2}^{k_1}} \frac{p_{s_2}^{k_2}}{p_{s_1}^{k_2}} \leq 1$$



Discussion

- Checking SARSEU
- $\exists$ data
- Prob. sophistication (Epstein)
- Maxmin
- Objective EU
- Savage

Checking SARSEU

Proposition

There is an algorithm that decides (in polynomial time) whether a dataset satisfies SARSEU.

Data

Need:

- obj. identifiable states
- complete asset markets (and no-arbitrage)

Turns out such data are routinely used in empirical finance.

Recent example: S. Ross "*The recovery theorem*" (J. of Finance, forth.). Such data is also used by Rubinstein (1998), Ait-Sahalia and Lo (1998) and many others.

Epstein (2000)

Necessary Condition for prob. sophistication: if $\exists$ (x, p) and (x', p')

$$\left[\begin{array}{l} \text{(i) } p_1 \geq p_2 \quad \text{and} \quad p'_1 \leq p'_2 \text{ with at least one strict ineq.} \\ \text{(ii) } x_1 > x_2 \;\; \text{and} \;\; x'_1 < x'_2 \end{array} \right]$$

$\Rightarrow$ Not Probability Sophisticated

Epstein (2000)

Necessary Condition for prob. sophistication: if $\exists$ (x, p) and (x', p')

$$\left[\begin{array}{l} \text{(i) } p_1 \geq p_2 \quad \text{and} \quad p_1' \leq p_2' \text{ with at least one strict ineq.} \\ \text{(ii) } x_1 > x_2 \;\; \text{and} \;\; x_1' < x_2' \end{array} \right]$$

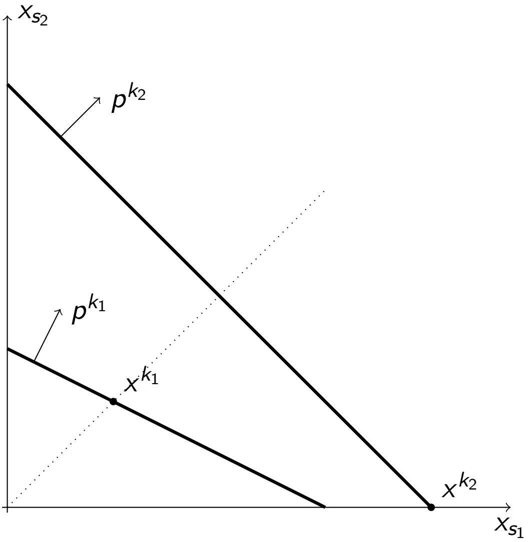
$\Rightarrow$ Not Probability Sophisticated

$\{(x_1, x_2), (x_2', x_1')\}$ satisfy conditions in SARSEU: so must have

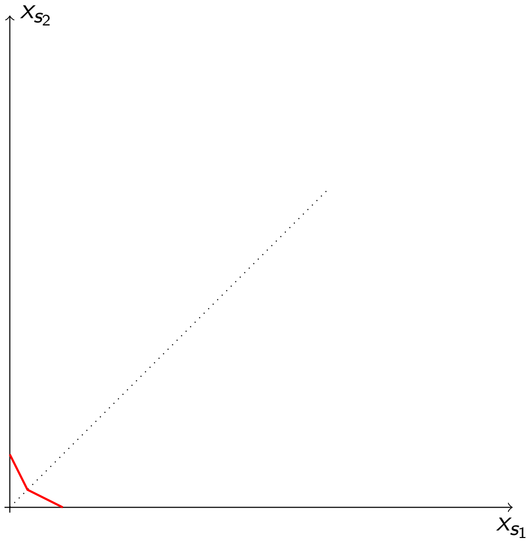
$$\frac{p_1}{p_2}\frac{p_2'}{p_1'} \leq 1,$$

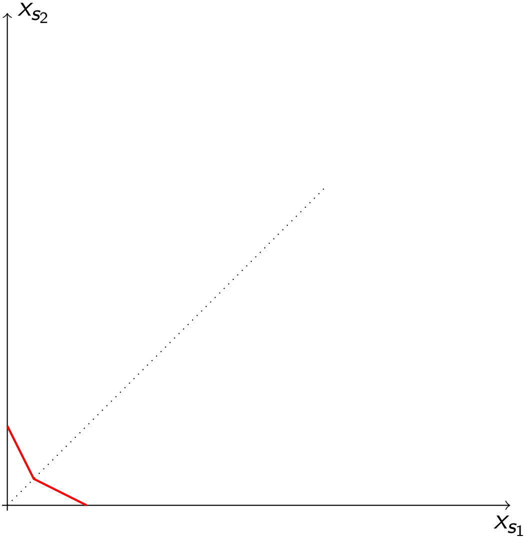
hence can't violate Epstein's condition.

A probabilistically sophisticated data set violating SARSEU.



x_{s_2}
p^{k_2}
p^{k_1}
x^{k_1}
x^{k_2}
x_{s_1}


x_{s_2}
x_{s_1}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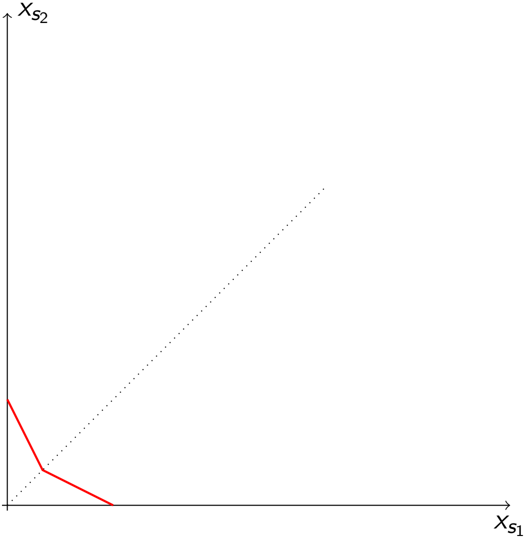
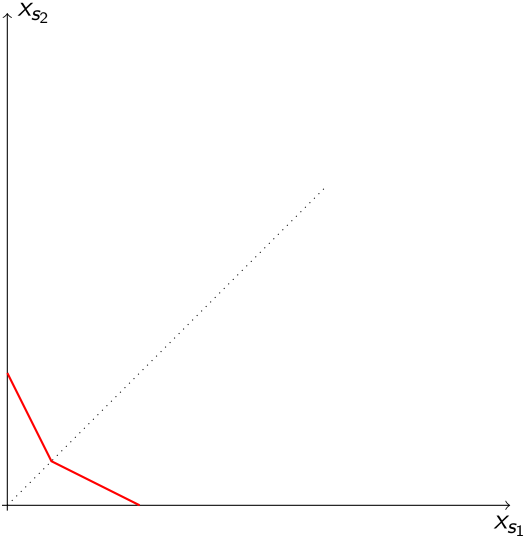
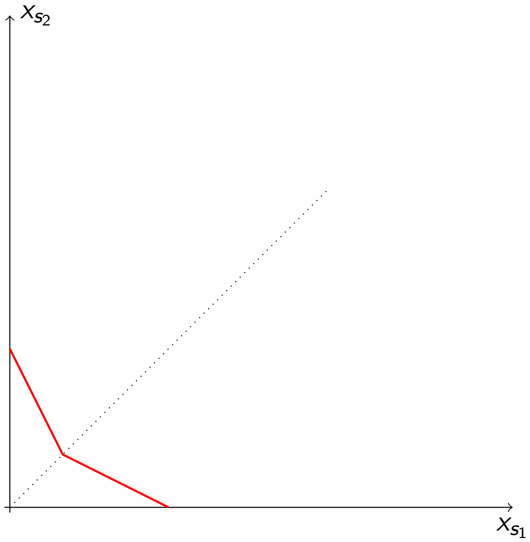


x_{s_2}
x_{s_1}



x_{s_2}
x_{s_1}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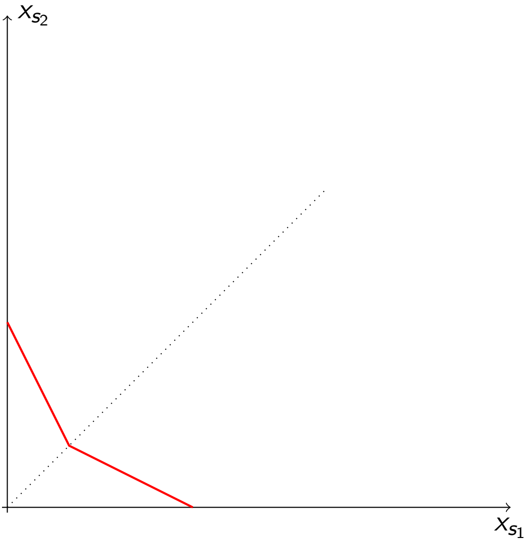
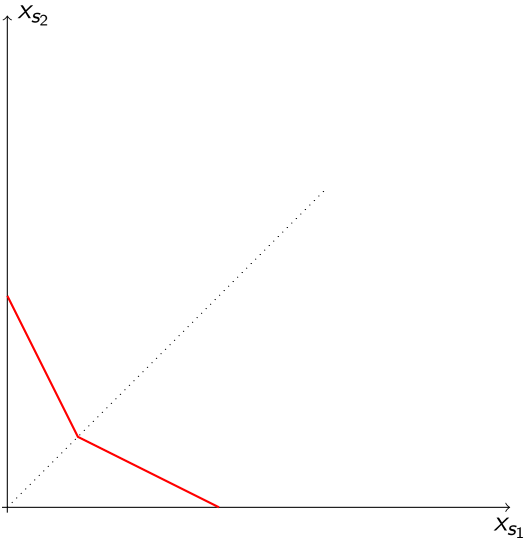
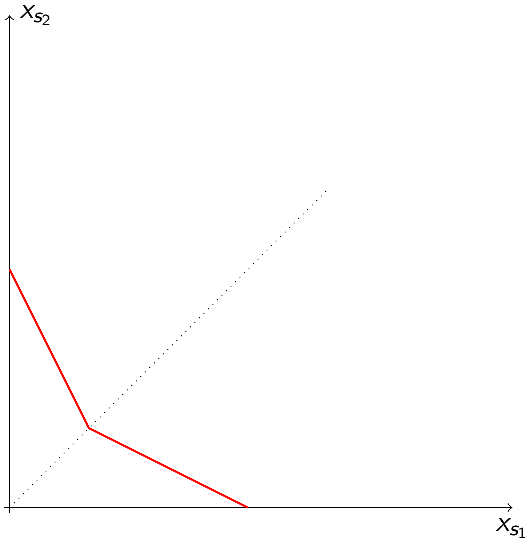
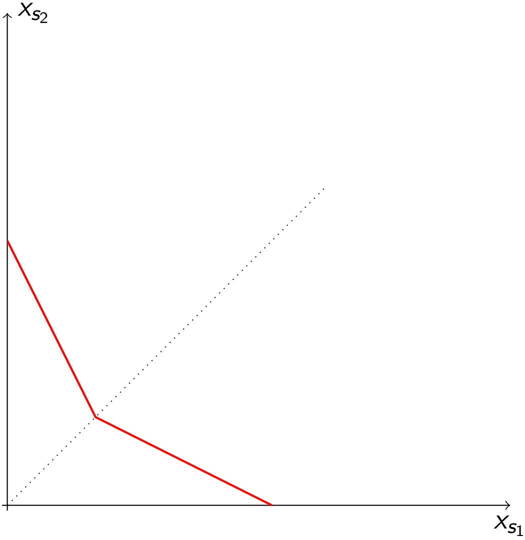


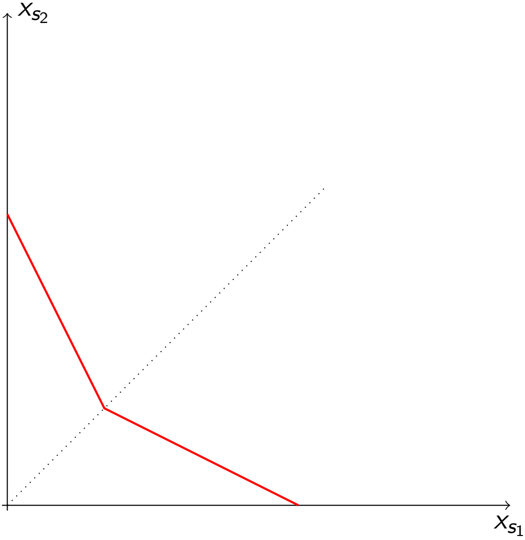
x_{s_2}
x_{s_1}

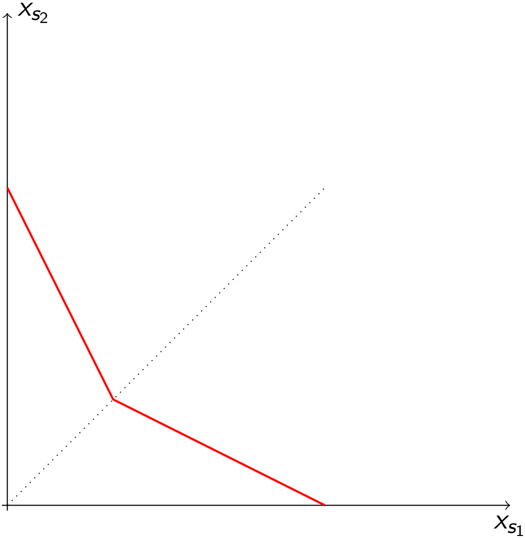

x_{s_2}
x_{s_1}


x_{s_2}
x_{s_1}



x_{s_2}
x_{s_1}


x_{s_2}
x_{s_1}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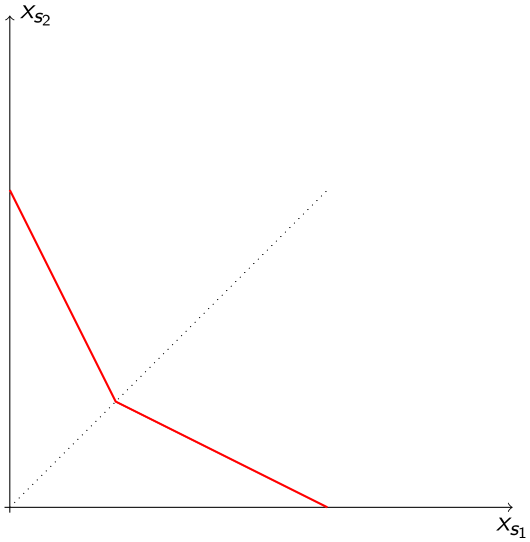
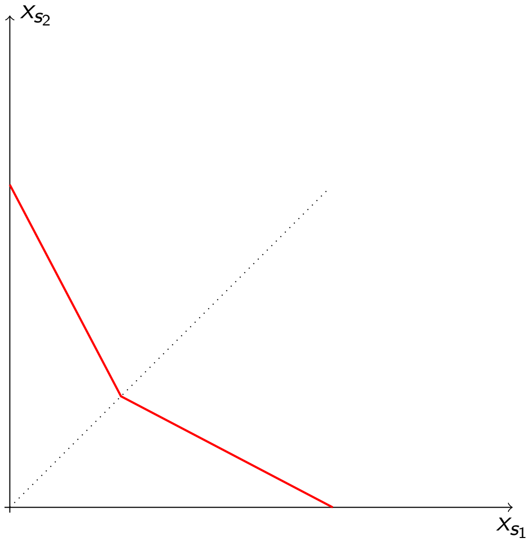
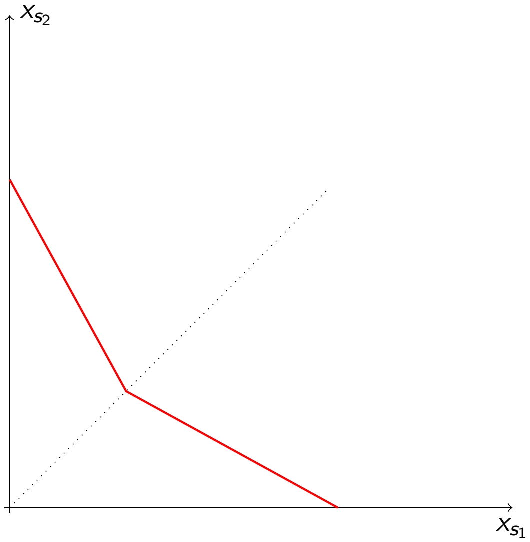
x_{s_1}



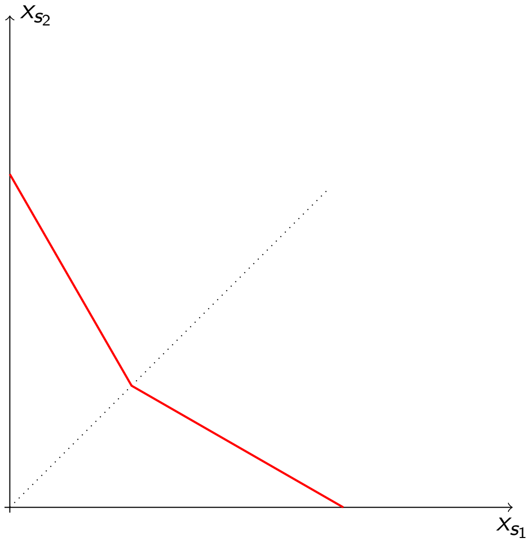
x_{s_2}
x_{s_1}

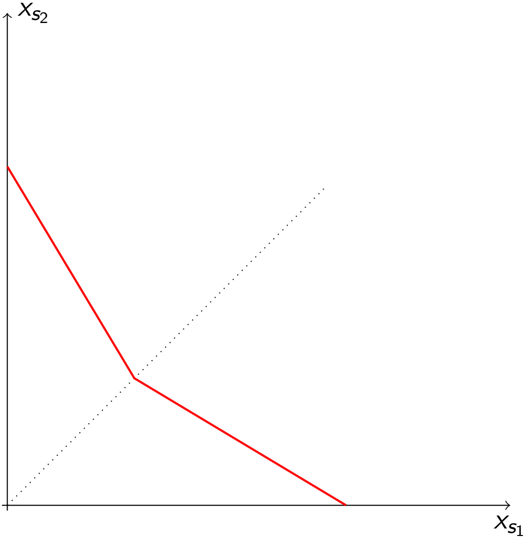


x_{s_2}
x_{s_1}

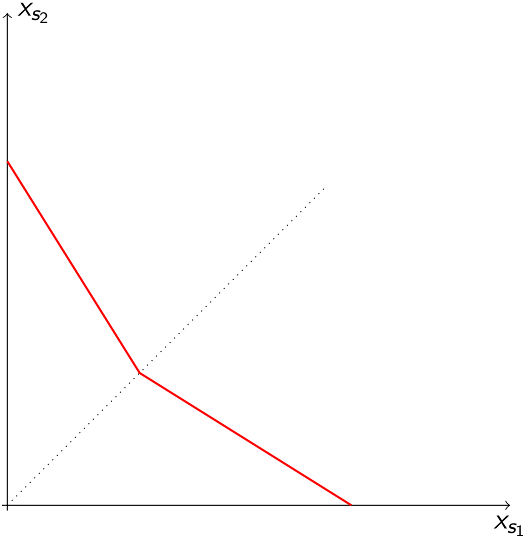


x_{s_2}
x_{s_1}

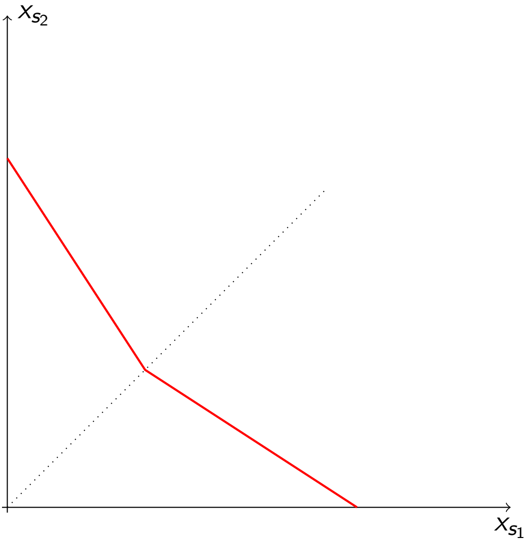

x_{s_2}
x_{s_1}



x_{s_2}
x_{s_1}



x_{s_2}
x_{s_1}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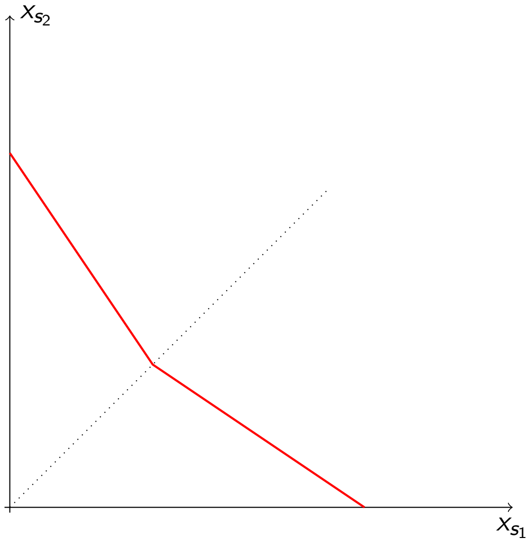
x_{s_1}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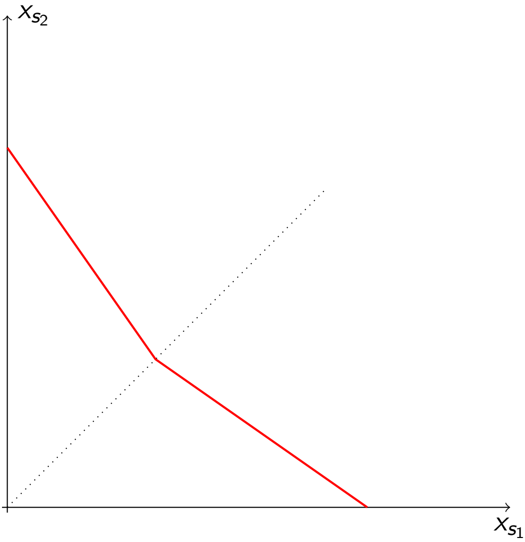
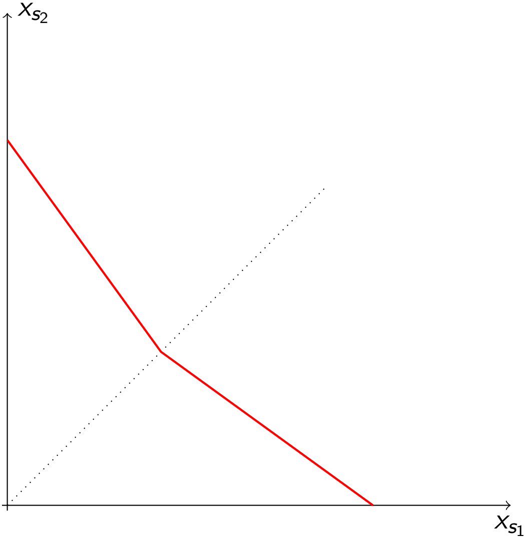

x_{s_2}
x_{s_1}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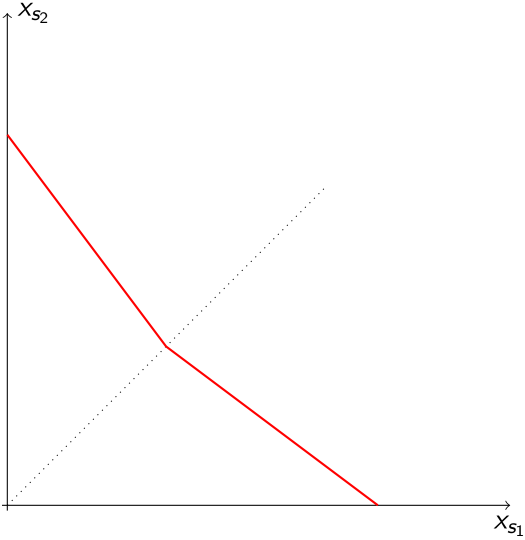
x_{s_1}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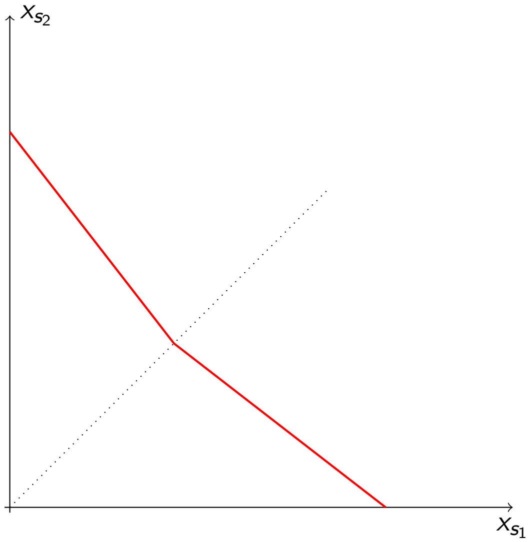
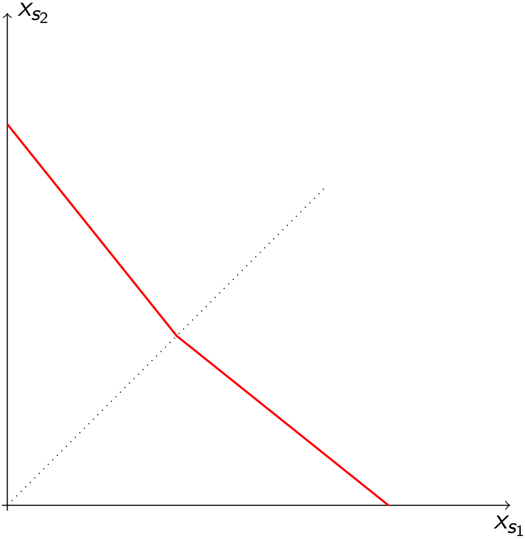


x_{s_2}
x_{s_1}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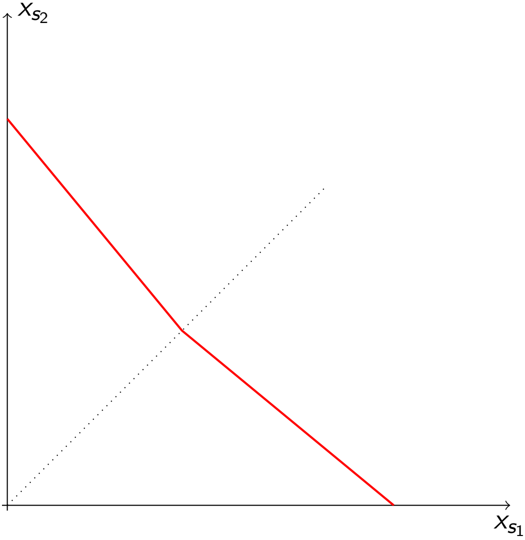
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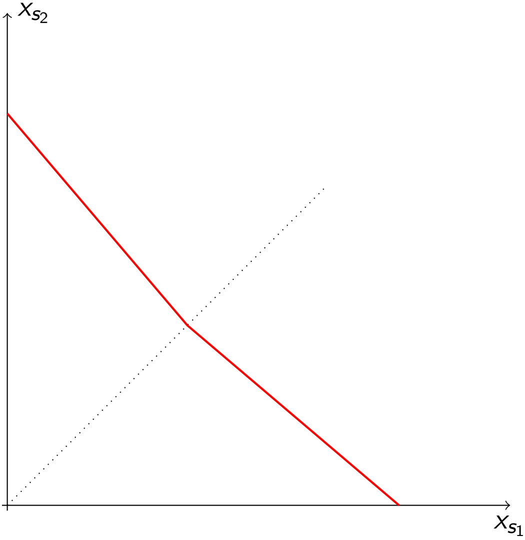
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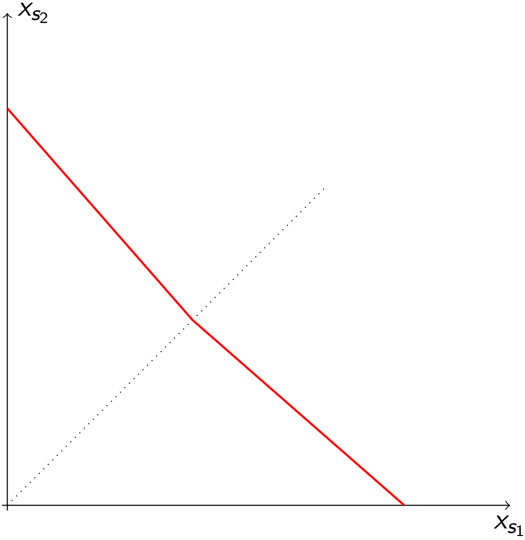
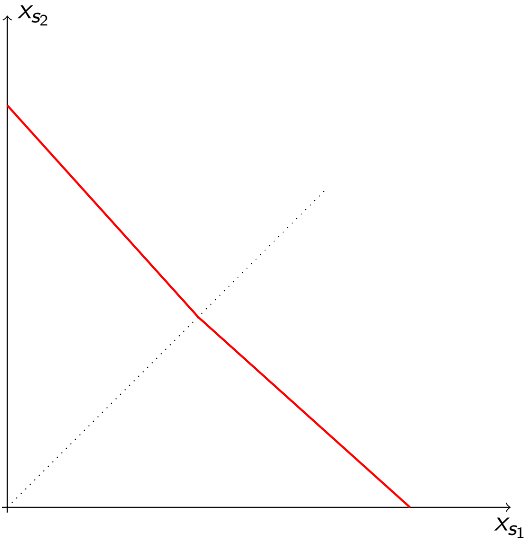
x_{s_1}



x_{s_2}
x_{s_1}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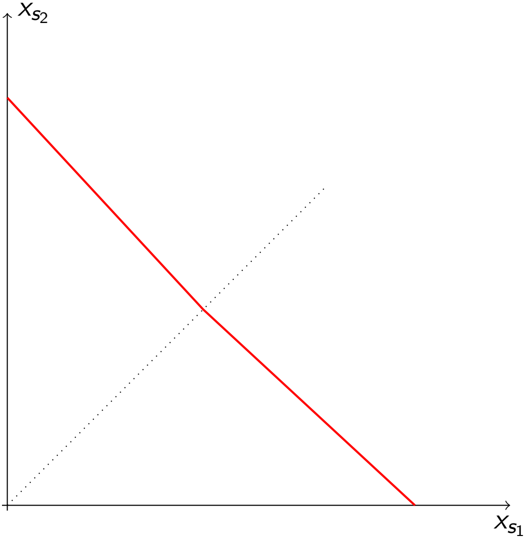
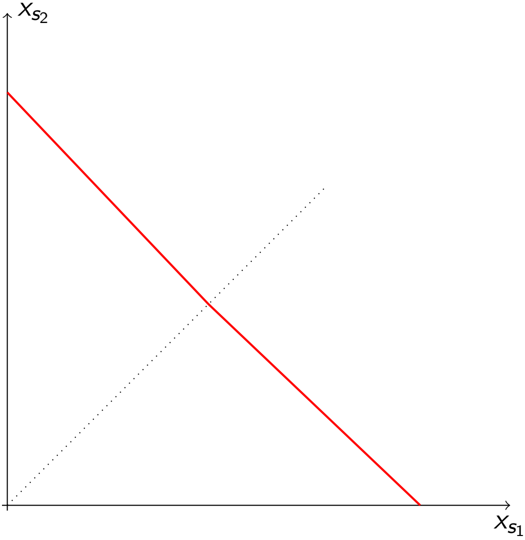
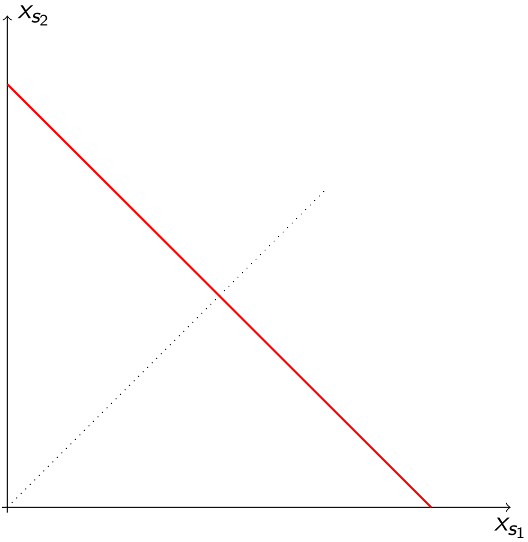
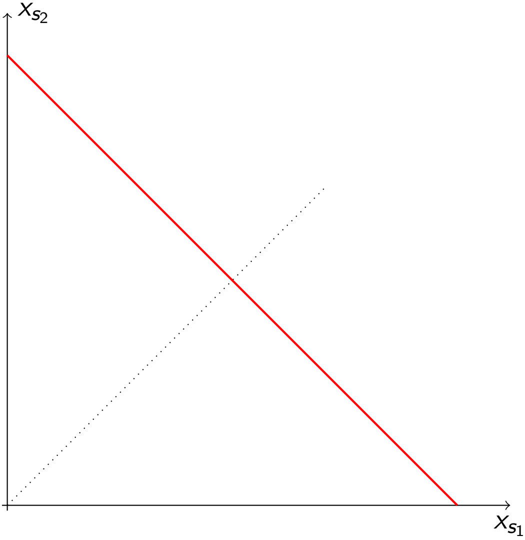

x_{s_2}
x_{s_1}



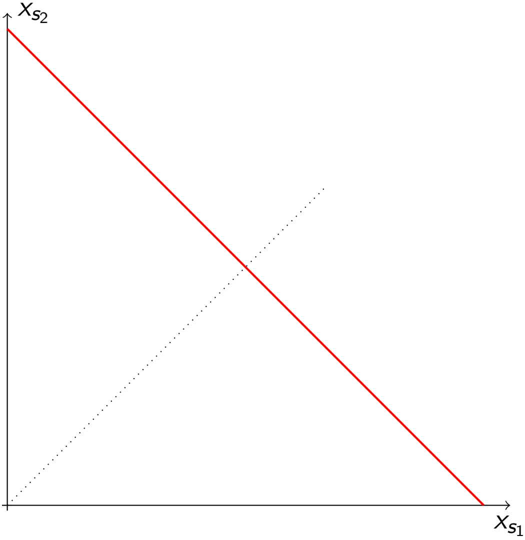
x_{s_2}
x_{s_1}

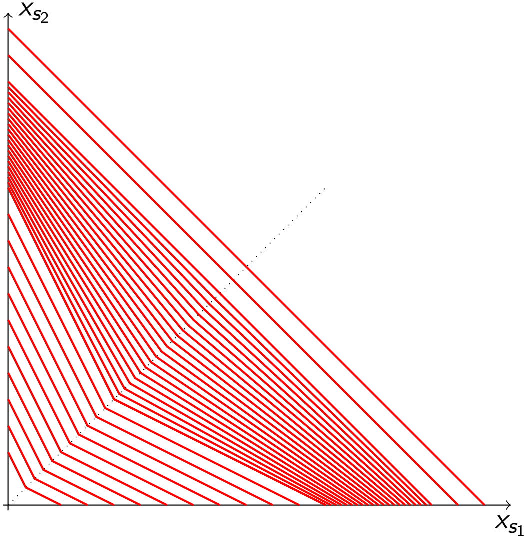


x_{s_2}
x_{s_1}

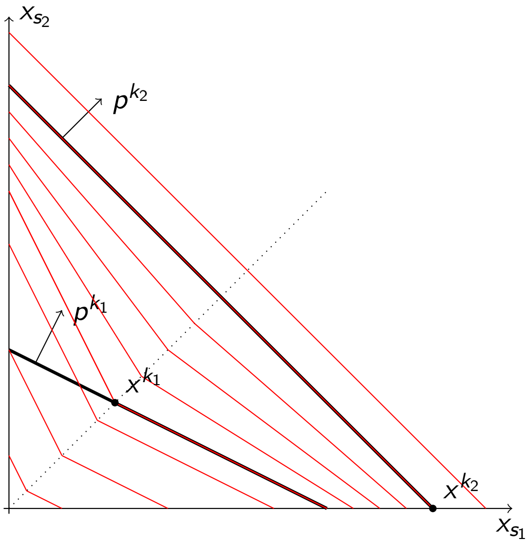


x_{s_2}
x_{s_1}


x_{s_2}
x_{s_1}



x_{s_2}
x_{s_1}



x_{s_2}
p^{k_2}
p^{k_1}
x^{k_1}
x^{k_2}
x_{s_1}

Maxmin

$$U(x) = \min_{\mu \in M} \sum_{s \in S} \mu_s u(x_s)$$

M is a convex set of priors.

Maxmin

$(x^k, p^k)_{k=1}^K$ is *maxmin rational* if $\exists$

- convex set $M \subseteq \Delta_{++}$
- and $u \in \mathcal{C}$ s.t.

$$y \in B(p^k, p^k \cdot x^k) \Rightarrow \min_{\mu \in M} \sum_{s \in S} \mu_s u(y_s) \leq \min_{\mu \in M} \sum_{s \in S} \mu_s u(x_s^k).$$

Maxmin

Proposition

Let $S = K = 2$. Then a dataset is max-min rational iff it is SEU rational.

Example with $S = 2$ and $K = 4$ of a dataset that is max-min rational and violates SARSEU.

Objective Probabilities

$$\max \quad \sum \mu_s u(x_s)$$
$$p \cdot x \leq I$$

- Observables: μ, p, x
- Unobservables: u

Varian (1983), Green and Srivastava (1986), and Kubler, Selden, and Wei (2013)

Objective Probabilities

Varian (1983), Green and Srivastava (1986): FOC

$$\mu_s u'(x_s) = \lambda p_s, \text{ (linear "Afriat" inequalities).}$$

Kubler, Selden, and Wei (2013): axiom on data.

Objective Probabilities

$$u'(x_s^k) = \lambda^k \frac{p_s^k}{\mu_s} = \lambda^k \rho_s^k,$$

- $\rho_s^k = p_s^k / \mu_s$ is a "risk neutral" price.

Objective Probabilities

(Strong Axiom of Revealed Exp. Utility (SAREU))

For any $(x_{s_i}^{k_i}, x_{s_i'}^{k_i'})_{i=1}^{n}$ *s.t.*

1. $x_{s_i}^{k_i} > x_{s_i'}^{k_i'}$
2. *each* k *appears in* k_i *(on the left of the pair) the same number of times it appears in* k_i' *(on the right):*

we have:

$$\prod_{i=1}^{n} \frac{\rho_{s_i}^{k_i}}{\rho_{s_i'}^{k_i'}} \leq 1.$$

Theorem

A dataset is EU rational if and only if it satisfies SAREU.

Savage

Primitives:
infinite S;
$\succeq$ on acts: information on all pairwise comparisons.

Define $\succeq$ to be the *rev. preference relation* defined from a finite dataset (x^k, p^k):

- $x^k \succeq y$ if $y \in B(p^k, p^k \cdot x^k)$
- $x^k \succ y$ if ...
- note: $\succeq$ is incomplete.

Savage

Axioms:

- P1
- P2
- P3
- P4
- P5
- P6
- P7

Proposition

If a data set violates P2, P4 or P7, then it violates SARSEU. No data can violate P3 or P5.

Ideas in the proof

$$
\begin{aligned}
\mu_s u'(x_s^k) &= \lambda^k p_s^k \\
x_s^k > x_s^k &\Rightarrow u'(x_s^k) \leq u'(x_s^k)
\end{aligned}
$$

quadratic equations $\Rightarrow$ linearize by logs.

$$
\begin{aligned}
\log \mu_s + \log u'(x_s^k) &= \log \lambda^k + \log p_s^k \\
x_s^k > x_{s'}^{k'} &\Rightarrow \log u'(x_{s'}^{k'}) \leq \log u'(x_s^k)
\end{aligned}
$$

When $\log p_s^k \in \mathbf{Q}$, the integer version of Farkas's lemma gives our axiom.
When $\log p_s^k \notin \mathbf{Q}$: approximation result.

$$\log v_s^k + \log \mu_s - \log \lambda^k - \log p_s^k = 0, \tag{1}$$

$$x_s^k > x_{s'}^{k'} \Rightarrow \log v_s^k \leq \log v_{s'}^{k'} \tag{2}$$

In the system (3)- (4), the unknowns are the real numbers $\log v_s^k$, $\log \mu_s$, $\log \lambda^k$, $k = 1, \ldots, K$ and $s = 1, \ldots, S$.

$$S1: \begin{cases} A \cdot u = 0, \\ B \cdot u \geq 0, \\ E \cdot u \gg 0. \end{cases}$$

Matrix A:

$$
\begin{array}{c}
\\
(1,1) \\ \vdots \\ (k,s) \\ \vdots \\ (K,S)
\end{array}
\begin{array}{c}
\begin{array}{ccccc|ccccc|cc}
(1,1) & \cdots & (k,s) & \cdots & (K,S) & 1 & \cdots & s & \cdots & S & 1 & \cdots
\end{array} \\
\left[
\begin{array}{ccccc|ccccc|cc}
1 & \cdots & 0 & \cdots & 0 & 1 & \cdots & 0 & \cdots & 0 & -1 & \cdots \\
\vdots & & \vdots & & \vdots & \vdots & & \vdots & & \vdots & \vdots & \\
0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots \\
\vdots & & \vdots & & \vdots & \vdots & & \vdots & & \vdots & \vdots & \\
0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots
\end{array}
\right.
\end{array}
$$

$$S2: \begin{cases} \theta \cdot A + \eta \cdot B + \pi \cdot E = 0, \\ \eta \geq 0, \\ \pi > 0. \end{cases}$$

Lemma

Let $(x^k, p^k)_{k=1}^K$ *be a dataset. The following statements are equivalent:*

1. $(x^k, p^k)_{k=1}^K$ *is SEU rational.*
2. $\exists$ *strictly positive numbers* v_s^k, λ^k, μ_s, *s.t.*

$$\mu_s v_s^k = \lambda^k p_s^k$$

$$x_s^k > x_{s'}^{k'} \Rightarrow v_s^k \leq v_{s'}^{k'}.$$

Lemma

Let data $(x^k, p^k)_{k=1}^k$ *satisfy SARSEU. Suppose that* $\log(p_s^k) \in \mathbf{Q}$ *for all* k *and* s*. Then there are numbers* v_s^k, λ^k, μ_s, *for* $s = 1, \ldots, S$ *and* $k = 1, \ldots, K$ *satisfying (2) in Lemma 3.*

Lemma

Let data $(x^k, p^k)_{k=1}^k$ *satisfy SARSEU. Then for all positive numbers* $\overline{\varepsilon}$*, there exists* $q_s^k \in [p_s^k - \overline{\varepsilon}, p_s^k]$ *for all* $s \in S$ *and* $k \in K$ *such that* $\log q_s^k \in \mathbf{Q}$ *and the data* $(x^k, q^k)_{k=1}^k$ *satisfy SARSEU.*

Lemma

Let data $(x^k, p^k)_{k=1}^k$ *satisfy SARSEU. Then there are numbers* v_s^k, λ^k, μ_s, *for* $s = 1, \ldots, S$ *and* $k = 1, \ldots, K$ *satisfying (2) in Lemma 3.*

Lemma

Let A be an $m \times n$ matrix, B be an $l \times n$ matrix, and E be an $r \times n$ matrix. Suppose that the entries of the matrices A, B, and E belong the a commutative ordered field $\mathbf{F}$. Exactly one of the following alternatives is true.

1. *There is $u \in \mathbf{F}^n$ such that $A \cdot u = 0$, $B \cdot u \geq 0$, $E \cdot u \gg 0$.*
2. *There is $\theta \in \mathbf{F}^r$, $\eta \in \mathbf{F}^l$, and $\pi \in \mathbf{F}^m$ such that $\theta \cdot A + \eta \cdot B + \pi \cdot E = 0$; $\pi > 0$ and $\eta \geq 0$.*

Proof

$$\log v_s^k + \log \mu_s - \log \lambda^k - \log p_s^k = 0, \tag{3}$$
$$x_s^k > x_{s'}^{k'} \Rightarrow \log v_s^k \leq \log v_{s'}^{k'} \tag{4}$$

In the system (3)- (4), the unknowns are the real numbers $\log v_s^k$, $\log \mu_s$, $\log \lambda^k$, $k = 1, \ldots, K$ and $s = 1, \ldots, S$.

Proof:

$$S1: \begin{cases} A \cdot u = 0, \\ B \cdot u \geq 0, \\ E \cdot u \gg 0. \end{cases}$$

Proof:

Matrix A:

$$
\begin{array}{c}
 \\ (1,1) \\ \vdots \\ (k,s) \\ \vdots \\ (K,S)
\end{array}
\begin{array}{c}
\begin{array}{ccccccccccccc}
(1,1) & \cdots & (k,s) & \cdots & (K,S) & 1 & \cdots & s & \cdots & S & 1 & \cdots
\end{array} \\
\left[\begin{array}{ccccc|ccccc|cc}
1 & \cdots & 0 & \cdots & 0 & 1 & \cdots & 0 & \cdots & 0 & -1 & \cdots \\
\vdots & & \vdots & & \vdots & \vdots & & \vdots & & \vdots & \vdots & \\
0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots \\
\vdots & & \vdots & & \vdots & \vdots & & \vdots & & \vdots & \vdots & \\
0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots
\end{array}\right.
\end{array}
$$

Proof:

$$S2: \begin{cases} \theta \cdot A + \eta \cdot B + \pi \cdot E = 0, \\ \eta \geq 0, \\ \pi > 0. \end{cases}$$

L. Savage

If I have seen less than other men, it is because I have walked in the footsteps of giants.

P. Chernoff