Preference identification

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This paper

- An experimenter and a subject.
- Subject makes choices according to some $\succeq^*$ on set X.
- Experimenter conducts a finite choice experiment of "size" k (k questions).
- Rationalizing preference: $\succeq_k$.

Question: When does $\succeq_k \to \succeq^*$?

Example 1

Subject chooses among alternatives: $X = \mathbf{R}^n_+$.

- Choices come from $\succeq^*$, a continuous preference.
- $B_k = \{x_k, y_k\}$.
- A *finite experiment:* choose an element from B_i, $i = 1, \ldots, k$.
- Assumption: $B = \cup_{k=1}^{\infty} B_k$ is dense.

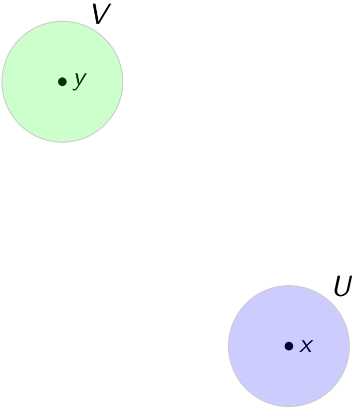
Example 1

• y

- $x \succ^* y$

• x

Example 1



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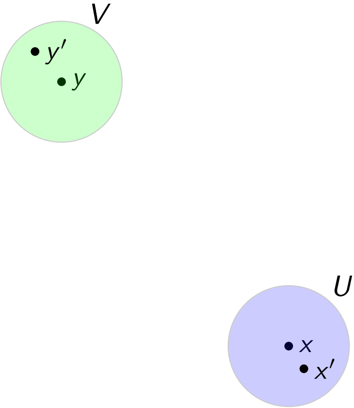

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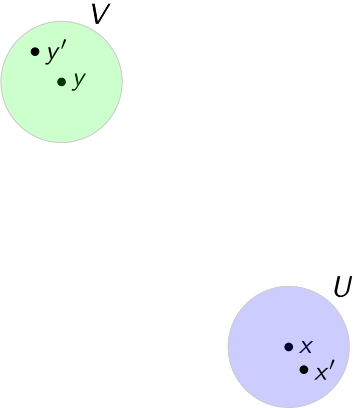

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- $\exists x' \in U$ and $y' \in V$ s.t $\forall k$ $\exists$ rationalizing $\succeq_k$, with $y' \succ_k x'$
- But $x' \succ y'$. $\forall \succeq$ s.t. $\succeq$ is cont. and $\succeq|_B = \succeq^*|_B$.

Example 1: a discontinuity.

- Infinite data (X):
- "Limiting" infinite data $(B = \cup_{k=1}^{\infty} B_k)$:
- Finite data: $(B_1 \ldots, B_k)$

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- Infinite data (X): observe $\succeq^*$; so $x \succ^* y$
- "Limiting" infinite data ($B = \cup_{k=1}^{\infty} B_k$):
  $x' \succ y' \ \forall \succeq$ s.t. $\succeq|_B = \succeq^*|_B$.
- Finite data: $(B_1 \ldots, B_k)$
  can't rule out $y' \succ_k x'$, no matter how large k.

Example 2: Complete indifference.

Let $X = \mathbf{R}^n_+$.
$X \times X$ is the *complete indifference* preference.

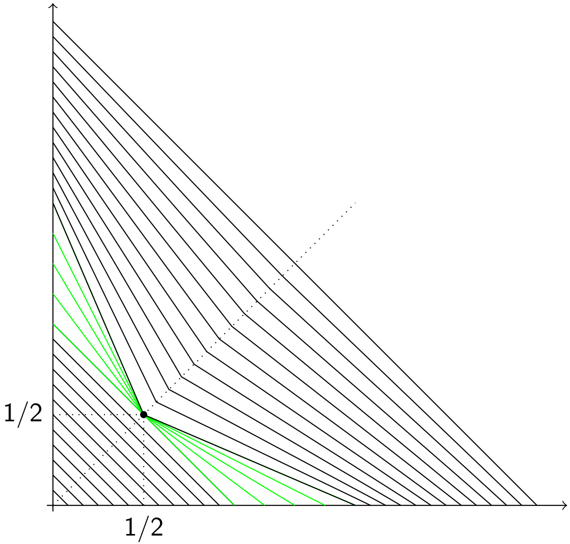
Fix a continuous preference $\succeq^*$ on X.

Proposition (informal)

There exists rationalizing $\succeq_k$ for each k s.t

$$\text{complete indifference} = \lim_{k \to \infty} \succeq_k$$

Example 3



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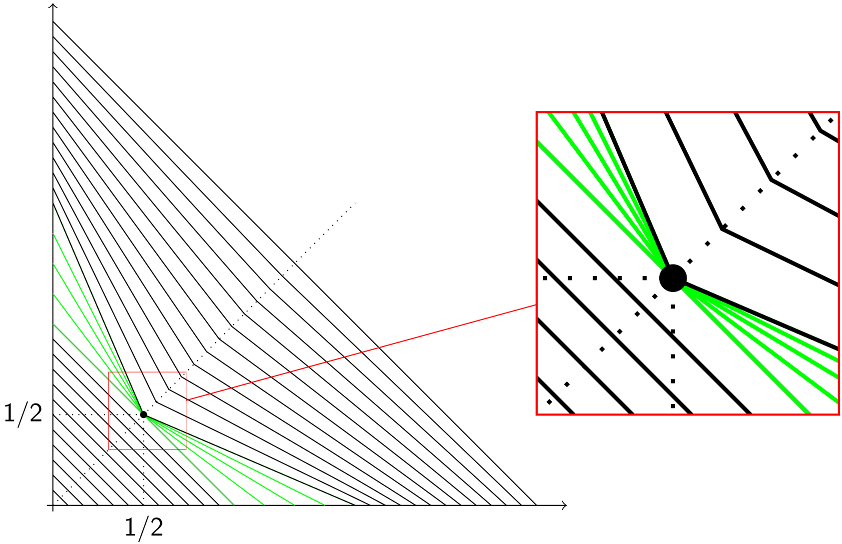

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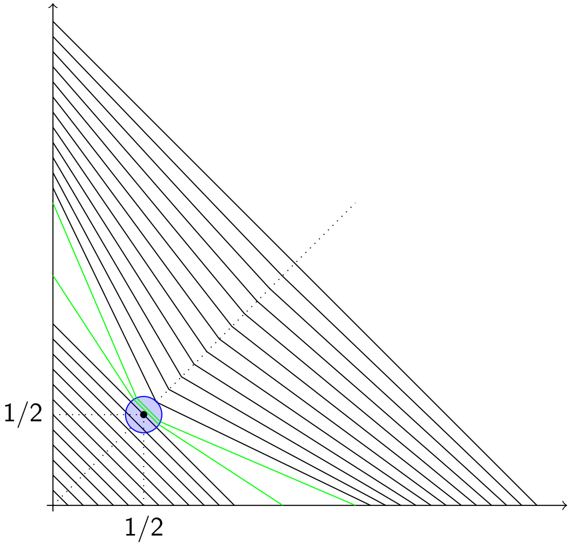
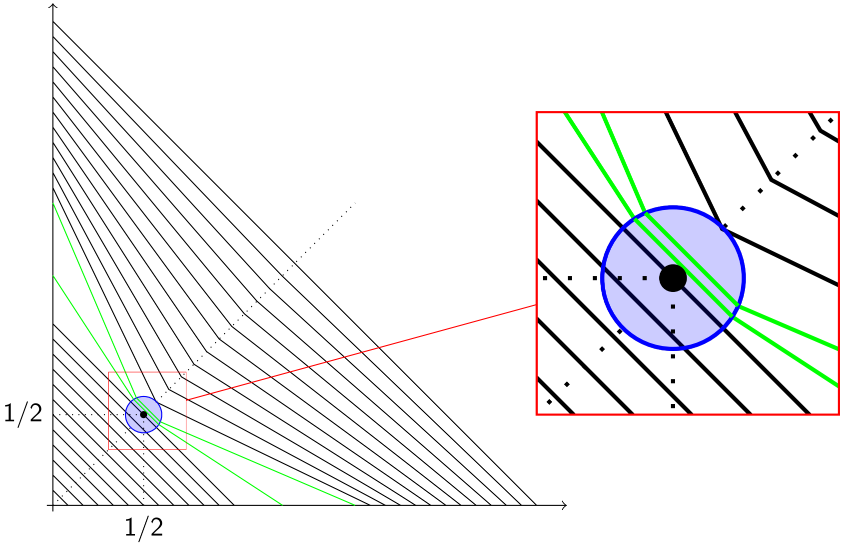


Example 3



Example 4

- Let $X = [0, 1]$, $\succeq^* = \geq$ and $u^*(x) = x$.
- For each k, let $\succeq_k = \geq$ and

$$u_k = \frac{x}{k}.$$

- Then $0 = \lim_n u_n$.
- But $\succeq_k = \succeq^*$ for all k!

Example 4

- Let $X = [0,1]$, $\succeq^* = \geq$ and $u^*(x) = x$.
- For each k, let $\succeq_k = \geq$ and

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- Then $0 = \lim_n u_n$.
- But $\succeq_k = \succeq^*$ for all k!

(For $\varepsilon > 0$, can choose u_n with $\|u_n\|_\infty = 1$ or $\|u_n\|_1 = 1$ and $0 = \lim_n u_n(x)$ for all $x \in [0, 1-\varepsilon]$.)

Lessons learned.

- Examples 1 and 2 speak to the role of theory as a *discipline* on data. Rationalizations shouldn't be arbitrary.
- Little less obviously: there's a role for some objective "known" component of preferences. This role is also provided by theory. Pure empiricism is problematic.
- Example 3: model of preferences may not be closed.
- Example 4: Utility estimates are more delicate than preferences.

Our results (informal)

Choices over X.
Let X be

- $X = \mathbf{R}^n$.
- or $X = \Delta([a, b])^{\Omega}$, set of Anscombe-Aumann acts;

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Choices over X.
Let X be

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- or $X = \Delta([a, b])^\Omega$, set of Anscombe-Aumann acts;

Two important features:

- Objective monotonicity.
- Connection between order and topology on X.

Our results (informal)

Theorem

Let

- $\succeq^*$ *be monotone and cont.;*
- $\succeq_k$ *strongly rationalize the* k*th finite choice experiment generated by* $\succeq^*$*.*

Then,

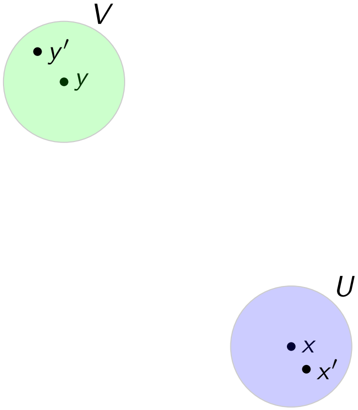
- $\succeq_k \to \succeq^*$ *(in the topology of closed convergence).*
- *For any utility* u^* *for* $\succeq^*$ $\exists$ u_k *for* $\succeq_k$ *s.t.* $u_k \to u^*$ *(in the topology of compact convergence).*

Discussion.

- Monotonicity.
- Convergence of *any arbitrary* nonparametric preference estimate. Choose $\succeq_k$ to conform to desired theory.
- Utility *can't be arbitrary*. Only get convergence of selected utility estimates. Require an identification theorem for each specific theory.

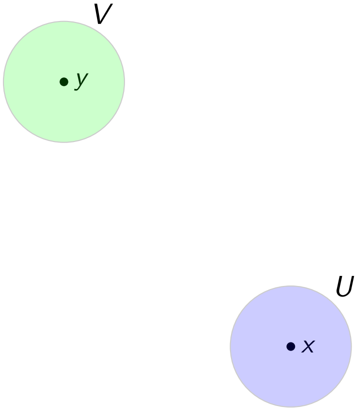
Why does monotonicity help?

Recall example 1



- $x \succ^* y$
- $U \succ^* V$
- $\exists x' \in U$ and $y' \in V$ s.t $y' \succ_k x'$ for some rationalizing $\succeq_k$
- But $x' \succ y'$. $\forall \succeq$ s.t. $\succeq$ is cont. and $\succeq |_B = \succeq^* |_B$.

Monotone rationalizations.



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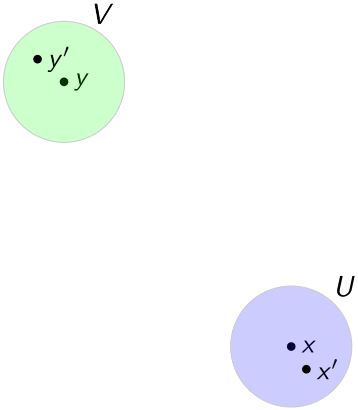

- $x \succ^* y$
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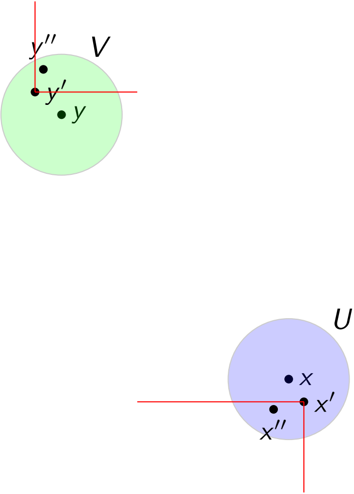
- Let $(x', y') \in U \times V$.

Monotone rationalizations.



- $x \succ^* y$
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- $\implies \exists x'', y'' \in B$
- $x'' \leq x'$
- $y' \leq y''$

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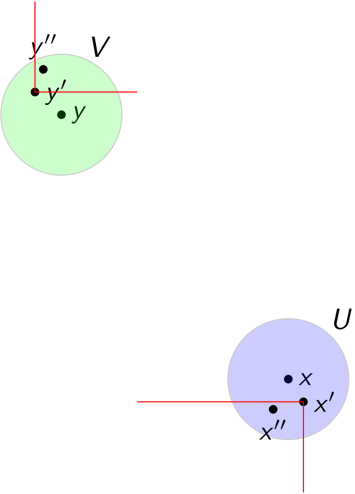

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- $\implies \exists x'', y'' \in B$
- $x'' \leq x'$
- $y' \leq y''$
  $\implies x' \geq x'' \succ_k y'' \geq y'$

Model

The set of alternatives is X.

Let X be

- Polish,
- locally compact,
- partially ordered by $\leq$.

Let $< \subseteq \leq$.

Definitions

$\succeq$, a binary relation on X, is

- a *preference relation* if it is complete and transitive;
- *locally strict* if $x \succeq y$ and V a nbd of (x, y) implies $\exists (x', y') \in V$ with $x' \succ y'$;
- *weakly monotone* if $x \geq y$ implies $x \succeq y$.
- *strictly monotonic* if $x \geq y$ implies $x \succeq y$, and $x > y$ implies $x \succ y$.
- *continuous* if $\{y \in X : y \succeq x\}$ and $\{y \in X : x \succeq y\}$ are closed.

Model

$B_1, B_2, \ldots$ is a collection of subsets of X of cardinality two.

Interpretation:

$B_i = \{x_i, y_i\}$; experimenter asks a subject to choose between x_i and y_i.

Model

Let $\Sigma_k = \{B_1, \ldots, B_k\}$.

A *finite experiment of order k* is a function

$$c : \Sigma_k \to 2^X \text{ s.t } \varnothing \neq c(B_i) \subseteq B_i$$

Model

Let $B = \cup_{k=1}^{\infty} B_k \subseteq X$.

Assume:

- Any subset of B of cardinality two is in some B_k.
- B is dense in X.

Model

A *choice sequence* is an increasing collection of experiments:

Denote by $\mathcal{C}^k$ the set of all finite experiments of order k.

Consider $c : \mathbf{N} \to \bigcup_k \mathcal{C}^k$ s.t.

- $\forall k,\ c^k \in \mathcal{C}^k$
- for all $k < l$, $c^l|_{\Sigma_k} = c^k$

The set of choice sequences is denoted $\mathcal{C}$.

$c \sqsubseteq c'$ if

$$(\forall k)(\forall B_i \in \Sigma_k) c^k(B_i) \subseteq c'^k(B_i).$$

Model

The *choice function of order k generated by* $\succeq$ is defined by

$$c_{\succeq}(B_i) = \arg\max_{B_i} \succeq = \{x \in B_i : x \succeq y \text{ for all } y \in B_i\}$$

Model: Rationalizability

$\succeq$ *weakly rationalizes* a finite experiment c of order k if $c(B_i) \subseteq c_{\succeq}(B_i)$ for all $B_i \in \Sigma_k$.

$\succeq$ *weakly rationalizes* a choice sequence c if $c \sqsubseteq c_{\succeq}$.

$\succeq$ *strongly rationalizes* a finite experiment c of order k if $c(B_i) = c_{\succeq}(B_i)$ for all $B_i \in \Sigma_k$.

Definitions

Order $>$ *has open intervals* if, for all $x, y \in X$,

$$\{(x, y) : x > y\}$$

is an open set.

Convergence 1

Theorem

Suppose that

1. $\succeq^*$ *is continuous and strictly monotone,*
2. $>$ *has open intervals,*
3. *every continuous and st. mon. preference relation is locally strict.*

Let $c \sqsubseteq c_{\succeq^*}$ *be a choice sequence, and let* $\succeq_k$ *be a continuous and strictly monotone preference that weakly rationalizes* c^k*. Then,* $\succeq_k \rightarrow \succeq^*$ *in the closed convergence topology.*

Basically the previous theorem applies to $X \subseteq \mathbf{R}^n$.

For $X = \Delta([0,1])$ does not have open intervals. (For example let $\bar{x} = \delta_1$ and $\underline{x}$ be the uniform distribution. Then if $z_\varepsilon = \frac{1+\varepsilon}{2}\delta_{1/2} + \frac{1-\varepsilon}{2}\delta_1$ we have $z_\varepsilon \notin [\underline{x}, \bar{x}]$ for any $\varepsilon > 0$ while $z_0 \in (\underline{x}, \bar{x})$.)

For a choice sequence c, let

$$\mathcal{P}^k(c) = \{\succeq : \succeq \text{ is cont. st. mon. weakly rationalizing } c^k\}.$$

For a set of binary relations S, define $\text{diam}(S) = \sup_{(\succeq,\succeq')\in S^2} \delta_C(\succeq,\succeq')$ to be the diameter of S according to the metric δ_C which generates the topology on preferences.

Theorem

Suppose that $<$ has open intervals. Let c be a choice sequence, and suppose that each strictly monotone continuous preference is also locally strict. Then

$$\lim_{k \to \infty} diam(\mathcal{P}^k(c)) \to 0.$$

Definitions

(X, B) has the *countable order property* if $\forall x \in X$ and $\forall V$ nbd of x $\exists x', x'' \in B \cap V$ with $x' \leq x \leq x''$.

X has the *squeezing* property if for any sequence $\{x_n\}$, if $x_n \to x^*$ then there is an increasing sequence $\{x_n'\}$, and an a decreasing sequence $\{x_n''\}$, such that $x_n' \leq x_n \leq x_n''$, and $\lim_{n\to\infty} x_n' = x^* = \lim_{n\to\infty} x_n''$.

Proposition

If X is

- $\mathbf{R}^n$, *or*
- $\Delta([a, b])$,

then X has the countable order and squeezing properties.

Convergence 2

Theorem

Suppose

1. $\succeq^*$ *is cont. and weakly monotone,*
2. (X, Σ_∞) *has the countable order property, and* X *the squeezing property.*

Let $\succeq_k$ *be a continuous and weakly monotone preference that strongly rationalizes the choice function of order k generated by* $\succeq^*$*. Then,* $\succeq_k \to \succeq^*$ *in the closed convergence topology.*

$\succeq^k \to \succeq$ (cpctness)
WTS: $\succeq = \succeq^*$.

proof ideas; $X = \mathbf{R}^n$

$\succeq^k \to \succeq$ (cpctness)
WTS: $\succeq = \succeq^*$.
Let $x \succ^* y$ and $x \in U \succ^* V \ni y$.

- Arbitrary convergent seq. $(x_n, y_n) \to (x, y)$; "squeeze" the sequence by (x'_n, y'_n).
- $x'_n = \inf\{x_m : m \geq n\}$ and $y'_n = \sup\{y_m : m \geq n\}$
- $(x'_n, y'_n) \to (x, y)$
- $x'_n \uparrow$ and $y'_n \downarrow$.

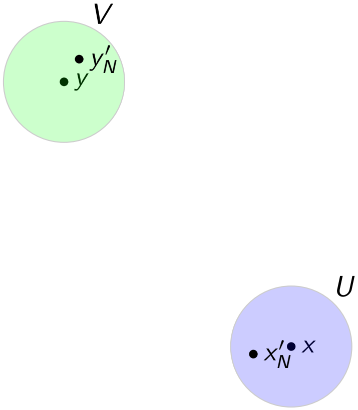
proof ideas

$\bullet\, y$

- $x \succ^* y$; $U \succ^* V$

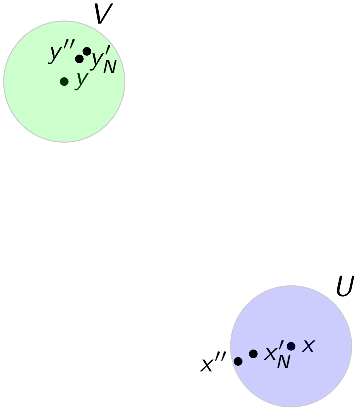
$\bullet\, x$

proof ideas



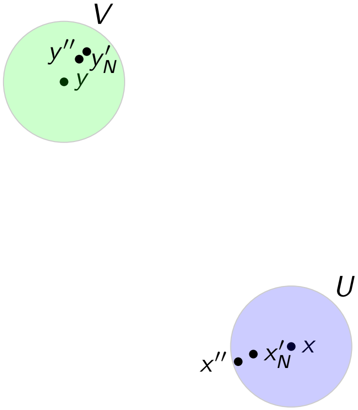
- $x \succ^* y$; $U \succ^* V$
- $\exists x'_N \in U$ and $y'_N \in V$

proof ideas



- $x \succ^* y$; $U \succ^* V$
- $\exists x'_N \in U$ and $y'_N \in V$
- $\exists x'' \in U \cap B$, $x'' \leq x'_N$.
- $y'' \in V \cap B$ $y'_N \leq y''$

proof ideas



- $x \succ^* y$; $U \succ^* V$
- $\exists x'_N \in U$ and $y'_N \in V$
- $\exists x'' \in U \cap B$, $x'' \leq x'_N$.
- $y'' \in V \cap B$ $y'_N \leq y''$
- Since $\succeq_k \uparrow$, $\exists N^* \geq N$ s.t. $(\forall n \geq N^*) x'' \succ_n y''$.

Then:

$$x_n \geq x'_n \geq x'_N \geq x'' \succ_n y'' \geq y'_N \geq y'_n \geq y_n$$

$\forall n \geq N^*$

- weak monotonicity
- hence avoids any sequence to have the wrong preference;
- so $x \succ y$ by completeness.

proof ideas

Conversely, show $x \succeq^* y \Longrightarrow x \succeq y$.

proof ideas

Conversely, show $x \succeq^* y \Longrightarrow x \succeq y$.

Let $x' \in N_x(1/k) \cap B$ with $x' \geq x$, and $y' \in N_y(1/k) \cap B$ with $y' \leq y$;
So

$$x' \succeq^* x \succeq^* y \succeq^* y'$$

Now,

- $\succeq_n \uparrow \Longrightarrow x' \succeq_n y'$ $\forall n$ large enough.
- $n_k \geq n_{k-1}$ such that $x' \succeq_{n_k} y'$; and let $x' = x_{n_k}$ and $y' = y_{n_k}$. Then $(x^{n_k}, y^{n_k}) \to (x, y)$ and $x_{n_k} \succeq_{n_k} y_{n_k}$.
- Thus $x \succeq y$.

Utility functions

Recall Examples 3 and 4.

Utility representations

We need a canonical utility representation.

Here we use the "equal coordinates" idea: a set M on which all preferences agree.

For $X = \mathbf{R}^n$ M, is the ray of equal coordinates.
For $X = \Delta([a, b])$, M is $[a, b]$.

Model

Let $M \subseteq X$ s.t

- M is connected and totally ordered by $<$.
- $\forall m \in M$ and nbd U of m in X $\exists \underline{m}, \overline{m} \in M$, with

$$m \in [\underline{m}, \overline{m}] \subseteq U.$$

- (If m is not the largest element of M we can choose $\overline{m}$ such that $m < \overline{m}$, and if m is not the smallest element we can choose $\underline{m}$ such that $\underline{m} < m$.)
- Any bd seq in X is bounded by elements of M.

Model

Let

- $\mathcal{U}$ be the set of st. monotone and cont. utility functions on X.
- $\mathcal{R}^{\text{mon}}$ be the set of preferences which are st. monotone and cont.

Homeomorphism

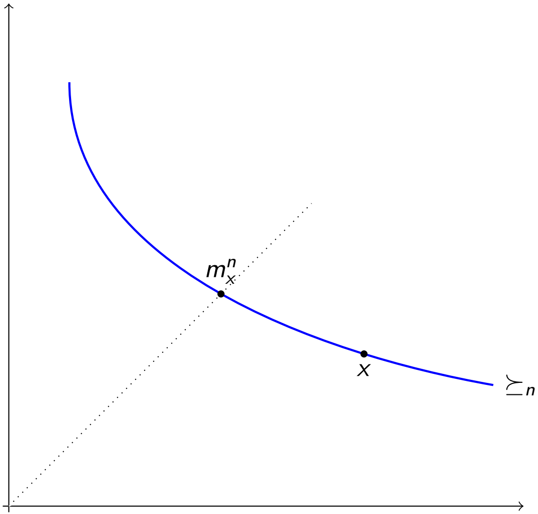
Let $\Phi : \mathcal{U} \to \mathcal{R}^{\text{mon}}$ such that $\Phi(u)$ is the preference represented by $u \in \mathcal{U}$.

Equivalence relation $\simeq$ on $\mathcal{U}$;

$\hat{\Phi} : \mathcal{U}/\simeq \to \mathcal{R}$ is defined in the natural way.

Theorem

$\hat{\Phi}$ *is a homeomorphism.*



$$u_n(x) = u^*(m_x^n) \Longrightarrow u_n \to u^*$$

Dated rewards

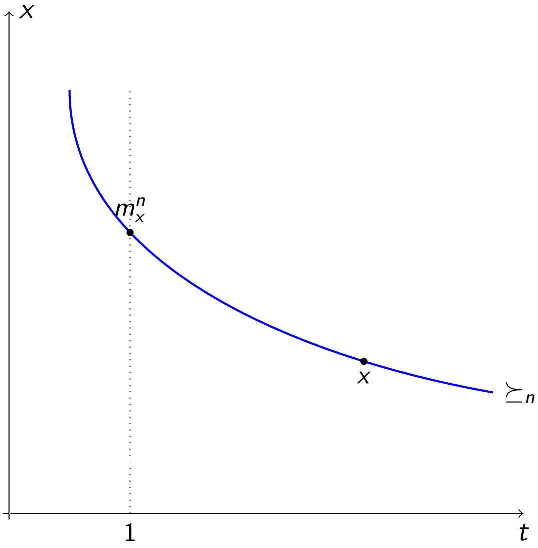
Model of intertemporal choice.

$X = \mathbf{R}^2_+$.

Interpret $(t, x) \in X$ has a monetary quantity x delivered on date t.

$(x, t) \leq (x', t')$ iff $x \leq x'$ and $t' \leq t$.

Dated rewards.



$u_n(x) = u^*(m_x^n) \Longrightarrow u_n \to u^*$

Compact set of utilities

Let $\mathcal{V} \subseteq \mathbf{R}^X$ be a set of cont. utility functions.
$\Phi(u)$ preference relation represented by u.

Theorem

Let

- $\mathcal{V}$ *be cpct. and all* $\succeq \in \Phi(\mathcal{V})$ *l.s.*
- $\succeq_k \in \Phi(\mathcal{V})$ *weakly rationalize* c^k

Then $\exists \succeq^* \in \Phi(\mathcal{V})$ *s.t.* $\succeq_k \to \succeq^*$ *(in the closed convergence topology.)*
Furthermore, if $\succeq'_k$ *also weakly rationalizes* c^k*, then* $\succeq'_k \to \succeq^*$.

Note contrast with Ex. 3 and 4.

Example: Expected utility

Let Π be finite set of *prizes* and $\Delta(\Pi)$ set of lotteries.

nonconstant expected utility preference are locally strict.

$\mathcal{V}$ is (homeomorphic to)

$$S = \{u \in \mathbf{R}^X : \sum_x u_x = 0, \|u\| = 1\};$$

so compact.

Identification "in the limit" $\Longrightarrow$ identification

Theorem

Suppose that $\succeq$ and $\succeq'$ are two complete and continuous binary relations. Suppose that $\succeq'$ is locally strict, and let $B \subseteq X$ be dense. If $\succeq|_{B\times B} \subseteq \succeq'|_{B\times B}$, then $\succeq = \succeq'$.

Identification "in the limit" $\Longrightarrow$ identification

Theorem

Suppose that $\succeq$ and $\succeq'$ are two continuous and complete binary relations. Suppose X is connected, and let $B \subseteq X$ be dense. If $\succeq|_{B\times B} = \succeq'|_{B\times B}$, then $\succeq = \succeq'$.

Identification “in the limit” $\Longrightarrow$ identification

Example

Let $X = [0,1] \cup [2,3]$ and $B = ([0,1) \cup (2,3])$, and observe that the rankings $x \succeq y$ iff $x \geq y$ and $x \succeq' y$ iff $x \geq y$ or $(x,y) = (1,2)$ have the same restriction to B. e.g. $\succeq$ can be represented by $u(x) = x$ and $\succeq'$ by $u(x) = x$ on $[0,1]$ and $u(x) = x - 1$ on $[2,3]$.

Topology on preferences

Basic requirement for a topology on preferences is that preferences that are close should have similar choice behavior.

If

- $(x_n, y_n) \to (x, y)$,
- $\succeq_n \to \succeq$,
- $x \succ y$,

then $x_n \succ_n y_n$ for all n large enough.

"Close preference have the same choice behavior for alternatives that are close to each other."

Topology on preferences

Let X be Polish and locally compact. Then the topology of closed convergence is the smallest topology for which the sets

$$\{(x, y, \succeq) : x \succ y\}$$

are open.

Topology on preferences

Let X be metrizable. Let F^n be a sequence of closed sets in X. $\mathrm{Li}(F^n)$ and $\mathrm{Ls}(F^n)$ are closed subsets of X defined by:

- $x \in \mathrm{Li}(F^n)$ iff for all nbd V of x there is $N \in \mathbf{N}$ such that $F^n \cap V \neq \emptyset$ for all $n \geq N$.
- $x \in \mathrm{Ls}(F^n)$ iff for all nbd V of x and all $N \in \mathbf{N}$ there is $n \geq N$ s.t $F^n \cap V \neq \emptyset$.

Note: $\mathrm{Li}(F^n) \subseteq \mathrm{Ls}(F^n)$.

Definition

F^n converges to F in the *topology of closed convergence* if $\mathrm{Li}(F^n) = F = \mathrm{Ls}(F^n)$.

Topology on preferences

Lemma

Let (X, d) be a locally compact separable metric space. The set of all closed subsets of X, endowed with the topology of closed convergence, is a compact metrizable space.

Related literature

With demand theory primitives:

- Mas-Colell (Restud 78); closest to us.
- Reny (Ecma 2015)
- Kübler and Polemarchaskis (Ecma forth)
- Polemarchakis, Selden and Song (2017)

Utility homeomorphisms.

- Mas-Colell (JET 74)
- Border and Segal (JET 92)

Topologies on preferences: Literature in the 70s (papers by Debreu, Kannai, Grodal and Hildenbrand).

Conclusion

- Convergence of finite-experiment rationalizing preferences.
- Some pathological examples.
- Sufficient conditions for convergence of preferences.
- Convergence of utilities is more subtle.