

Fairness and efficiency for probabilistic allocations with endowments

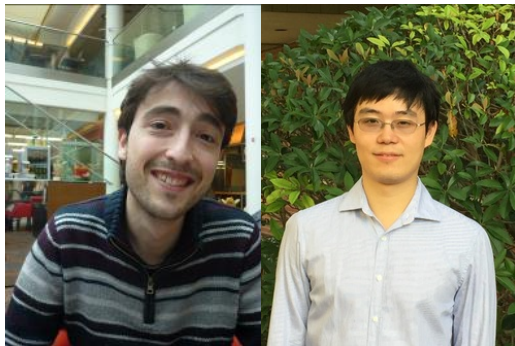
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U. Montreal — March 18th, 2019

Antonio and Jun:



Discrete allocation

Agents: I

Resources: S (finite, indivisible).

1 ●

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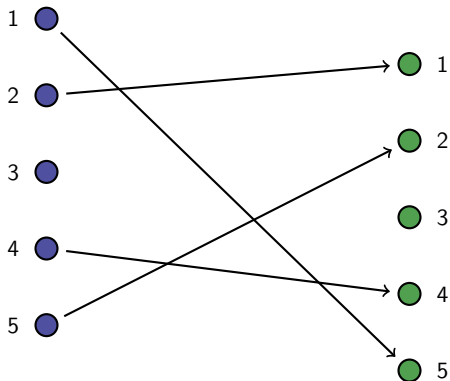
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Discrete allocation

Agents: I

Resources: S (finite, indivisible).



School choice:

- ▶ Property rights are captured by priorities.
- ▶ As property rights, priorities are equivocal; not transparent.
- ▶ We investigate the use of explicit endowments instead.

Problem:

Priorities do not unambiguously map to property rights.

A solution?

- ▶ Endowments are **explicit** property rights.
- ▶ For example, guarantee a:
 1. chance at a good school;
 2. neighborhood school;
 3. slot for a sibling.

Motivation

*“... school choice has not delivered on a central promise: to give every student a **real chance** to attend a good school. Fourteen years into the system, black and Hispanic students are just as isolated in segregated high schools as they are in elementary schools — a situation that school choice was supposed to ease.”*

“The Broken Promises of Choice in New York City Schools” *New York Times*, May 5th, 2017.

Eric Nadelstern at Columbia University, deputy school chancellor when school choice system was implemented, proposed to have a lottery decide the allocations. (“Confronting Segregation in New York City Schools”, *New York Times*, May 15th, 2017.)

Arguably also why Boston abandoned the initial school choice program.

I focus on school choice.

Another application: Time banks.

Three general objectives:

- ▶ Efficiency
- ▶ Fairness
- ▶ Respect property rights

Important challenge: how to talk about fairness for unequal agents.

Standard model:

- ▶ Property rights: school priorities
- ▶ Fairness among unequal agents: lack of justified envy.
- ▶ Efficiency and fairness are incompatible.

This paper:

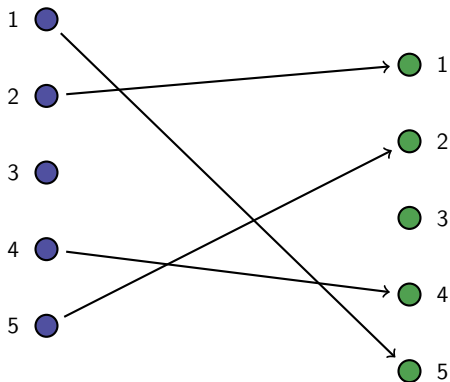
- ▶ Property rights: endowments
- ▶ New notion of fairness among unequally endowed agents
- ▶ Efficiency and fairness (and respect for property rights) are compatible.
- ▶ We can handle quotas and additional constraints.

- ▶ Mkts. & fairness: Varian (1974), Hylland-Zeckhauser (1979), Budish (2011).
- ▶ Justified envy w/endowments: Yilmaz (2010)
- ▶ Exog. and endog. budgets: Mas-Colell (1992), McLennan (2017) and Le (2017)
- ▶ Endowments in school choice: Hamada, Hsu, Kurata, Suzuki, Ueda, and Yokoo (2017)

More references in the paper...

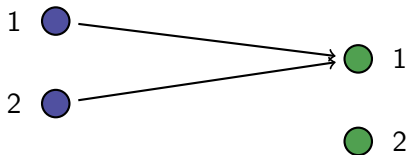
Discrete allocation

Agents: I
Resources: S .



Fairness

Agents: I
Resources: S .



1 and 2 want school s_1 .
Fairness: toss a coin.

- ▶ Agents: $I = \{1, \dots, N\}$.
- ▶ Schools: $S = \{s_1, \dots, s_L\}$.
- ▶ A *lottery* is an element of

$$\Delta_- = \{x \in \mathbf{R}_+^L : \sum_{j=1}^L x_j \leq 1\}$$

- ▶ $u^i : \Delta_- \rightarrow \mathbf{R}$ (cont. & mon.)

An *allocation* is $x = (x^i)_{i=1}^N$, with $x^i \in \Delta_-^L$, s.t

$$\sum_{i \in I} x_s^i = q_s$$

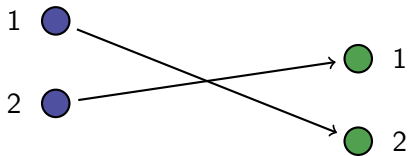
An *allocation* is $x = (x^i)_{i=1}^N$, with $x^i \in \Delta_-^L$, s.t

$$\sum_{i \in I} x_s^i = 1$$

i *envies* j at x if $u^i(x^j) > u^i(x^i)$

$$x^i = (1/N, \dots, 1/N) \implies \text{no envy}$$

What if?



1 and 2 would like to *trade* probability shares.

An allocation x is *Pareto optimal* (PO) if there is no allocation y
s.t

$$u^i(y^i) \geq u^i(x^i) \text{ for all } i \text{ and } u^j(y^j) > u^j(x^j)$$

for some j .

Theorem (Hylland and Zeckhauser (1979))

There is a PO and envy-free allocation. It is a market equilibrium allocation.

An *HZ-equilibrium* is a pair (x, p) , with $x \in \Delta_-^N$ and $p = (p_s)_{s \in S} \geq 0$ s.t.

1. $\sum_{i=1}^N x^i = (1, \dots, 1)$
2. x^i solves

$$\text{Max } \{u^i(z^i) : z^i \in \Delta_- \text{ and } p \cdot z^i \leq 1\}$$

Condition (1): supply = demand.

Condition (2): x^i is i 's demand at prices p and income = 1.

Observe:

- ▶ Income is independent of prices
- ▶ No endowments.

Suppose that each u^i is linear (expected utility).

Theorem (Hylland and Zeckhauser (1979))

There is a HZ equilibrium allocation. It is envy-free and PO.

Fairness among unequals

No envy makes sense for “equals.”

What if agents are unequal?

In an economy with endowments, agents start from different positions.

Fairness among unequals

- ▶ Each i has an *endowment* $\omega^i \in \Delta$.
- ▶ ω^i is an initial lottery.
- ▶ Suppose that $\sum_i \omega^i = (1, \dots, 1)$.

For example, suppose schools are allocated via a lottery. Admission probabilities reflect: neighborhood school (walk-zone priority), sibling priority, or test scores.

A *Walrasian equilibrium* is a pair (x, p) with $x \in \Delta_-^N$, $p \geq 0$ s.t

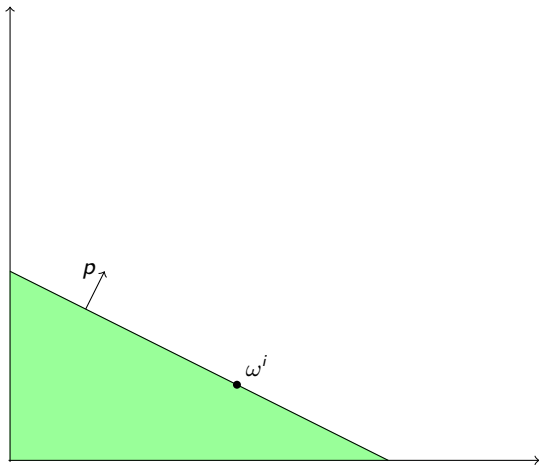
1. $\sum_{i=1}^N x^i = \sum_{i=1}^N \omega^i$; and
2. x^i solves

$$\text{Max } \{u^i(z^i) : z^i \in \Delta_- \text{ and } p \cdot z^i \leq p \cdot \omega^i\}$$

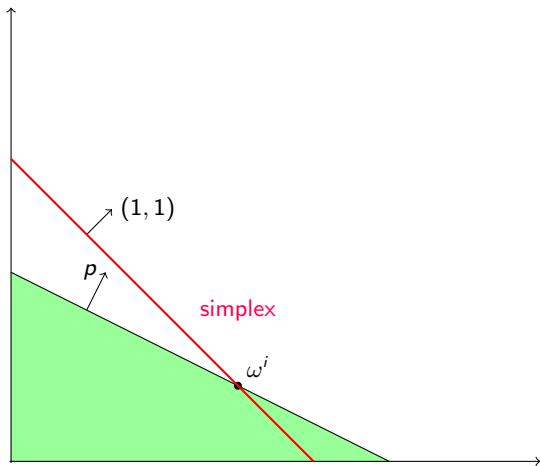
Proposition (Hylland and Zeckhauser (1979))

There are economies in which all agents' utility functions are expected utility, that possess no Walrasian equilibria.

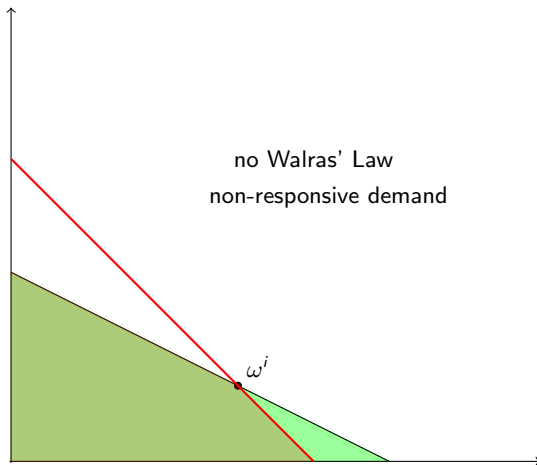
Budget set



Budget set



Budget set



HZ Example

3 agents; exp. utility

	u^1	u^2	u^3
s_A	10	10	1
s_B	1	1	10

Endowments: $\omega^i = (1/3, 2/3)$.

HZ Example

3 agents; exp. utility

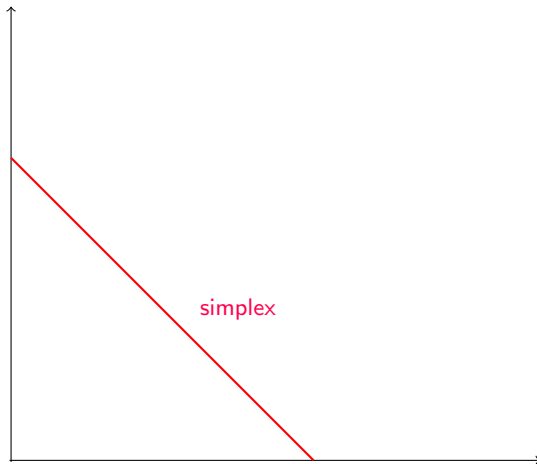
	u^1	u^2	u^3
s_A	10	10	1
s_B	1	1	10

Endowments: $\omega^i = (1/3, 2/3)$.

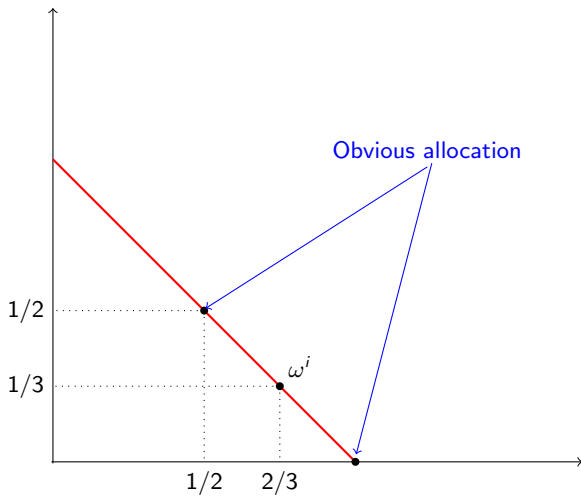
Obvious allocation:

$$\begin{aligned}x^1 &= x^2 = (1/2, 1/2) \\x^3 &= (0, 1)\end{aligned}$$

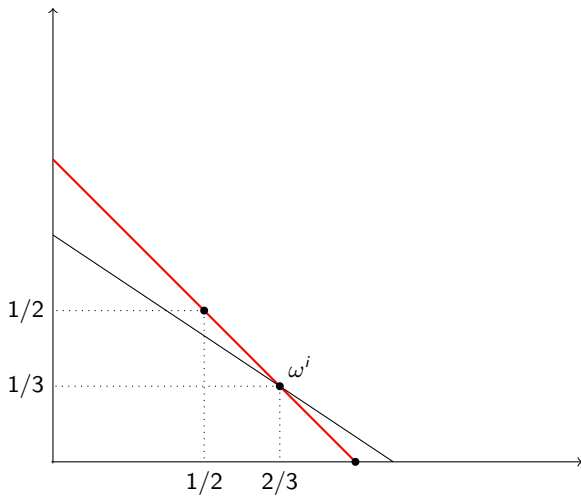
HZ Example



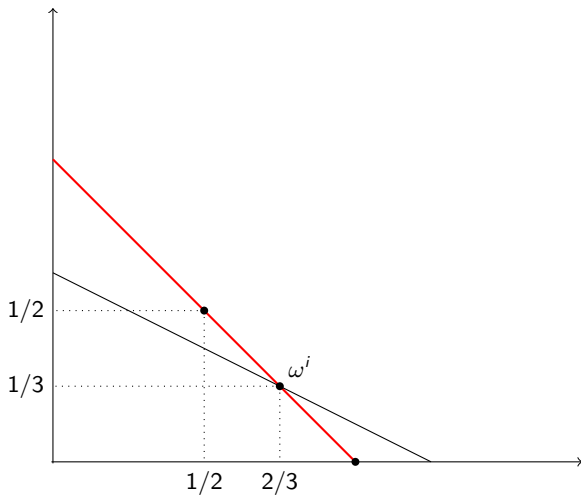
HZ Example



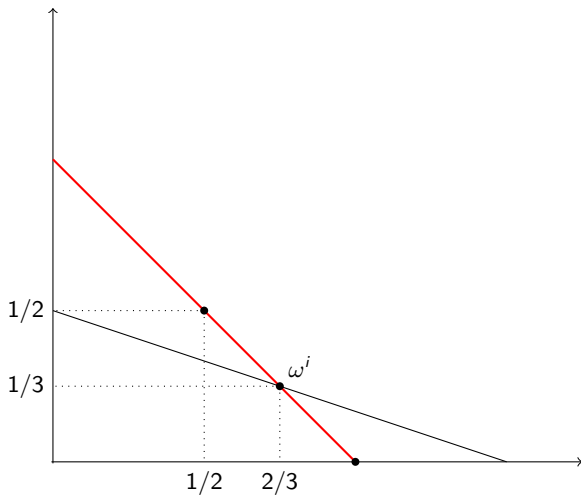
HZ Example



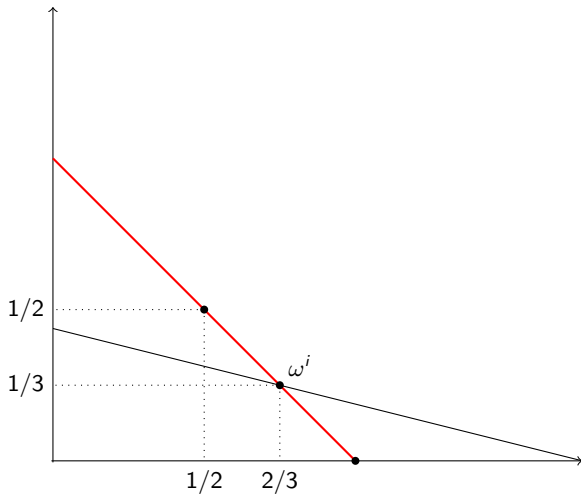
HZ Example



HZ Example

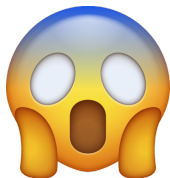


HZ Example



Moreover, . . .

- ▶ the first welfare theorem fails.
- ▶ There are Pareto ranked Walrasian equilibria.



Our results

A function $u : \Delta_-^n \rightarrow \mathbf{R}$ is

- ▶ *concave* if $\forall x, z \in \Delta_-$, and $\forall \lambda \in (0, 1)$,
 $\lambda u(z) + (1 - \lambda)u(x) \leq u(\lambda z + (1 - \lambda)x)$;
- ▶ *quasi-concave* if, $\forall x \in \Delta_-$,

$$\{z \in \Delta_- : u(z) \geq u(x)\}$$

is a convex set.

- ▶ *semi-strictly quasi-concave* if $\forall x, z \in \Delta_-$,

$$u(z) < u(x) \text{ and } \lambda \in (0, 1) \implies u(z) < u(\lambda z + (1 - \lambda)x)$$

- ▶ *expected utility* if it is linear.

Let x be an allocation.

- ▶ x is *weak Pareto optimal* (wPO) if \nexists an allocation y s.t.
 $u^i(y^i) > u^i(x^i)$ for all i
- ▶ ε -*weak Pareto optimal* (ε -PO), for $\varepsilon > 0$, if \nexists an allocation y
s.t $u^i(y^i) > u^i(x^i) + \varepsilon$ for all i .

Let x be an allocation.

- ▶ x is *acceptable* to i if $u^i(x^i) \geq u^i(\omega^i)$.
- ▶ x is *individually rational* (IR) if it is acceptable to all agents.

i envies j at x if $u^i(x^j) > u^i(x^i)$.

Such envy will be tolerated (i.e not be justified) only if j 's endowment is "good enough."

i envies j at x if $u^i(x^j) > u^i(x^i)$.

Such envy will be tolerated (i.e not be justified) only if j regards x^i as *unacceptable*.

i envies j at x if $u^i(x^j) > u^i(x^i)$.

Such envy will be tolerated (i.e not be justified) only if
 $u^j(\omega^j) > u^j(x^i)$

i has *justified envy* towards j at allocation x if

$$u^i(x^j) > u^i(x^i) \text{ and } u^j(x^i) \geq u^j(\omega^j).$$

Let x be an allocation.

x has *no justified envy* (NJE) if no agent has justified envy towards any other agent at x .

Observe: NJE and IR imply *equal treatment of equals*.

Let x be an allocation.

x has *no justified envy* (NJE) if no agent has justified envy towards any other agent at x .

- ▶ i has *strong justified envy* (SJE) towards j at x if $u^i(x^j) > u^i(x^i)$ and $v^j(x^i) > v^j(\omega^j)$.
- ▶ For $\varepsilon > 0$, i has *ε -justified envy* (ε -JE) towards j at x if $u^i(x^j) > u^i(x^i)$ and $v^j(x^i) > v^j(\omega^j) - \varepsilon$.

no ε -justified envy \implies no justified envy \implies no strong just. envy

Theorem

Suppose all u^i are concave, and let $\varepsilon > 0$.

- 1. \exists an allocation that is ε -IR, ε -PO and has no ε -justified envy;*
- 2. \exists an allocation that is IR, wPO and has no strong justified envy.*

Consider problem

$$\text{Max } \sum_i \lambda_i u^i(x_i)$$

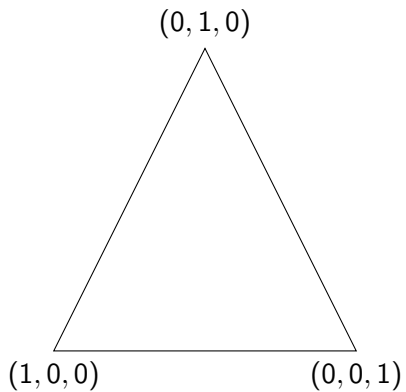
s.t. x is an allocation.

Obtain a NJE allocation from this problem

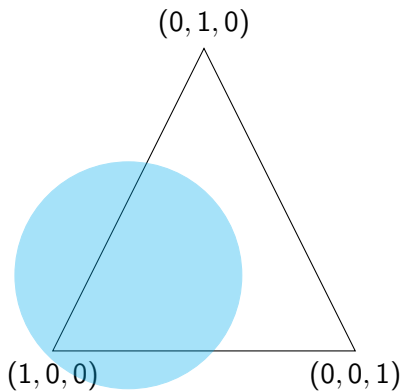
by choosing right welfare weights, $(\lambda_i) \in \Delta^N$.

(Actual proof uses an approximation to this problem, hence the ε).

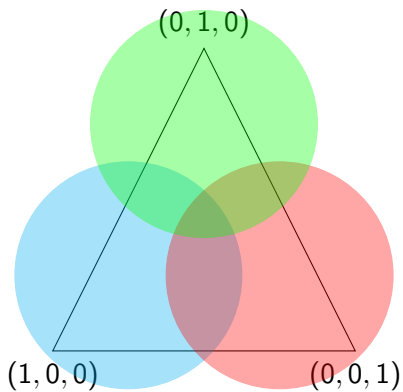
KKM Lemma



KKM Lemma



KKM Lemma



Proposal No. 2: Walrasian equilibrium.

Let $\alpha \in [0, 1]$

An *α -slack Walrasian equilibrium* is (x, p) with $x \in \Delta_-^N$, $p \geq 0$ s.t

1. $\sum_{i=1}^N x^i = \sum_{i=1}^N \omega^i$; and
2. x^i solves

$$\text{Max } \{u^i(z^i) : z^i \in \Delta_- \text{ and } p \cdot z^i \leq \alpha + (1 - \alpha)p \cdot \omega^i\}$$

$$x_{i \leftrightarrow j}^i = x^j, x_{i \leftrightarrow j}^j = x^i, \text{ and } x_{i \leftrightarrow j}^k = x^k \text{ for all } k \in I \setminus \{i, j\}.$$

A constraint structure \mathcal{H} is *anonymous* if for all $x \in \mathcal{A}^{\mathcal{H}}$ and all distinct $i, j \in I$, $x_{i \leftrightarrow j} \in \mathcal{A}^{\mathcal{H}}$.

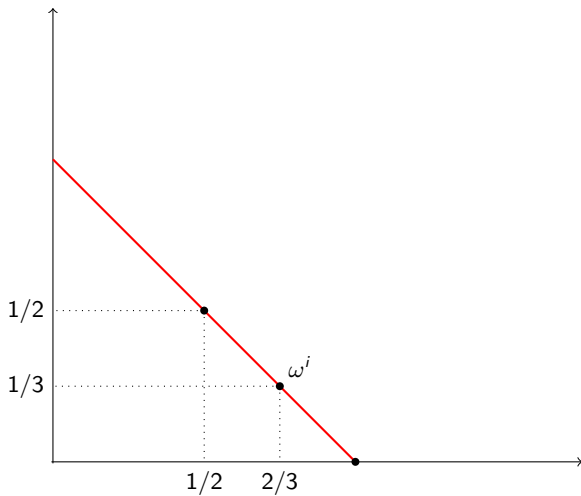
Suppose all u^i are quasi-concave.

Theorem

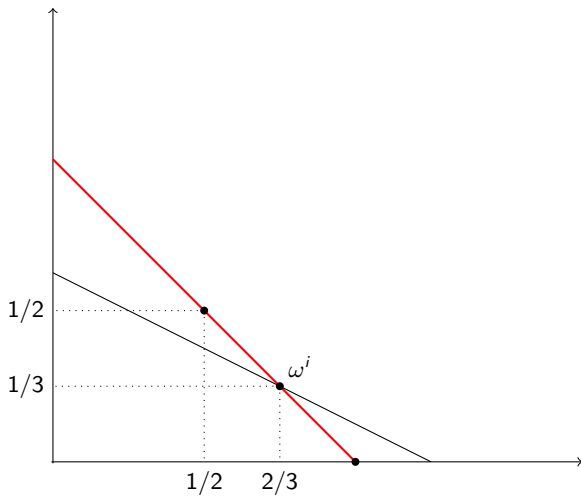
$\forall \alpha \in (0, 1], \exists$ an α -slack Walrasian equilibrium (x, p) .

Moreover, if all u^i are semi-strictly quasi-concave, then x is PO.

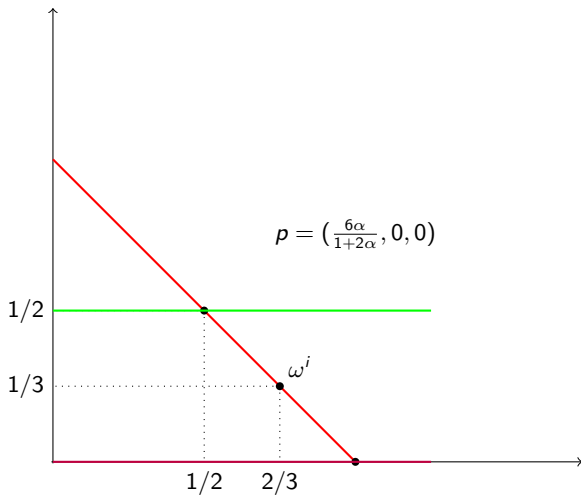
HZ Example



HZ Example



HZ Example



Having $\alpha > 0$ gives an “extra role” for prices, allowing agents 1 and 2 to spend above the income they get from $p \cdot \omega^i$.

That doesn't mean that ω^i don't matter ...

Suppose all u^i are semi-strictly quasi-concave.

Theorem

$\forall \varepsilon > 0 \exists \alpha \in (0, 1]$ and an α -slack Walrasian equilibrium (x, p) , s.t x is PO and

$$\max\{u^i(y) : y \in \Delta_- \text{ and } p \cdot y \leq p \cdot \omega^i\} - u^i(x) < \varepsilon.$$

In particular, x is ε -individually rational.

Let (x, p) be an α -slack Walrasian equilibrium.

If i envies j at x , then $p \cdot \omega^j > p \cdot \omega^i$.

“Society” values j 's endowment more than i 's.

Suppose all u^i are concave and C^1 . Let (x, p) be an α -slack Walrasian equilibrium. Denote by $S = \{i : u^i(x^i) = \max\{u^i(z^i) : z^i \in \Delta_-\}\}$ the set of *satiated* consumers, and by $U = [N] \setminus S$. Suppose that $\sum_{i \in U} x^i \gg 0$.

Theorem

If i envies j at x then $p \cdot \omega^j > p \cdot \omega^i$,

Suppose all u^i are concave and C^1 . Let (x, p) be an α -slack Walrasian equilibrium. Denote by $S = \{i : u^i(x^i) = \max\{u^i(z^i) : z^i \in \Delta_-\}\}$ the set of *satiated* consumers, and by $U = [N] \setminus S$. Suppose that $\sum_{i \in U} x^i \gg 0$.

Theorem

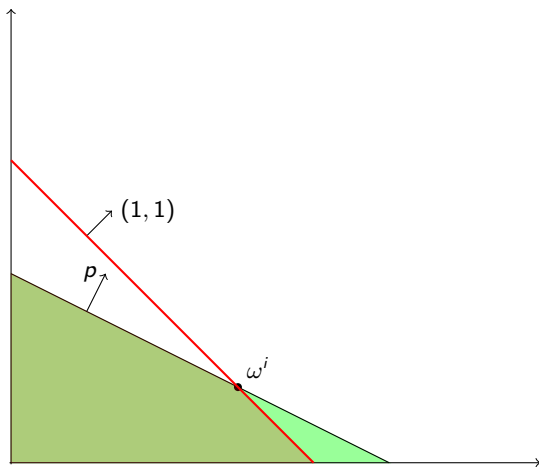
If i envies j at x then $p \cdot \omega^j > p \cdot \omega^i$, and \exists welfare weights $\theta \in \mathbf{R}_{++}^U$ s.t

$$v(t) = \sup \left\{ \sum_{i \in U} \theta^i u^i(\tilde{x}^i) : (\tilde{x}^i) \in \Delta_-^U \text{ and} \right. \\ \left. \sum_{i \in U} \tilde{x}^i \leq \bar{\omega} + t(\omega^i - \omega^j) - \sum_{i \in S} x^i \right\},$$

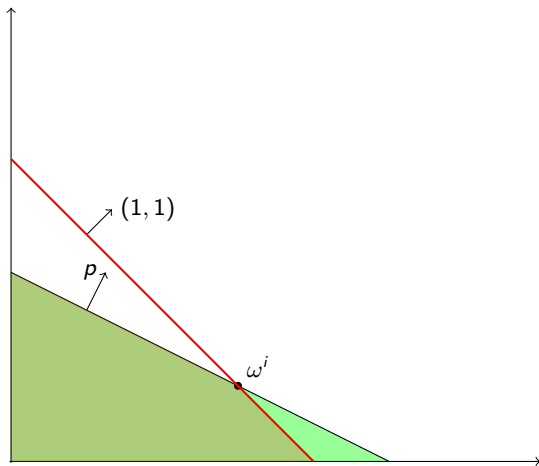
then $(x^i)_{i \in U}$ solves the problem for $v(0)$, and $v(t) < v(0)$ for all t small enough.

Idea

Classical result relies on Walras Law: $p \cdot z(p) = 0$ for all p . Walras Law does not hold in our model because. . .



Demand is not responsive to price once boundary is reached.



Budget constraint:

$$p \cdot x^i \leq \alpha + (1 - \alpha)p \cdot \omega^i$$

Budget constraint:

$$p \cdot (x^i - \omega^i) \leq \alpha(1 - p \cdot \omega^i).$$

This allows prices to matter: large prices imply that the value of excess demand is < 0 .

Consider $\phi : [0, \bar{p}]^L \rightarrow [0, \bar{p}]^L$ defined by

$$\phi_I(p) = \{\min\{\max\{0, \zeta_I + p_I\}, \bar{p}\} : \zeta \in z(p)\}.$$

where \bar{p} is a large price.

Lemma

ϕ is upper hemi-continuous, convex- and compact- valued.

(In paper deal with a different ϕ , which ensures PO.)

By Kakutani, $\exists p^*$ and $\zeta \in z(p^*)$ s.t

$$p_l^* = \min\{\max\{0, \zeta_l + p_l^*\}, \bar{p}\}.$$

Lemma

$$p^* \cdot \zeta \geq 0.$$

This is sort of a “weak Walras law.”

$$\text{Pf: } \zeta_l < 0 \implies p_l^* = 0$$

Lemma

$p_l^* < \bar{p}$ for all $l \in [L]$

Pf: Suppose $p_l^* = \bar{p}$. \bar{p} is large $\implies 1 - p \cdot \omega^i < 0$; so
 $p \cdot (x^i - \omega^i) < 0$.

By adding up we get that

$$p \cdot \zeta \leq \alpha(N - p \cdot \bar{\omega}) < 0,$$

in contradiction to prev. lemma.

Now think about:

$$p_l^* = \min\{\max\{0, \zeta_l + p_l^*\}, \bar{p}\}.$$

when $p_l^* < \bar{p}$.

we have

$$p_l^* = \max\{0, \zeta_l + p_l^*\}.$$

$$p_l^* = \max\{0, \zeta_l + p_l^*\}.$$

For all l , $\zeta_l = 0$, or $\zeta_l < 0$ and $p_l^* = 0$.

Latter case is not possible.

Pareto ranked equilibria

Two agents and two schools.

i	$u_{S_1}^i$	$u_{S_2}^i$
1	1	1
2	1	100

Endowments $\omega^i = (1/2, 1/2)$.

Consider the allocations

$x = ((1, 0), (0, 1))$ and

$y = ((1/2, 1/2), (1/2, 1/2))$.

Note that x Pareto dominates y .

Pareto ranked equilibria

α	allocation	p	$\alpha + (1 - \alpha)p \cdot \omega^i$
0	x	(1, 1)	1
1/2	x	(1, 1)	1
0	y	(0, 1)	1/2
1/2	y	(0, 2)	1

Theorem

Any Walrasian equilibrium with slack is weakly Pareto optimal, and any Walrasian equilibrium with slack and the cheapest-bundle property is Pareto optimal.

- ▶ Mkts. & fairness: Varian (1974), Hylland-Zeckhauser (1979), Budish (2011).
- ▶ Justified envy w/endowments: Yilmaz (2010)
- ▶ Exog. and endog. budgets: Mas-Colell (1992), McLennan (2017) and Le (2017)
- ▶ Endowments in school choice: Hamada, Hsu, Kurata, Suzuki, Ueda, and Yokoo (2017)

- ▶ New criteria for discrete (probabilistic) allocation.
- ▶ Model with *explicit* property rights.
- ▶ Fairness among unequals.
- ▶ Market equilibrium
- ▶ Fairness, efficiency and property rights.