A characterization of combinatorial demand

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This paper

Literature on matching (e.g Kelso-Crawford) and combinatorial auctions (e.g Milgrom):

$$D(p) = \text{argmax}\{v(A) - \sum_{a \in A} p_a : A \subseteq X\} \qquad (*)$$

When is * true?
What is the behavioral content of the combined assumptions of *rationality* and *quasilinearity*?

Notation

- Let X be a finite set (of *items*).
- Let S be the set of all nonempty subsets of 2^X
- (so the empty set is not in S, but $\{\varnothing\}$ is).
- Identify $A \subseteq X$ with $1_A \in \mathbf{R}^X$.
- If $p \in \mathbf{R}^X$ then $\langle p, A \rangle = \sum_{x \in A} p_x$.

Demand

A *demand function* is

$$D : \mathbf{R}^X_{++} \to S$$

s.t. $\exists \bar{p} \in \mathbf{R}^X_{++}$ with $D(p) = \{\varnothing\}$ for all $p \geq \bar{p}$.

($\bar{p}$ a choke price)

Demand

D is *quasilinear rationalizable* if

$\exists\ v : 2^X \to \mathbf{R}$ s.t

$$D(p) = \text{argmax}_{A \subseteq X} v(A) - \langle p, A \rangle$$

Suppose D is QL-rationalizable

Let $A \in D(p)$ and $B \in D(q)$.

$$v(A) - \langle p, A \rangle \geq v(B) - \langle p, B \rangle$$
$$v(B) - \langle q, B \rangle \geq v(A) - \langle q, A \rangle.$$

Thus: $\langle p - q, A - B \rangle \leq 0$.

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The law of demand!

Demand

A demand function D

- satisfies the *law of demand* if for all $p, q \in \mathbf{R}^X_{++}$, and all $A \in D(p)$ and $B \in D(q)$,

$$\langle p - q, A - B \rangle \leq 0;$$

- is *upper hemicontinuous* if, $\forall p \in \mathbf{R}^X_{++}$, $\exists$ nbd V of p s.t. $D(q) \subseteq D(p)$ when $q \in V$.

Main result

Theorem

A demand function is quasilinear rationalizable iff it is upper hemicontinuous and satisfies the law of demand.

Identification

Theorem

For any quasilinear rationalizable D, there is a unique monotone $v : 2^X \to \mathbf{R}$ for which $v(\varnothing) = 0$ which rationalizes D.

Utility is identified up to an additive constant.

Monotone rationalization

D is *monotone, concave, quasilinear rationalizable (MCQ-rationalizable)* if $\exists$ a monotone, concave $g : \mathbf{R}_+^X \to \mathbf{R}$ s.t $v(A) = g(1_A)$, and

$$D(p) = \text{argmax}\{v(A) - \langle p, A\rangle : A \subseteq X\}.$$

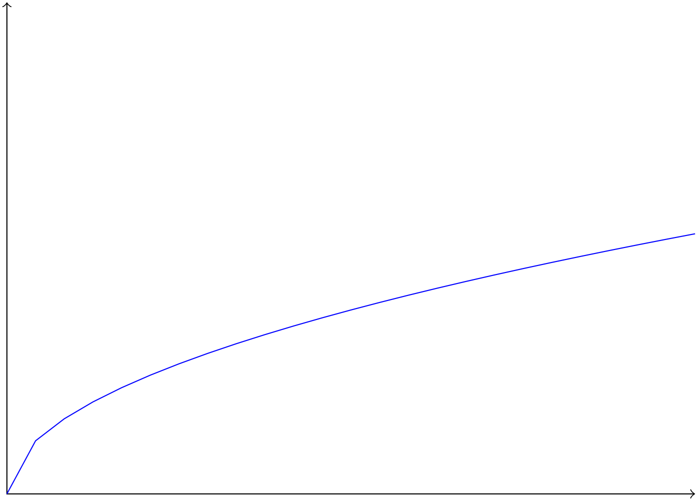
Corollary

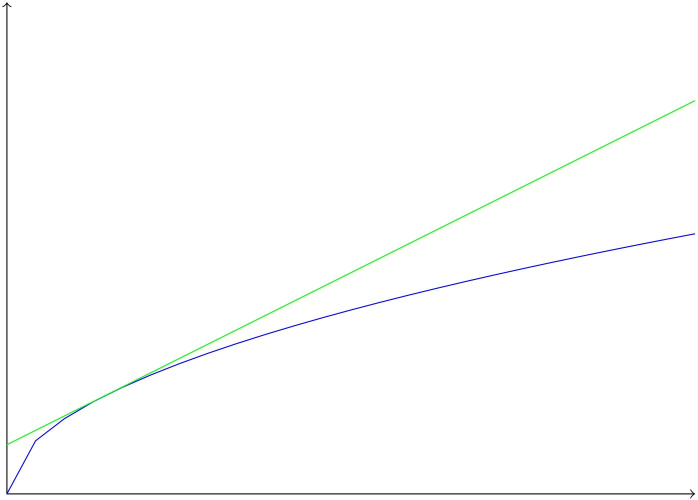
If a demand function is quasilinear rationalizable, then it is MCQ-rationalizable.

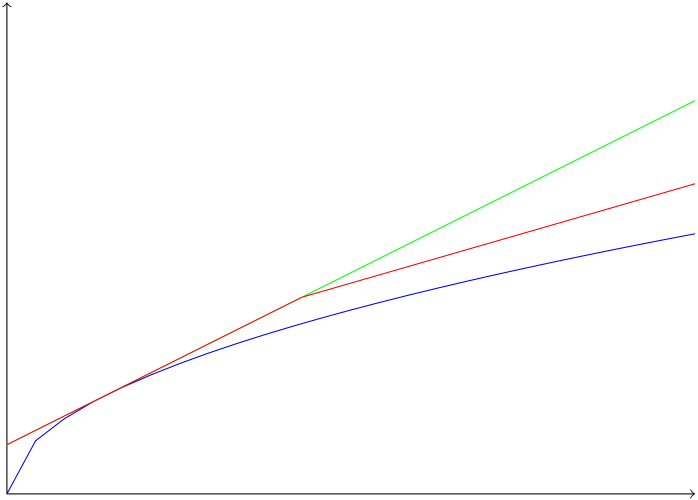
Proof ideas

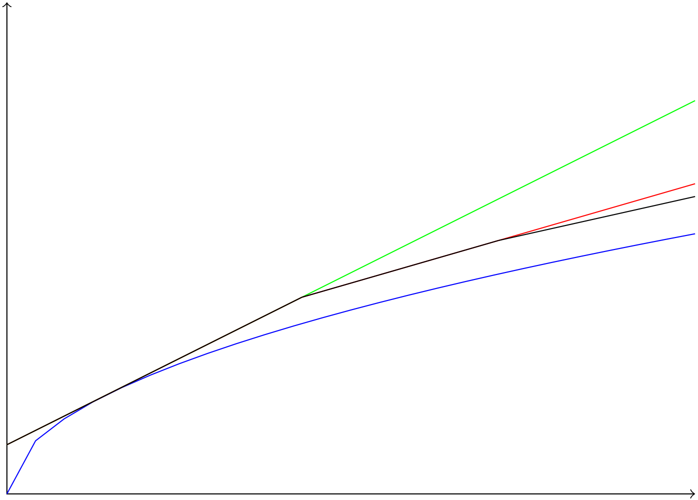
$$D(p) = \text{argmax}_{A \subseteq X} v(A) - \langle p, A \rangle$$

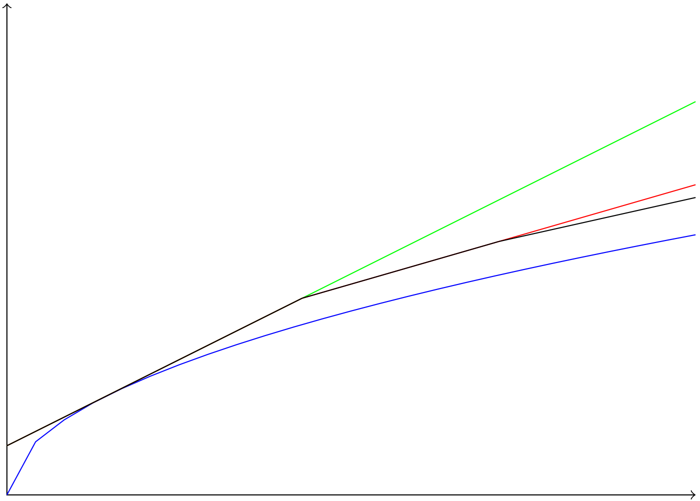
If $A \in D(p)$ then we want p to be the "gradient of v at A."
Can recover v by "integrating" over p.











Cyclic monotonicity

D satisfies *cyclic monotonicity* if, for all n (using summation mod n),

$$\sum_{i=1}^{n} \langle p_i, A_i - A_{i+1} \rangle \leq 0,$$

where $A_i \in D(p_i)$, for all sequences $\{p_i\}_{i=1}^{n}$.

Cyclic monotonicity

Define:

$$v(A) = \inf \langle p_1, A - A_1 \rangle + \ldots + \langle \bar{p}, A_k - \varnothing \rangle,$$

inf is taken over all finite seq. $(p_i, A_i)_{i=1}^{k}$ with $A_i \in D(p_i)$.

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Observe, by CM,

$$-\{\langle p_1, A - A_1 \rangle + \ldots + \langle \bar{p}, A_k - \varnothing \rangle\} + \langle p, A - \varnothing \rangle \leq 0.$$

So $v(A)$ is well defined (and ≥ 0).

Let $A \in D(p)$ and $B \subseteq X$ ($B \in D(\mathbf{R}^X_{++})$ need a different arg. otherwise).
By defn. of v,

$$v(B) \leq \langle p, B - A \rangle + v(A).$$

Thus $v(A) - \langle p, A \rangle \geq v(B) - \langle p, B \rangle$.

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Proof that if $A \in D(p)$ and $B \notin D(p)$ then $v(A) - \langle p, A \rangle > v(B) - \langle p, B \rangle$ requires more.

D satisfies condition ♠ if
$\forall p$ and $B \notin D(p)$ $\exists A \in D(p)$ and p' s.t

$$A \in D(p') \text{ and } \langle p', A - B \rangle > \langle p, A - B \rangle.$$

Lemma

If D is upper hemicontinuous, then it satisfies condition ♠.

Cyclic monotonicity

Lemma

If D satisfies cyclic monotonicity, and condition ♠, then it is quasilinear rationalizable.

Based on ideas in Rochet/Rockafellar (but ♠ plays a technical role).

Lemma

A demand function satisfies cyclic monotonicity if it satisfies the law of demand.

Follows from recent results in mech. design (Lavi, Mu'alem, and Nisan; Saks and Yu; and Ashlagi, Braverman, Hassidim, and Monderer).

Related literature

- Rochet/Rockafeller
- Brown and Calsamiglia
- Sher and Kim
- Lavi, Mu'alem, and Nisan;
- Saks and Yu;
- Ashlagi, Braverman, Hassidim, and Monderer

Conclusions

- Quasilinear rational demand is a ubiquitous assumption.
- Our result is the first characterization in terms of observable behavior.
- Identification enables welfare analysis.
- New use for recent results in mech. design.