Average choice

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Path independence

This paper:
An exploration of path independence in stochastic choice.

Plott path independence

Plott (Ecma 1973) in response to Arrow's impossibility theorem:

$$c(A \cup B) = c(c(A) \cup c(B))$$

For example: if $c(x, y) = x$,
then

$$c(x, y, z) = c(c(x, y) \cup c(z)) = c(x, z)$$

$c =$ "choice"

Plott path independence

- Kalai-Megiddo (Ecma 1980)
- Machina-Parks (Ecma 1981)

NO stochastic choice can be continuous and Plott Path Indep.*

Restore the impossibility.

Primitive: stochastic* choice.
(I'll explain what I mean by stochastic*).

Path independence

We: allow the path to affect choice.

Choice from $A \cup B$ is a *lottery* between choice from A and choice from B.

Who A and B are may affect the lottery.

Our main result

Theorem

A stochastic choice is cont. and path independent iff it is a cont. Luce (or Logit) rule.*

Our main result

Kalai-Megiddo and Machina-Parks Impossibility thm.:

NO stochastic choice can be cont. and PPI.*

Our paper: tweaking PPI avoids impossibility and characterizes the Luce model.

Stochastic choice

Stochastic choice: for each A,

given prob. of choosing x out of A.

Average ($=$ stochastic*) choice:

given the average (or mean) stochastic choice from A.

Stochastic choice at McDonalds

	Burger	Cheese burger	Fries	Drink	prob
Combo 1	1	0	2	50	.5
Combo 2	0	2	1	50	.2
Kids menu	0	1	.5	25	.3
avg.	.5	.7	1.35	42.5	

For ex. standard IO models (Berry-Levinsohn-Pakes).

Why average choice?

- Aggregate data can be available when choice frequencies are not.
- Aggregate data can be more reliably estimated.
- Allows us to understand how utility depends on object characteristics. This is how economists use the Logit model.

Main thm.

Luce or *Logit:*

$$\rho(x, A) = \frac{u(x)}{\sum_{y \in A} u(y)}$$

Average choice:

$$\rho^*(A) = \sum_{x \in A} x \rho(x, A)$$

Luce model iff

- Path independence
- Continuity

Results - II

Characterization of the *(ordinally) linear Luce model:*

$$\rho(x, A) = \frac{u(x)}{\sum_{y \in A} u(y)} = \frac{f(v \cdot x)}{\sum_{y \in A} f(v \cdot y)}$$

Average choice:

$$\rho^*(A) = \sum_{x \in A} x \rho(x, A)$$

An avg. choice is cont. PI, and independent iff it is a linear Luce rule.

Results - III

Characterization of the *(cardinally) affine Luce model:*

$$\rho(x, A) = \frac{u(x)}{\sum_{y \in A} u(y)} = \frac{v \cdot x + \beta}{\sum_{y \in A}(v \cdot y + \beta)}$$

Average choice:

$$\rho^*(A) = \sum_{x \in A} x \rho(x, A)$$

An avg. choice is cont. PI, independent, and calibrated iff it is an affine Luce rule.

Small sample advantage.

Luce's IIA

$$\frac{\rho(x,\{x,y\})}{\rho(y,\{x,y\})} = \frac{\rho(x,\{x,y,z\})}{\rho(y,\{x,y,z\})}$$

- Theory: ρ is observed.
- Reality: ρ is estimated.

Small sample advantage.

Estimating frequencies can require large samples.

Luce (1959): need 1000s of observations to test his model.

"It is clear that rather large sample sizes are required from each subset to obtain sensitive direct tests of axiom 1."

Average choice avoids the problem.

Primitive

- Let X be a compact and convex subset of $\mathbf{R}^n$, with $n \geq 2$.
  For ex. $X = \Delta(P)$ and P set of *prizes*

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- Let $\mathcal{A}$ be the set of all finite subsets of X.

Primitive

- Let X be a compact and convex subset of $\mathbf{R}^n$, with $n \geq 2$. For ex. $X = \Delta(P)$ and P set of *prizes*
- Let $\mathcal{A}$ be the set of all finite subsets of X.
- An *average choice* is a function

$$\rho^* : \mathcal{A} \to X,$$

such that, for all $A \in \mathcal{A}$, $\rho^*(A) \in \text{conv}A$.

Luce model

A *stochastic choice* is a function $\rho : \mathcal{A} \to \Delta(X)$ s.t. $\rho(A) \in \Delta(A)$.

$\rho : \mathcal{A} \to \Delta(X)$ is a *continuous Luce rule* if $\exists$ a cont. $u : X \to \mathbf{R}_{++}$ s.t.

$$\rho(x, A) = \frac{u(x)}{\sum_{y \in A} u(y)}.$$

Luce rationalizable

ρ^* is *continuous Luce rationalizable* if $\exists$ a cont. Luce rule ρ s.t.

$$\rho^*(A) = \sum_{x \in A} x \rho(x, A).$$

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$$\rho^*(A) = \sum_{x \in A} x\rho(x, A).$$

i.e. if $\exists$ cont. $u : X \to \mathbf{R}_{++}$ s.t.

$$\rho^*(A) = \sum_{x \in A} \left(\frac{u(x)}{\sum_{y \in A} u(y)} \right) x.$$

Path independence

If $A \cap B = \emptyset$ then

$$\rho^*(A \cup B) = \lambda \rho^*(A) + (1 - \lambda)\rho^*(B),$$

for some $\lambda \in (0, 1)$.

Path independence

Contrast with Plott P.I.:

$$\rho^*(A \cup B) = \rho^*(\{\rho^*(A), \rho^*(B)\}).$$

Path independence

Contrast with Plott P.I.:

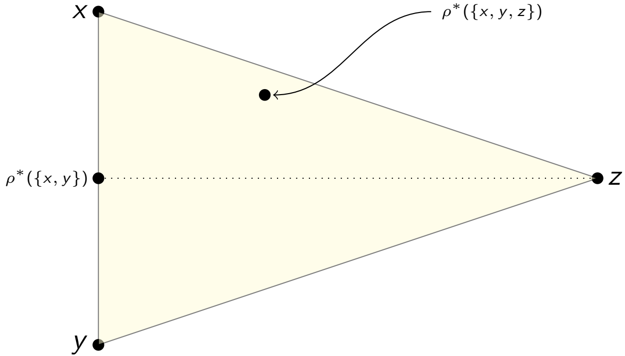
$$\rho^*(A \cup B) = \rho^*(\{\rho^*(A), \rho^*(B)\}).$$

Let $\rho^*(A) = \rho^*(A')$.

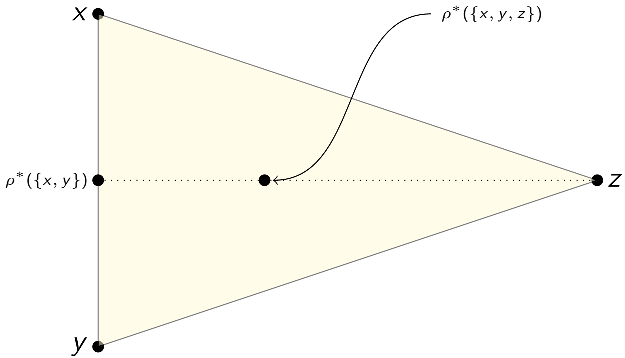
Then PPI demands: $\rho^*(A \cup B) = \rho^*(A' \cup B)$.

We allow for the "path" to matter *through the weights* on $\rho^*(A) = \rho^*(A')$ and $\rho^*(B)$.

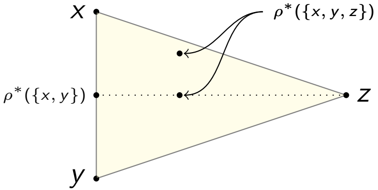
Path independence



Path independence



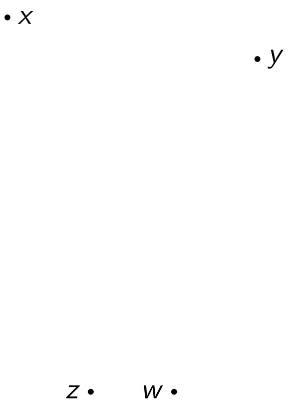
Path independence



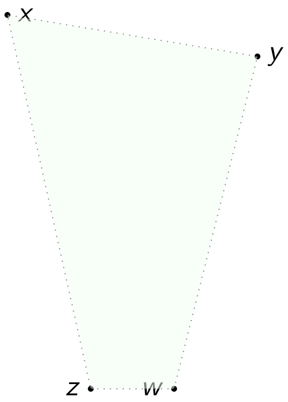
Luce's IIA:

$$\frac{\rho(x, \{x, y\})}{\rho(y, \{x, y\})} = \frac{\rho(x, \{x, y, z\})}{\rho(y, \{x, y, z\})}$$

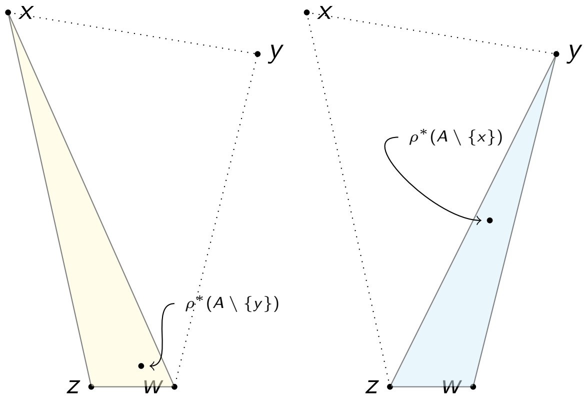
Path independence



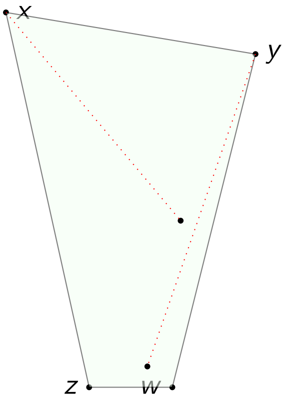
Path independence



Violation of path independence



Violation of path independence



$\rho^*(A)$?

Continuity

Let $x \notin A$.
For any sequence x_n in X, if $x = \lim_{n \to \infty} x_n$, then

$$\rho^*(A \cup \{x\}) = \lim_{n \to \infty} \rho^*(A \cup \{x_n\}).$$

Theorem

An average choice is continuous Luce rationalizable iff it satisfies continuity and path independence.

Proof sketch

Necessity:

$$\rho^*(A) = \sum_{x \in A} \left(\frac{u(x)}{\sum_{y \in A} u(y)} \right) x$$

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$$\left(\sum_{x \in A} u(x) + \sum_{y \in B} u(x) \right) \rho^*(A \cup B) = \sum_{x \in A} u(x)x + \sum_{x \in B} u(x)x$$

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Plott path independence

If $\text{conv}A \cap \text{conv}B = \emptyset$ then

$$\rho^*(A \cup B) = \rho^*\left(\{\rho^*(A), \rho^*(B)\}\right).$$

Plott path independence

Kalai-Megiddo (*Ecma* 1980) and Machina-Parks (*Ecma* 1981):

Theorem

If ρ^* is continuous then it cannot satisfy Plott path independence.

Plott path independence

Proposition

If an average choice is continuous Luce rationalizable, then it cannot satisfy Plott path independence.

Plott path independence

Let $x, y, z \in X$ be aff. indep..
PPI $\Rightarrow$

$$\begin{aligned}\rho^*(\{x, y, z\}) &= \rho^*(\rho^*(\{x, y\}), \{z\}) \\ &= \frac{u(\rho^*(\{x, y\}))\rho^*(\{x, y\}) + u(z)z}{u(\rho^*(\{x, y\})) + u(z)}.\end{aligned}$$

Plott path independence

Let $x, y, z \in X$ be aff. indep..
PPI $\Rightarrow$

$$\begin{aligned}\rho^*(\{x,y,z\}) &= \rho^*(\rho^*(\{x,y\}),\{z\}) \\ &= \frac{u(\rho^*(\{x,y\}))\rho^*(\{x,y\}) + u(z)z}{u(\rho^*(\{x,y\})) + u(z)}.\end{aligned}$$

But

$$\rho^*(\{x,y,z\}) = \frac{u(x)x + u(y)y + u(z)z}{u(x) + u(y) + u(z)}.$$

Plott path independence

By aff. indep.:

$$\frac{u(z)}{u(x)+u(y)+u(z)} = \frac{u(z)}{u(\rho^*(\{x,y\}))+u(z)}.$$

$$\Rightarrow u(\rho^*(\{x,y\})) = u(x)+u(y).$$

Plott path independence

So:

$$u(x) + u(y) = u(\rho^*(\{x, y\})) = u\left(\frac{u(x)x + u(y)y}{u(x) + u(y)}\right).$$

Choose y arbitrarily close to x while satisfying aff. indep.
Then $u(x) = 2u(x)$, a contradiction as $u(x) > 0$.

Plott path independence

Proposition

No average choice satisfies Plott path independence and (our) path independence.

Linear Luce

ρ is a *linear Luce rule* if

- $\exists\ v \in \mathbf{R}^n$;
- and a monotone and cont. $f : \mathbf{R} \to \mathbf{R}_{++}$

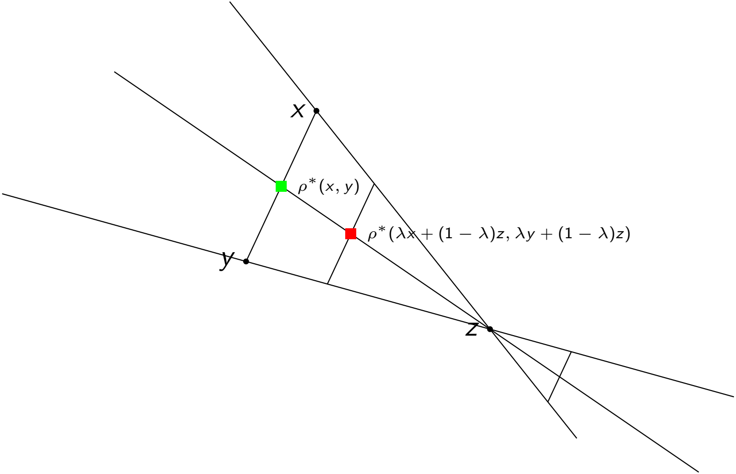
s.t.

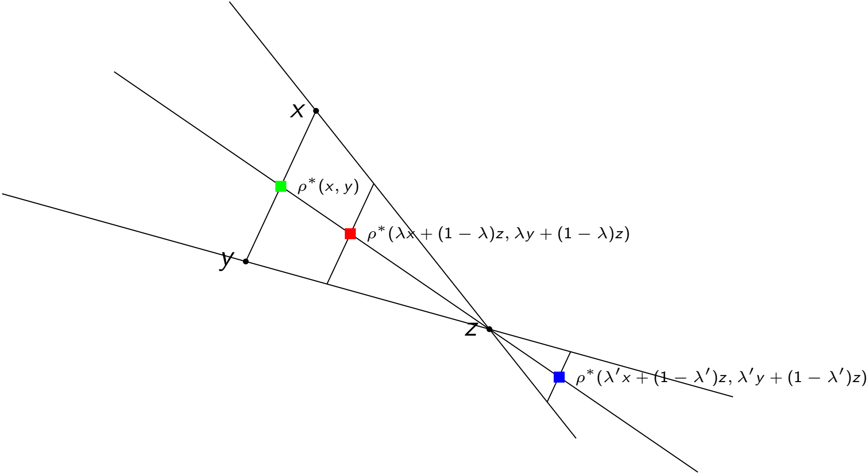
$$\rho(x, A) = \frac{f(v \cdot x)}{\sum_{y \in A} f(v \cdot y)}.$$

Independence

$u(x) = u(y)$ iff $\forall \lambda, z$

$$\rho^*(\{\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)z\}) = \lambda \rho^*(\{x, y\}) + (1-\lambda)z$$


x
$\rho^*(x, y)$
$\rho^*(\lambda x + (1 - \lambda)z, \lambda y + (1 - \lambda)z)$
y
z


x
y
z
$\rho^*(x, y)$
$\rho^*(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)z)$
$\rho^*(\lambda' x + (1-\lambda')z, \lambda' y + (1-\lambda')z)$

Linear Luce

Theorem

An average choice is continuous linear Luce rationalizable iff it satisfies independence, continuity and path independence.

Linear Luce

Let ρ^* be cont. Luce rationalizable.

Lemma

If ρ^* satisfies independence then $u(x) = u(y)$ iff

$$u(\lambda x + (1-\lambda)z) = u(\lambda y + (1-\lambda)z) \quad \forall \lambda, z$$

Linear Luce

Let ρ^* be cont. Luce rationalizable.

Lemma

If ρ^* satisfies independence, then

$$u(x) \geq u(y) \textit{ iff } u(\lambda x + (1-\lambda)z) \geq u(\lambda y + (1-\lambda)z) \quad \forall \lambda, z.$$

Strictly affine Luce

ρ is a *strictly affine Luce rule* if $\exists$

- $v \in \mathbf{R}^n$;
- and $\beta \in \mathbf{R}$

s.t.

$$\rho(x, A) = \frac{v \cdot x + \beta}{\sum_{y \in A}(v \cdot y + \beta)}.$$

Calibration

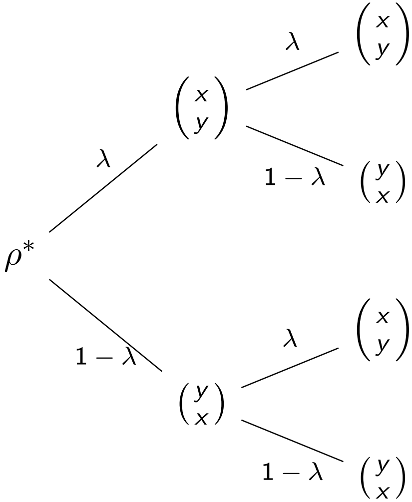
$$\rho^*(\{\lambda x + (1-\lambda)y, \lambda y + (1-\lambda)x\}) = \rho^*(\{x,y\}) \\ + 2\lambda(1-\lambda)\left(x + y - 2\rho^*(\{x,y\})\right)$$

Calibration

Interpret $\lambda x + (1-\lambda)y$ and $\lambda y + (1-\lambda)x$ as two perfectly correlated lotteries.

$$\rho^*(\{\lambda x + (1-\lambda)y, \lambda y + (1-\lambda)x\}) = (\lambda^2 + (1-\lambda)^2)\frac{u(x)x + u(y)y}{u(x)+u(y)} + (2\lambda(1-\lambda))\frac{u(y)x + u(x)y}{u(x)+u(y)}$$

Calibration



Strictly affine Luce

Theorem

An average choice is strictly affine Luce rationalizable iff it satisfies calibration, independence, continuity and path independence.

On continuity and Debreu's example

Debreu's example:

$$\frac{\rho(t, \{t, b\})}{\rho(b, \{t, b\})} \neq \frac{\rho(t, \{t, b, b'\})}{\rho(b, \{t, b, b'\})}$$

On continuity and Debreu's example

Let ρ^* be continuous Luce rationalizable.
$z_n \to x$ then:

$$\rho^*(\{x,y\}) \neq \frac{2u(x)x + u(y)y}{2u(x)+u(y)} = \lim_{n\to\infty} \rho^*(\{x,y,z_n\}).$$

Thus ρ^* must be discontinuous.

Finite sample test

- Theory: ρ is observed.
- Reality: ρ is estimated.

Finite sample test

Fix A. Estimate $\rho(x, A)$ by sampling from ρ.
Luce (1959):
"It is clear that rather large sample sizes are required from each subset to obtain sensitive direct tests of axiom 1."

Finite sample test

Fix A.
Population choices from A are given by a Luce rule $p(A)$.
Observe iid sample $X_1, \ldots, X_k$ of choices: $X_i \in A$ for $i = 1, \ldots, k$.

$$p_x^k = \frac{|i : X_i = x|}{k}.$$

Finite sample test

Two possibilities to test the Luce model:

1. Use Luce's IIA. Requires:

$$\frac{p_x^k}{p_y^k},$$

   for $x, y \in A$.

2. Use average choice:

$$\mu^k = \sum_{x \in A} x p_x^k$$

$$\sqrt{k}\left(\frac{p_x^k}{p_y^k} - \frac{p_x}{p_y}\right) \xrightarrow{d} N\left(0, 2\frac{p_x^2}{p_y^2}\right).$$

Recall that

$$\frac{p_x}{p_y} = \frac{u(x)}{u(y)}$$

On the other hand,

$$\sqrt{k}(\mu^k - \mu) \xrightarrow{d} N(0, \Sigma),$$

where

$$\Sigma = (\sigma_{l,h}) \text{ and } |\sigma_{l,h}| \leq \max\{x_l x_k : x \in A\}.$$

Proposition

For any M, $\exists$ a Luce model s.t. asymptotic variance of p_a^k / p_b^k relative to $\max\{\sigma_{l,h}\}$*, the largest element of Σ, is greater than M.*

The inefficiency in using ratios relative to means can be arbitrarily large.

Proof sketch.

Recall:

Theorem

An average choice is continuous Luce rationalizable iff it satisfies continuity and path independence.

Proof sketch

Necessity:

$$\rho^*(A) = \sum_{x \in A} \left(\frac{u(x)}{\sum_{y \in A} u(y)} \right) x$$

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Proof sketch

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Proof sketch

Sufficiency:

First determine ρ on A with cardinality 2 and 3.

- $A = \{x, y\}$
- $A = \{x, y, z\}$ with x, y, z affinely indep.

Proof sketch

Sufficiency:

Determine $\rho(x,\{x,y\})$ and $\rho(y,\{x,y\})$ from

$$\rho^*(\{x,y\}) = x\rho(x,\{x,y\}) + y\rho(y,\{x,y\}).$$

Proof sketch

Sufficiency:

Determine $\rho(x, \{x, y\})$ and $\rho(y, \{x, y\})$ from

$$\rho^*(\{x, y\}) = x\rho(x, \{x, y\}) + y\rho(y, \{x, y\}).$$

For *affinely indep.* x, y, z, $\rho(x, \{x, y, z\})$, $\rho(y, \{x, y, z\})$ and $\rho(z, \{x, y, z\})$ are also determined from $\rho^*(\{x, y, z\})$.

Proof sketch

By path independence, $\exists\ \theta$ s.t.

$$\begin{aligned}\rho^*(A) &= \theta z + (1-\theta)\rho^*(A \setminus \{z\}) \\ &= \theta z + (1-\theta)[x\rho(x,\{x,y\}) + y\rho(y,\{x,y\})].\end{aligned}$$

x, y and z are affinely indep., $\Rightarrow \rho(x,A)$, $\rho(y,A)$ and $\rho(z,A)$ are unique; thus
$\rho(x,A) = (1-\theta)\rho(x,\{x,y\})$ and $\rho(y,A) = (1-\theta)\rho(y,\{x,y\})$.

Proof sketch

$\rho(x, A) = (1-\theta)\rho(x, \{x, y\})$ and $\rho(y, A) = (1-\theta)\rho(y, \{x, y\})$.
Hence

$$\frac{\rho(x, A)}{\rho(y, A)} = \frac{\rho(x, \{x, y\})}{\rho(y, \{x, y\})}.$$

Luce's IIA!

Proof sketch

We can define a Luce rule with utility u.

Fix $x^* \in X$. Let $u(x^*) = 1$, and

$$u(x) = \frac{\rho(x, \{x, x^*\})}{\rho(x^*, \{x, x^*\})}.$$

Then,

$$\frac{u(x)}{u(y)} = \frac{\rho(x, \{x, y\})}{\rho(y, \{x, y\})}$$

so u defines a Luce rule.
Let $\bar{\rho}$ be the implied avg. choice.

We need to show $\bar{\rho} = \rho^*$.

Proof sketch

The proof is by induction on $|A|$.

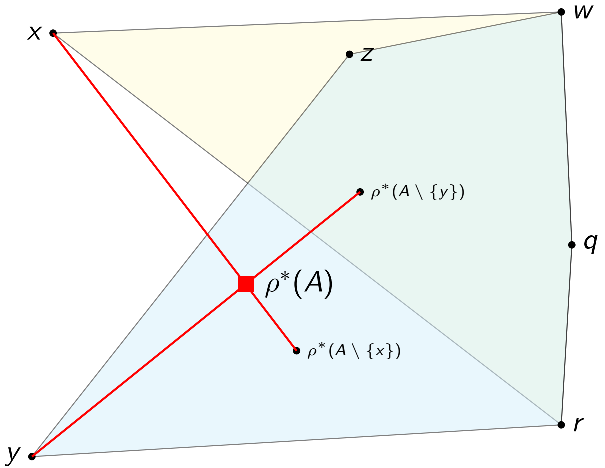
We know that $\bar{\rho}(A) = \rho^*(A)$ when $|A| \leq 3$.

Proof sketch

Crucial lemma (roughly stated):
Let $A \in \mathcal{A}$ be "generic" then there is $x, y \in A$ s.t.

$$\mathrm{conv}^0(\{x, \rho^*(A \setminus \{x\})\}) \cap \mathrm{conv}^0(\{y, \rho^*(A \setminus \{y\})\})$$

is a singleton.


x
w
z
$\rho^*(A \setminus \{y\})$
q
$\rho^*(A)$
$\rho^*(A \setminus \{x\})$
r
y

Proof sketch

$\bar{\rho}(A \setminus \{z\}) = \rho^*(A \setminus \{z\})$ So

$$\text{conv}^0(\{x, \bar{\rho}(A \setminus \{x\})\}) = \text{conv}^0(\{x, \rho^*(A \setminus \{x\})\})$$
$$\text{conv}^0(\{y, \bar{\rho}(A \setminus \{y\})\}) = \text{conv}^0(\{y, \rho^*(A \setminus \{y\})\})$$

$$\text{conv}^0(\{x, \rho^*(A \setminus \{x\})\}) \cap \text{conv}^0(\{y, \rho^*(A \setminus \{y\})\})$$

being a singleton,
and path independence Implies $\bar{\rho}(A) = \rho^*(A)$.

Conclusion

- Plott's path independent choice leads to an impossibility.
- A simple modification of path independence, allowing the path to affect the weights of stochastic choice, avoid the impossibility.
- Our modified path independence and cont. pins down a unique choice: Luce's rule.