# How to control controlled school choice

Federico Echenique — Caltech
M. Bumin Yenmez — Carnegie Mellon

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# School choice

# Example

Two schools/colleges: $c_1$, $c_2$
Two students: $s_1$, $s_2$.

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|---|---|---|---|
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# Main results

Tension between:

- "package" preferences,
- "item" preferences,

Pure item preferences $\rightarrow$ GS.
Pure package preferences $\rightarrow$ complements.

# Main results

- GS + Axioms on how to resolve tension
  $\iff$ specific "utility function" (or procedure) for schools.
- Implications for matching: some procedures are better for students than others (Pareto ranking of school choice procedures).

# Literature

- School choice: Abdulkadiroğlu and Sönmez (2003), Abdulkadiroğlu, Pathak, Sönmez and Roth (2005) (Boston), Abdulkadiroğlu, Pathak and Roth (2005) (NYC)
- Controlled school choice: Abdulkadiroğlu and Sönmez (2003), Abdulkadiroğlu (2005), Kamada and Kojima (2010), Kojima (2010), Hafalir, Yenmez, Yildirim (2011), Ehlers, Hafalir, Yenmez, Yildirim (2011), Budish, Che, Kojima, and Milgrom (2011), Erdil and Kurino (2012), Kominers and Sönmez (2012), Aygün and Bo (2013), Westkamp (2013).

# One School

# The model: primitives

- A finite set $\mathcal{S}$ of *students*.
- A *choice rule* $C$ on $\mathcal{S}$.
- A strict *priority* $\succ$ on $\mathcal{S}$.
- Students partitioned into *types.*

# The model: primitives

1. A finite set $\mathcal{S}$ of *students*.
2. A *choice rule*: $C : 2^{\mathcal{S}} \setminus \{\emptyset\} \to 2^{\mathcal{S}}$

   $$\text{s.t. } C(S) \subseteq S.$$
3. A number $q > 0$ s.t. $|C(S)| \leq q$.

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   s.t. $C(S) \subseteq S$.
3. A number $q > 0$ s.t. $|C(S)| \leq q$.

Note:

1. Choice $C(S)$ is a "package"
2. Allow $C(S) = \emptyset$.
3. $q =$ school capacity.

# The model: primitives

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- A strict *priority* $\succ$ on $\mathcal{S}$.

# Axioms: Gross substitutes

**Axiom (Gross Substitutes (GS))**

$$s \in S \subseteq S' \text{ and } s \in C(S') \Rightarrow s \in C(S).$$

# Axioms: Gross substitutes

Equivalently:

**Axiom (Gross Substitutes (GS))**

$$S \subseteq S' \text{ and } s \in S \setminus C(S) \Rightarrow s \in S' \setminus C(S').$$

Here: substitutes = absence of complements.
When schools satisfy GS, there is a stable matching & the DA algorithm finds one.

# Example

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| $s_1, s_2$ | $s_1$ | $c_1$ | $c_2$ |
| | $s_2$ | $c_2$ | $c_1$ |

$s_1 \notin C_{c_1}(\{s_1\})$ while $s_1 \in C_{c_1}(\{s_1, s_2\})$.

# The model: primitives

1. $\mathcal{S}$ is partitioned into students of different types.
2. Set $T \equiv \{t_1, \ldots, t_d\}$ of *types*,
3. $\tau : \mathcal{S} \rightarrow T$

# The model: primitives

1. $\mathcal{S}$ is partitioned into students of different types.
2. Set $T \equiv \{t_1, \ldots, t_d\}$ of *types*,
3. $\tau : \mathcal{S} \to T$

Define function $\xi : 2^{\mathcal{S}} \to \mathbf{Z}_+^d$.
Let

$$\xi(S) = (|S \cap \tau^{-1}(t)|)_{t \in T};$$

$\xi(S)$ is the *type distribution* of students in $S$.

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over a low priority student?

First resolution of this tension: never when of different types.
Put package (distributional) preferences first.

## Axiom (Monotonicity)

$$\xi(S) \leq \xi(S') \text{ implies that } \xi(C(S)) \leq \xi(C(S')).$$

Axiom (Within-type $\succ$-compatibility)

$$s \in C(S), s' \in S \setminus C(S) \textit{ and } \tau(s) = \tau(s') \Rightarrow s \succ s'.$$

# Ideal point

$(\mathcal{S}, C, \succ)$ is *generated by an ideal point* if:

Given an ideal $z^* \in \mathbf{Z}_+^d$,

1. Chose closest feasible distribution of types to $z^*$.
2. For each type, chose "best" (highest priority) available students.

# Ideal point

$(\mathcal{S}, C, \succ)$ is *generated by an ideal point* if:

$\exists z^* \in \mathbf{Z}_+^d$ such that $\|z^*\| \leq q$ s.t,

1. $\xi(C(S))$ min. Euclidean distance to $z^*$ in $B(\xi(S))$ where

$$B(x) = \{z \in \mathbf{Z}_+^d : z \leq x \text{ and } |z| \leq q\};$$

2. students of type $t$ in $C(S)$ have higher priority than students of type $t$ in $S \setminus C(S)$.

# Ideal point

**Theorem**

$(\mathcal{S}, C, \succ)$ *satisfies*

- *GS*
- *Monotonicity*
- *and within-type* $\succ$*-compatibility*

*iff it is generated by an ideal point.*

Ideal point rule may be wasteful.

**Axiom (Acceptance)**

*A student is rejected only when all seats are filled.*

$$|C(S)| = \min\{|S|, q\}.$$

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When will you admit a high priority student over a low priority student?

Second resolution of tension: some times; depending on the number of students of each type.

$t \in T$ is *saturated* at $S$ if there is $S'$ such that $|S^t| = |S'^t|$ with $S'^t \setminus C(S')^t \neq \emptyset$.

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**Axiom (Saturated $\succ$-compatibility)**

*$s \in C(S)$, $s' \in S \setminus C(S)$ and $\tau(s)$ is saturated at $S$ imply $s \succ s'$.*

# Reserves

$(\mathcal{S}, C, \succ)$ is *generated by reserves* if:

Lower bound on each student type that school tries to fill: "painted seats."

Students compete openly for the unfilled seats.

# Reserves

$(\mathcal{S}, C, \succ)$ is *generated by reserves* if:

$\exists$ vector $(r_t)_{t \in T} \in \mathbf{Z}_+^d$ with $\|r\| \leq q$ such that for any $S \subseteq \mathcal{S}$,

1. $|C(S)^t| \geq r_t \wedge |S^t|$;
2. if $s \in C(S)$, $s' \in S \setminus C(S)$ and $s' \succ s$, then it must be the case that $\tau(s) \neq \tau(s')$ and $|C(S)^{\tau(s)}| \leq r_{\tau(s)}$; and
3. if $\emptyset \neq S \setminus C(S)$, then $|C(S)| = q$.

# Reserves

**Theorem**

$(\mathcal{S}, C, \succ)$ *satisfies*

- *GS,*
- *acceptance,*
- *saturated* $\succ$*-compatibility,*

*iff it is generated by reserves.*


- Pathak - Sönmez
- Kominers - Sönmez


- First assign open seats based on priorities.
- Second, assign reserved seats based on priorities.

The *opposite order* to Reserves.

# Chicago

Ex:

- $\mathcal{S} = \{s_1, s_2, s_3, s_4\}$
- $s_1, s_2$ of type 1
- $s_3, s_4$ of type 2
- one school with three seats: one reserved for each type and one open.
- priorities are

$$s_1 \succ s_3 \succ s_4 \succ s_2.$$

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- $\mathcal{S} = \{s_1, s_2, s_3, s_4\}$
- $s_1, s_2$ of type 1
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- one school with three seats: one reserved for each type and one open.
- priorities are

$$s_1 \succ s_3 \succ s_4 \succ s_2.$$

Reserves assign: $s_1$, $s_3$ and $s_4$
Chicago: $s_1$, $s_2$ and $s_3$
a violation of saturated $\succ$-compatibility

# Quotas

Achieve diversity by upper bound:

$$|C(S)^t| \leq r_t$$

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Achieve diversity by upper bound:

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New axiom:

**Axiom**

*Choice rule $C$ satisfies rejection maximality (RM) if $s \in S \setminus C(S)$ and $|C(S)| < q$ imply for every $S'$ such that $|S'^{\tau(s)}| \leq |S^{\tau(s)}|$ we have $|C(S)^{\tau(s)}| \geq |C(S')^{\tau(s)}|$.*

**Theorem**

$(\mathcal{S}, C, \succ)$ *satisfies*

- *GS,*
- *RM,*
- *demanded* $\succ$*-compatibility,*

*iff it is generated by quotas.*

Proofs:
Idea is to map $C$ into $f : \mathbf{Z}_+^d \to \mathbf{Z}_+^d$.
Translate axioms into properties of $f$.

# Proof sketch:

**Theorem**

*$(\mathcal{S}, C, \succ)$ satisfies*

- *GS*
- *Monotonicity*
- *and within-type $\succ$-compatibility*

*iff it is generated by an ideal point.*

Under Mon, $\{\xi(C(S)) : \xi(S) = x\}$ is a singleton.
So, map $C$ into a function $f : \mathbf{Z}^d_+ \to \mathbf{Z}^d_+$ by

$$f(x) = \{\xi(C(S)) : \xi(S) = x\}.$$

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Then $C$ satisfies GS iff

$$y \le x \Rightarrow f(x) \wedge y \le f(y).$$

(proof: ...)

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(proof: ...)
Then $C$ satisfies GS and Mon iff

$$y \leq x \Rightarrow f(x) \wedge y = f(y).$$

# Proof sketch:

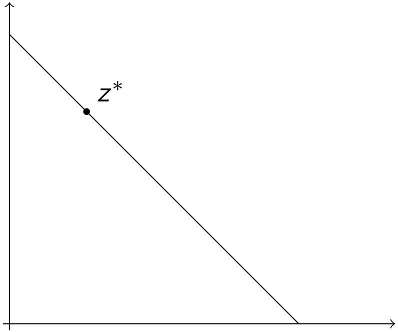
$C$ satisfies GS and Mon iff

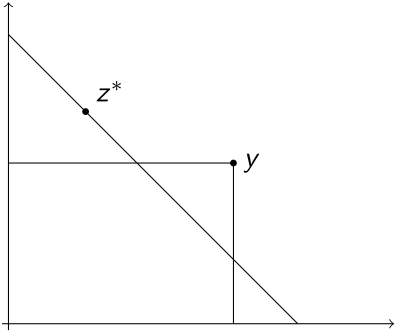
$$y \leq x \Rightarrow f(x) \wedge y = f(y).$$

Let $z^* = \xi(C(\mathcal{S}))$. For any $x$, $x \leq \xi(\mathcal{S})$ implies

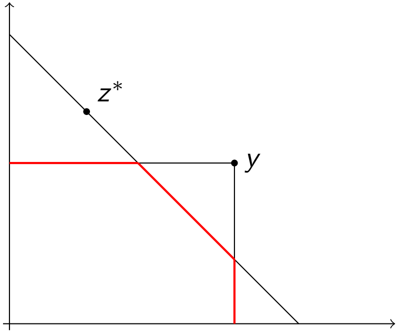
$$f(x) = x \wedge f(\xi(\mathcal{S})) = x \wedge z^*.$$

A "projection," hence min. Euclidean distance.


$z^*$

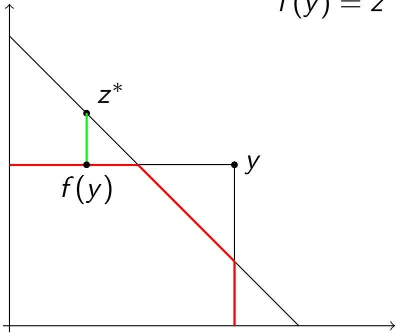


z*
y



$z^*$
$y$

$$f(y) = z^* \wedge y$$



# Proof sketch:

**Theorem**

$(\mathcal{S}, C, \succ)$ *satisfies*

- *GS,*
- *acceptance,*
- *saturated* $\succ$*-compatibility,*

*iff it is generated by reserves.*

Map $C$ into a function $f : \mathbf{Z}^d_+ \to \mathbf{Z}^d_+$ by

$$f(x) = \bigwedge \{\xi(C(S)) : \xi(S) = x\}.$$

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**Lemma**

*Let $C$ satisfy GS. If $y \in \mathbf{Z}^d_+$ is such that $f(y)_t < y_t$ then $f(y + e_{t'})_t < y_t + 1_{t=t'}$*

Lemma $\Rightarrow$ construct the vector $r$ of minimum quotas as follows. Let $\bar{x} = \xi(\mathcal{S})$.
The lemma implies that if $f(y_t, \bar{x}_{-t})_t < y_t$ then $f(y'_t, \bar{x}_{-t})_t < y'_t$ for all $y'_t > y_t$. Then there is $r_t \in \mathbf{N}$ such that $y_t > r_t$ if and only if $f(y_t, \bar{x}_{-t}) < y_t$.

# Overview

Basic tension: when to trade off students of different types. GS disciplines this tradeoff.

| Model | GS | Diversity | | | | Priorities | | |
|---|---|---|---|---|---|---|---|---|
| | | Mon | Dep | Eff | RM | t-WARP | A-SARP | E-SARP |
| Ideal point | ✓ | ✓ | | | | ✓ | | |
| Schur | ✓ | | ✓ | ✓ | | ✓ | | |
| Reserves | ✓ | | | ✓ | | | ✓ | |
| Quotas | ✓ | | | | ✓ | | | ✓ |

Rules in red are rigid. Rules in blue are flexible.

# Conclusion

- Gross substitutes & diversity & rationality axioms pin down precise choice rules:
  1. ideal-point and Schur-generated generated rules,
  2. choice rules generated by quotas and reserves.
- Procedures are Pareto ranked.

# Axioms: rationality

**Axiom**

*C satisfies the type-WARP if,* $\forall s, s'$, *S and S' s.t.* $\tau(s) = \tau(s')$ *and* $s, s' \in S \cap S'$,

$$s \in C(S) \text{ and } s' \in C(S') \setminus C(S) \Rightarrow s \in C(S').$$

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# Axioms: diversity

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*C satisfies distribution-monotonicity (Mon) if*

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- Strong assumption.
- Doesn't restrict the *form* of diversity.

# Law of aggregate demand

**Axiom**

*$C$ satisfies the law of aggregate demand if*

$$S \subseteq S' \Rightarrow |C(S)| \leq |C(S')|.$$

If $C$ satisfies monotonicity, then it also satisfies the law of aggregate demand. Therefore, if $C$ is generated by an ideal point then it satisfies the law of aggregate demand.

# Quotas

Choice rule $\mathcal{C}$ is *generated by quotas* if:

there exists an upper bound on each student type

but otherwise students compete openly for the seats.

# Quotas

Choice rule $\mathcal{C}$ is *generated by quotas* if:

$\exists$ a strict priority $\succeq$ over $\mathcal{S}$ and a vector $(r_t)_{t \in T} \in \mathbf{Z}^d_+$ such that for any $S \subseteq \mathcal{S}$,

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1. $|C(S)^t| \leq r_t$;
2. if $s \in C(S)$, $s' \in S \setminus C(S)$ and $s' \succ s$, then it must be the case that $\tau(s) \neq \tau(s')$ and $|C(S)^{\tau(s')}| = r_{\tau(s')}$; and
3. if $s \in S \setminus C(S)$, then either $|C(S)| = q$ or $|C(S)^{\tau(s)}| = r_{\tau(s)}$.

# Quota-generated choice

**Theorem**

*A choice $C$ satisfies*

- *gross substitutes,*
- *E-SARP,*
- *and rejection maximality*

*if and only if it is quota-generated.*

# Matching Market

# Matching market

A *matching market* is a tuple $\langle \mathcal{C}, \mathcal{S}, (\succ_s)_{s \in \mathcal{S}}, (C_c)_{c \in \mathcal{C}} \rangle$,

- $\mathcal{C}$ is a finite set of *schools*
- $\mathcal{S}$ is a finite set of *students*
- $\succeq_s$ is a strict preference order over $C \cup \{s\}$
- $C_c$ is a choice rule over $\mathcal{S}$.

# Matching market

A *matching* in a market $\langle \mathcal{C}, \mathcal{S}, (\succ_s)_{s \in \mathcal{S}}, (C_c)_{c \in \mathcal{C}} \rangle$ is a function $\mu$ defined on $\mathcal{C} \cup \mathcal{S}$ s.t.

- $\mu(c) \subseteq \mathcal{S}$
- $\mu(s) \in \mathcal{C} \cup \{s\}$
- $s \in \mu(c)$ iff $c = \mu(s)$.

# Matching market

A matching $\mu$ is *stable* if

- (*individual rationality*) $C_c(\mu(c)) = \mu(c)$ and $\mu(s) \succeq_s \{s\}$;
- (*no blocking*) there's no $(c, S')$ s.t
  - $S' \not\subseteq \mu(c)$
  - $S' \subseteq C_c(\mu(c) \cup S')$
  - $c \succeq_s \mu(s)$ for all $s \in S'$.

# Gale-Shapley deferred acceptance algorithm (DA)

**Deferred Acceptance Algorithm (DA)**

Step 1 Each student applies to her most preferred school. Suppose that $S_c^1$ is the set of students who applied to school $c$. School $c$ tentatively admits students in $C_c(S_c^1)$ and permanently rejects the rest. If there are no rejections, stop.

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Step $k$ Each student who was rejected at Step $k-1$ applies to their next preferred school. Suppose that $S_c^k$ is the set of new applicants and students tentatively admitted at the end of Step $k-1$ for school $c$. School $c$ tentatively admits students in $C_c(S_c^k)$ and permanently rejects the rest. If there are no rejections, stop.

# Standard results

**Theorem**

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**Theorem**

- *Suppose that choice rules satisfy gross substitutes, then DA produces the stable matching that is simultaneously the best stable matching for all students.*
- *Suppose, furthermore, that choice rules satisfy the law of aggregate demand then DA is group incentive compatible for students and each school is matched with the same number of students in any stable matching.*

The student-proposing deferred-acceptance algorithm = SOSM

# Pareto comparisons-1

**Theorem**

*Consider profiles $(C)_{c\in\mathcal{C}}$ and $(C')_{c\in\mathcal{C}}$ that satisfy GS.Suppose that $C_c(S) \subseteq C'_c(S)$ for every $S \subseteq \mathcal{S}$ and $c \in C$. Let $\mu$ and $\mu'$ be the SOSM's with $(C)_{c\in\mathcal{C}}$ and $(C')_{c\in\mathcal{C}}$, respectively. Then $\mu'(s) \succeq_s \mu(s)$ for all $s$.*

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Under some assumptions, then:
Reserves are better than quotas for all students.
Schur concave is better than ideal point for all students.

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  2. choice rules generated by quotas and reserves.
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# Proofs

# Proof of Ideal Point-1

Let $f : \mathbf{Z}_+^d \to \mathbf{Z}_+^d$.

- $f$ is *monotone increasing* if $y \leq x$ implies that $f(y) \leq f(x)$;

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- $f$ is *within budget* if

$$f(x) \in B(x) \equiv \{z \in \mathbf{Z}_+^d : z \leq x \text{ and } |z| \leq q\}.$$

**Lemma**

*$f$ is monotone increasing, within budget, and satisfies gross substitutes if and only if there exists $z^* \in \mathbf{Z}_+^d$ s.t. $|z^*| \leq q$, and $f(x) = x \wedge z^*$.*

# Proof of Ideal Point-2

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Let $f : A \subseteq \mathbf{Z}_+^d \to \mathbf{Z}_+^d$ be defined by $f(x) = \xi(C(S))$ for $S$ with $\xi(S) = x$.

**Lemma**

*$f$ is well defined, within budget, and monotone increasing.*

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**Lemma**

*$f$ is well defined, within budget, and monotone increasing.*

**Lemma**

*If $C$ satisfies gross substitutes, then*

$$y \leq x \Rightarrow f(y) \geq y \wedge f(x).$$

# Proof of Ideal Point-2

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Let $f : A \subseteq \mathbf{Z}_+^d \to \mathbf{Z}_+^d$ be defined by $f(x) = \xi(C(S))$ for $S$ with $\xi(S) = x$.

**Lemma**

*$f$ is well defined, within budget, and monotone increasing.*

**Lemma**

*If $C$ satisfies gross substitutes, then*

$$y \leq x \Rightarrow f(y) \geq y \wedge f(x).$$

$\Rightarrow$ There exists $z^*$ s.t. $f(x) = x \wedge z^*$.

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- $s'' \notin C(S' \cup \{s\})$ (GS)

Define $\succ$ to be the linear extension of $\succ$.

Back

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**Lemma**

*$f$ is efficient and satisfies gross substitutes if and only if it is Schur-generated.*

# Schur-generated choice-2

Let $f : A \subseteq \mathbf{Z}_+^d \to \mathbf{Z}_+^d$ be defined by $f(x) = \xi(C(S))$ for $S$ with $\xi(S) = x$.

# Proof: Pareto comparisons

**Theorem**

*There are $z_c^* \in Z$, $c \in \mathcal{C}$, s.t*

- *if $\mu^i$ results from SOSM using the $C_c$ that minimize the Euclidean distance to $z_c^*$*
- *and if $\mu^s$ is the matching resulting from SOSM using Schur-generated choices from $z_c^*$,*

*then $\forall s \in \mathcal{S}$,*

$$\mu^s(s) \succeq_s \mu(s) \succeq_s \mu^i(s).$$

### Proposition

*Suppose that $C_c$ and $C'_c$ satisfy gross substitutes and that $C_c(S) \subseteq C'_c(S)$. Then the student-optimal stable matching in $\langle \mathcal{C}, \mathcal{S}, (\succ_s)_{s \in \mathcal{S}}, (C'_c)_{c \in \mathcal{C}} \rangle$ Pareto dominates the student-optimal stable matching in $\langle \mathcal{C}, \mathcal{S}, (\succ_s)_{s \in \mathcal{S}}, (C_c)_{c \in \mathcal{C}} \rangle$ .*