Constrained Pseudo-market Equilibrium

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Allocation problems

- Jobs to workers
- Courses to students
- Chores to family members.
- Organs to patients
- Schools to children
- Offices to professors.

Normative desiderata

- Efficiency: Pareto
- Fairness (no envy): randomization
- Property rights
- First part of the talk: Pareto and fairness.

Pseudomarkets

- Provide agents with a fixed budget in "Monopoly money."
- Allow purchase of (fractions of) objects at given prices.

Hylland-Zeckhauser (1979)

Assign workers to jobs.

An *economy* is a tuple $\Gamma = (I, L, (u_i)_{i \in I})$, where

- I is a finite set of *agents*;
- L is the number of *objects*.
- Suppose $L = |I|$.
- $u_i : \Delta_- = \{x \in \mathbf{R}^L_+ : \sum_l x_l \leq 1\} \to \mathbf{R}$ is i's *utility function*.

Hylland-Zeckhauser (1979)

An *assignment* in Γ is $x = (x_i)_{i \in I}$ with $x_i \in \Delta_-$ and $\sum_i x_i \leq \mathbf{1} = (1, \dots, 1)$.

Hylland and Zeckhauser (1979)

An *HZ-equilibrium* is a pair (x, p), with $x \in \Delta_-^N$ and $p = (p_l)_{l \in [L]} \geq 0$ s.t.

1. $\sum_{i=1}^N x_i = (1, \ldots, 1) = \mathbf{1}$
2. x_i solves

$$\text{Max } \{u_i(z_i) : z_i \in \Delta_- \text{ and } p \cdot z_i \leq 1\}$$

Condition (1): supply $=$ demand.
Condition (2): x_i is i's demand at prices p and income $= 1$.

Fairness and efficiency

Suppose that each u_i is linear (expected utility).

Theorem (Hylland and Zeckhauser (1979))

There is an efficient HZ equilibrium. All HZ equilibrium assignments are fair.

Hylland-Zeckhauser (1979)

- The textbook model has endowments ω_i
- Income at prices p is $p \cdot \omega_i$
- w/endowments, eqm. may not exist.

This paper:

- Study efficient and fair allocations via pseudomarkets.
- With *general constraints*.
- With and without *endowments*.

Key idea

Price the constraints

For example: in HZ the price of good l is the price of the supply constraint.

More generally, constraints $\rightarrow$ pecuniary externalities. Can be internalized via prices.

Example: Rural hospitals

- Agents: doctors
- Objects: positions in hospitals
- Constraints: each doctor gets at most one position.
- Constraints: UB on available positions.
- Constraints: LB on number of doctors/region.

Problem: Some hospitals are undesirable.

Challenge is to meet the LB on certain regions.

Solution: "price" UB so that most desirable hospitals are too expensive. Demand "overflows" to meet the LB on undesirable hospitals.

Example: Course bidding in B-schools

- Agents: MBA students.
- Objects: Courses.
- Constraints: UB on course enrollment.
- Constraints: LB on mandatory courses.

Problem: Want efficiency; reflect student pref Solution: “price”

UB so that most desirable courses are expensive. Demand “overflows” to meet the LB on less desirable.
vspace.5cm
Properties: efficiency and fairness.

Example: Roomates in college

- Agents: students
- Objects: students
- Constraints: At most one roommate (= "unit demand").
- Constraints: symmetry (i's purchase of j = j's purchase of i).

Problem: Non-existence of stable matchings.

Equilibrium (a form of stability) + efficiency.

Example: Endowments

- Agents: faculty.
- Objects: office.
- Constraints: Exactly one office for each faculty.
- Status quo: offices are currently assigned.

New challenge: existing tenants must buy into the re-assignment
$\Longrightarrow$ individual rationality constraints.

Example: School choice

- Agents: children.
- Objects: slots in schools.
- Constraints: unit demand and school capacities.
- Endowment: neighborhood school (or sibling priority; etc.)

New challenge:
Respect option to attend neighborhood school $\Longrightarrow$ individual rationality constraints.

What we *don't* do:

- Max SWF (e.g utilitarian) subject to constraints.
- Outcome can be decentralized (think 2nd Welfare Thm - Miralles and Pycia, 2017).
- Dual variables $\rightarrow$ prices.

Related Literature

- Mkts. & fairness: Varian (1974), Hylland-Zeckhauser (1979), Budish (2011).
- Allocations with constraints: Ehlers, Hafalir, Yenmez and Yildrim (2014), Kamada and Kojima (2015, 2017).
- Markets and constraints: Kojima, Sun and Yu (2019), Gul, Pesendorfer and Zhang (2019).
- Endowments: Mas-Colell (1992), He (2017) , and McLennan (2018).

(Many) more references in the paper...

Definitions

- A pair $(a, b) \in \mathbf{R}^n \times \mathbf{R}$ defines a *linear inequality* $a \cdot x \leq b$.
- A linear inequality (a, b) has *non-negative coefficients* if $a \geq 0$.
- A linear inequality (a, b) defines a (closed) *half-space*:

$$\{x \in \mathbf{R}^n : a \cdot x \leq b\}.$$

Definitions

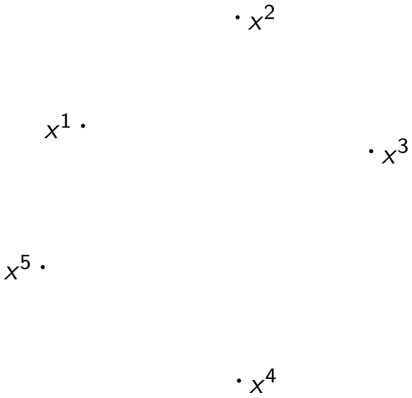
- A *polyhedron* in $\mathbf{R}^n$ is a set that is the intersection of a finite number of closed half-spaces.
- A *polytope* in $\mathbf{R}^n$ is a bounded polyhedron.
- Two special polytopes are the *simplex* in $\mathbf{R}^n$:

$$\Delta^n = \{x \in \mathbf{R}^n_+ : \sum_{l=1}^{n} x_l = 1\},$$

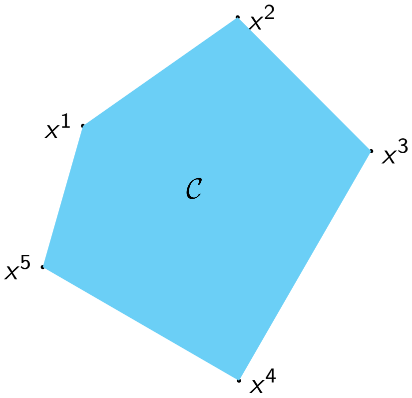
  and the *subsimplex*

$$\Delta^n_- = \{x \in \mathbf{R}^n_+ : \sum_{l=1}^{n} x_l \leq 1\}.$$

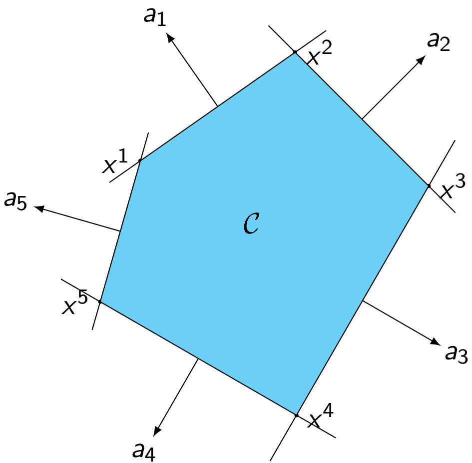
- When n is understood, we use the notation Δ and Δ_-.



x^2
x^1
x^3
x^5
x^4



x^2
x^1
x^3
$\mathcal{C}$
x^5
x^4



a_1
a_2
a_3
a_4
a_5
x^1
x^2
x^3
x^4
x^5
$\mathcal{C}$

Preliminary defns

A function $u : \Delta_- \to \mathbf{R}$ is

- *concave* if $\forall x, z \in \Delta_-$, and $\forall \lambda \in (0,1)$, $\lambda u(z) + (1-\lambda)u(x) \le u(\lambda z + (1-\lambda)x)$;
- *quasi-concave* if, $\forall x \in \Delta_-$,

$$\{z \in \Delta_- : u(z) \ge u(x)\}$$

  is a convex set.
- *semi-strictly quasi-concave* if $\forall x, z \in \Delta_-$,

$$u(z) < u(x) \text{ and } \lambda \in (0,1) \Longrightarrow u(z) < u(\lambda z + (1-\lambda)x)$$

- *expected utility* if it is linear.

The economy

An *economy* is a tuple $\Gamma = (I, O, (Z_i, u_i)_{i \in I}, (q_l)_{l \in O})$, where

- I is a finite set of *agents*;
- O is a finite set of *objects*, with $L = |O|$;
- $Z_i \subseteq \mathbf{R}^L_+$ is i's *consumption space*;
- $u_i : Z_i \to \mathbf{R}$ is i's *utility function*;
- $q_l \in \mathbf{R}_{++}$ is the amount of $l \in O$.

Assignments

An *assignment* in Γ is a vector

$$x = (x_{i,l})_{i \in I, l \in O} \text{ with } x_i \in Z_i.$$

$\mathcal{A}$ denotes the set of all assignments in Γ.

$x \in \mathcal{A}$ is *deterministic* if $(\forall i, j)(x_{i,l} \in \mathbf{Z}_+)$.

Constraints in the literature

Constraints are often imposed on deterministic assignments.

For example:

- *unit-demand constraints* require $\sum_{l \in O} x_{i,l} \leq 1 \ \forall i \in I$
- *supply constraints* require $\sum_{i \in I} x_{i,l} \leq q_l \ \forall l \in O$.

Constraints in the literature

Floor constraints may be used to capture distributional objectives. For example:

- A minimum number of doctors to be assigned to hospitals in rural areas,
- Lower bound on the number minority students that are assigned to a particular school.
- All students take at least two math courses.

Constraints in the literature

A deterministic assignment is *feasible* if it satisfies all exogenous constraints.

An (random) assignment is *feasible* if it belongs to the convex hull of feasible deterministic assignments.

The convex hull is a polytope since the number of feasible deterministic assignments is usually bounded, and therefore finite.

Constraints in our paper

We don't start from an explicit model of constraints.

We introduce constraints *implicitly* through a *primitive* nonempty set $\mathcal{C} \subseteq \mathcal{A}$.

The elements of $\mathcal{C}$ are the *feasible assignments*.

Constrained allocation problems

A *constrained allocation problem* is a pair $(\Gamma, \mathcal{C})$ in which

- Γ is an economy and
- $\mathcal{C} \subseteq \mathcal{A}$, a polytope, is the set of *feasible assignments* in Γ.

Normative properties

- $x \in \mathcal{C}$ is *weakly $\mathcal{C}$-constrained Pareto efficient* if there is no $y \in \mathcal{C}$ s.t. $u_i(y_i) > u_i(x_i)$ for all i.
- $x \in \mathcal{C}$ is *$\mathcal{C}$-constrained Pareto efficient* if there is no $y \in \mathcal{C}$ s.t. $u_i(y_i) \geq u_i(x_i)$ for all i with at least one strict inequality for one agent.

Fairness

- No envy among "equals" (agents that the constraints treat the same).
- Fairness rules out envy among agents who are treated symmetrically by the primitive constraints.

Formal defn. soon. . .

Pre-processing of constraints

- Recall that a pair
$$(a, b) \in \mathbf{R}^{NL} \times \mathbf{R}$$
defines a linear constraint $a \cdot x \leq b$.
- It has non-negative coefficients when $a \geq 0$.

Pre-processing of constraints

The *lower contour set* of $\mathcal{C}$ is

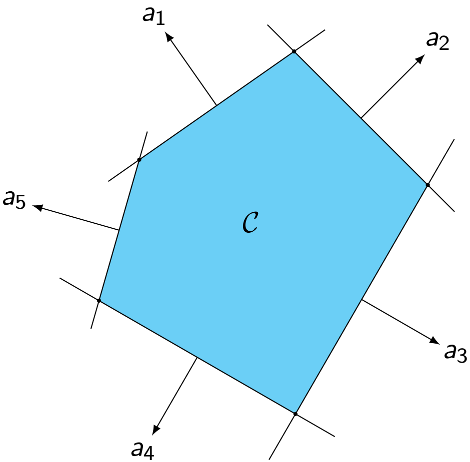
$$\text{lcs}(\mathcal{C}) = \{x \in \mathbf{R}_+^{NL} : \exists x' \in \mathcal{C} \text{ such that } x \leq x'\}.$$

Lemma

There exists a finite set Ω of linear inequalities with non-negative coefficients such that

$$lcs(\mathcal{C}) = \bigcap_{(a,b)\in\Omega} \{x \in \mathbf{R}_+^{LN} : a \cdot x \leq b\}.$$

Used by Ivan Balbuzanov (2019)


a_1
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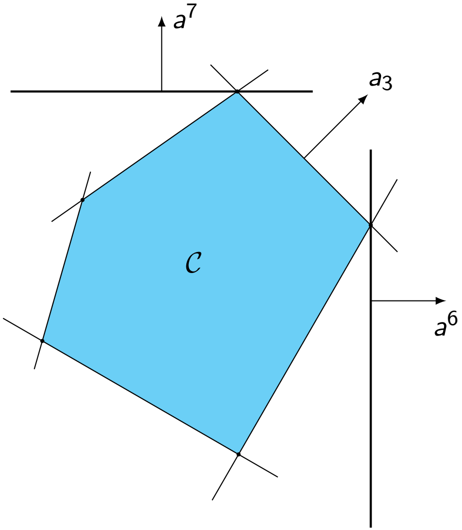
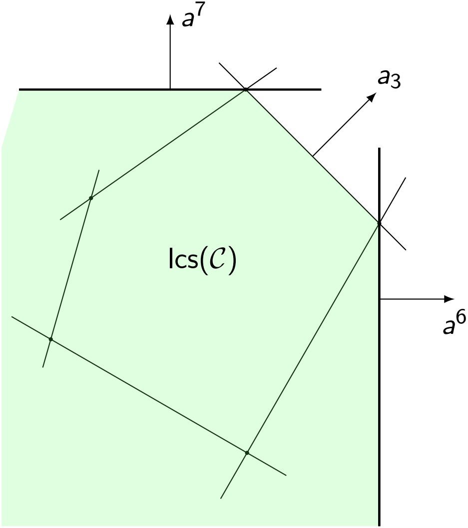
$\mathcal{C}$



a^7
a_3
$\mathcal{C}$
a^6


a^7
a_3
$\mathsf{lcs}(\mathcal{C})$
a^6

Pre-processing of constraints

For any $c = (a, b) \in \Omega$, define

$$\text{supp}(c) = \{(i, l) \in I \times O : a_{i,l} > 0\}.$$

Two types of inequalities $(a, b) \in \Omega$:

- those with $b = 0$ and
- those with $b > 0$.

If $b = 0$, then for any $x \in \mathcal{C}$ we must have $x_{i,l} = 0$ for all $(i, l) \in \text{supp}(c)$. Wlog assume there's a unique such ineq.

Say that l is a *forbidden object* for agent i when $a^0_{i,l} > 0$.

Pre-processing of constraints

Say that $(a, b) \in \Omega \setminus \{(a^0, 0)\}$ is an *individual constraint* for i if for all $j \neq i$ and $l \in O$, $a_{j,l} = 0$.

In words, (a, b) only restricts i's consumption.

Let Ω^i denote the set of all individual constraints for i.

Pre-processing of constraints

Let $\Omega^* = \Omega \setminus \left(\{(a^0, 0)\} \bigcup \cup_{i \in I} \Omega^i \right)$ collect remaining inequalities.

The elements of Ω^* will be "priced."

Constraints in Ω^* give rise to pecuniary externalities.

Pre-processing of constraints

Individual consumption space:

All x_i that satisfy forbidden object and individual constraints for i.

$$\mathcal{X}_i = \{x_i \in \mathbf{R}^L_+ : a^0_i \cdot x_i \leq 0 \text{ and } a_i \cdot x_i \leq b \text{ for all } (a, b) \in \Omega^i\}.$$

Example

- Unit demand constraints are individual and go into $\mathcal{X}_i$
- Supply constraints go into Ω^*. These will be "priced."'

For each $c = (a, b) \in \Omega^*$, we introduce a price p_c.

Given $p = (p_c)_{c \in \Omega^*} \in \mathbf{R}^{\Omega^*}$, the *personalized price vector* faced by $i \in I$ is

$$p_{i,I} = \sum_{(a,b) \in \Omega^*} a_{i,I} p_{(a,b)}.$$

Note: analogous the shadow prices for constraints.

Fairness

- i and j are of *equal type* if $\mathcal{X}_i = \mathcal{X}_j$ and, for all $(a, b) \in \Omega^*$, $a_i = a_j$.
- x is *envy-free* if $u_i(x_i) \geq u_i(x_j)$.
- x is *equal-type envy-free* $u_i(x_i) \geq u_i(x_j)$ whenever i and j are of equal type.

Equilibrium

A pair (x^*, p^*) is a *pseudo-market equilibrium* for $(\Gamma, \mathcal{C})$ if

1. $x_i^* \in \arg\max_{x_i \in \mathcal{X}_i} \{u_i(x_i) : p_i^* \cdot x_i \leq 1\}$.
2. $x^* \in \mathcal{C}$.
3. For any $c = (a, b) \in \Omega^*$, $\sum_{(i,l)} a_{i,l} x_{i,l}^* < b$ implies that $p_c^* = 0$.

Main result

Suppose each u_i is cont., quasi-concave, and st. increasing.

Theorem

- *$\exists$ a pseudo-market eqm. (x^*, p^*) in which x^* is weakly $\mathcal{C}$-constrained Pareto efficient.*
- *If each u_i is semi-strictly quasi-concave, $\exists$ a pseudo-market eqm. (x^*, p^*) in which x^* is $\mathcal{C}$-constrained Pareto efficient.*
- *Every pseudo-market eqm. assignment is equal-type envy-free.*

Endowments

Endowments

Each agent i is described by

- A utility u_i
- An *endowment vector* $\omega_i \in \mathbf{R}^L_+$

Assume: $\sum_i \omega_{i,l} = q_l$

Walrasian equilibrium

A *Walrasian equilibrium* is a pair (x, p) with $x \in \Delta_-^N$, $p \geq 0$ s.t

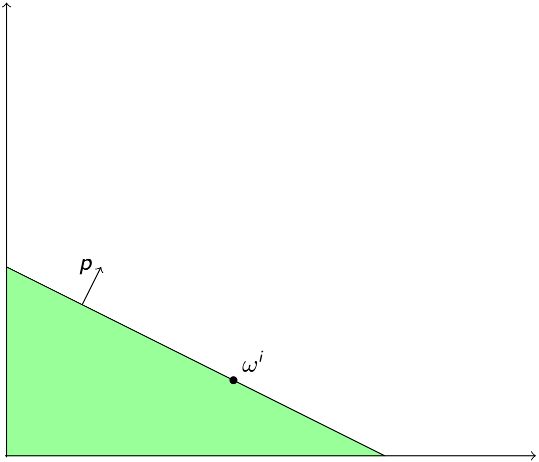
1. $\sum_{i=1}^N x_i = \sum_{i=1}^N \omega_i$; and
2. x_i solves

$$\text{Max } \{u_i(z_i) : z_i \in \Delta_- \text{ and } p \cdot z_i \leq p \cdot \omega_i\}$$

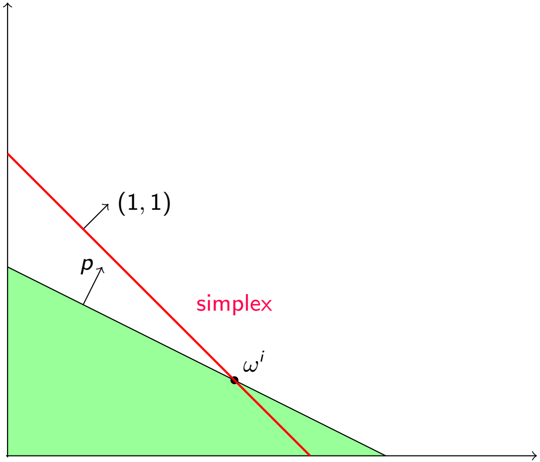
Proposition (Hylland and Zeckhauser (1979))

There are economies in which all agents' utility functions are expected utility, that posses no Walrasian equilibria.

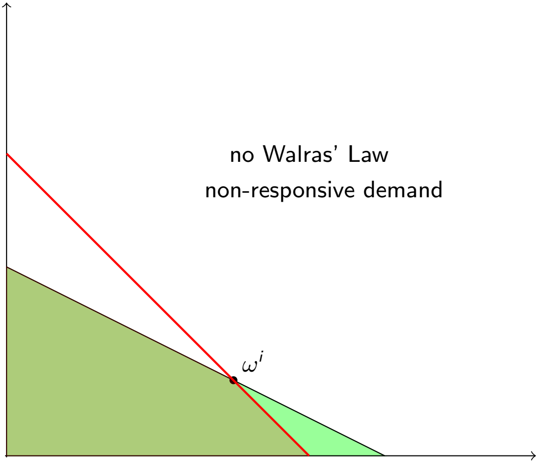
Budget set



Budget set



Budget set



HZ Example

3 agents; exp. utility

	u_1	u_2	u_3
s_A	10	10	1
s_B	1	1	10

Endowments: $\omega_i = (1/3, 2/3)$.

HZ Example

3 agents; exp. utility

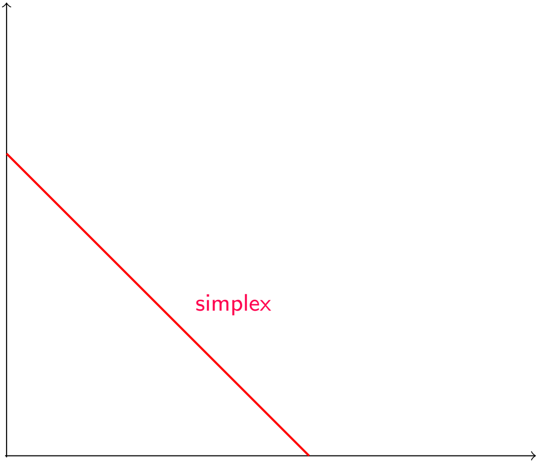
	u_1	u_2	u_3
s_A	10	10	1
s_B	1	1	10

Endowments: $\omega_i = (1/3, 2/3)$.

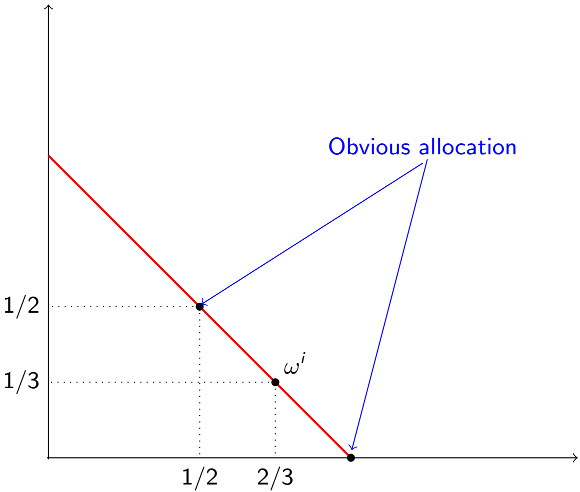
Obvious allocation:

$$x^1 = x^2 = (1/2, 1/2)$$
$$x^3 = (0, 1)$$

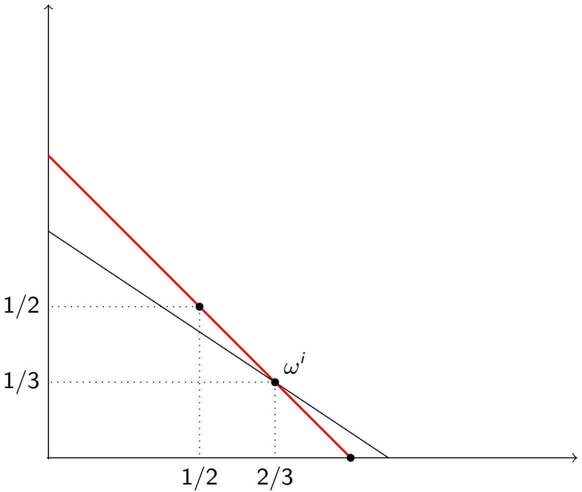
HZ Example



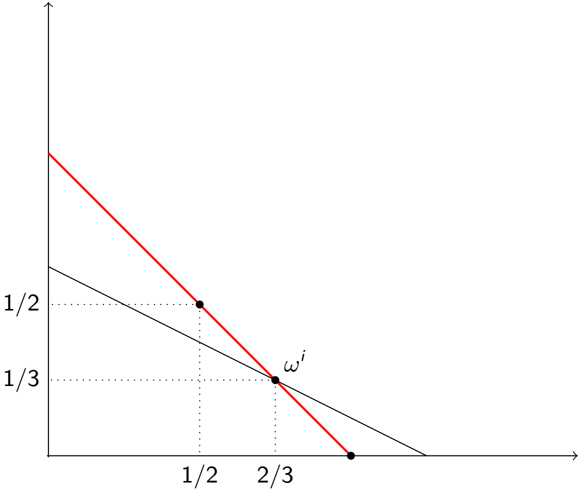
HZ Example



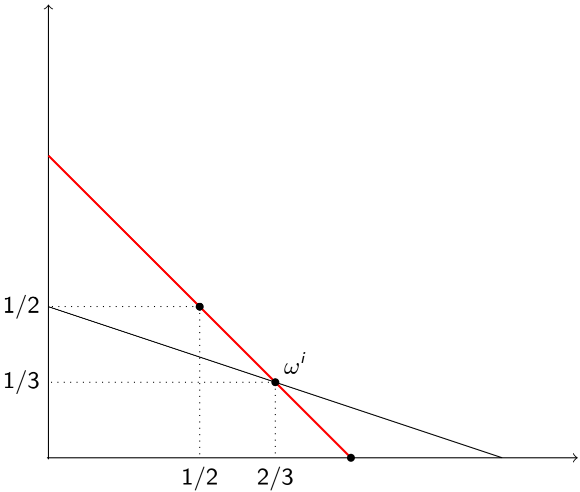
HZ Example



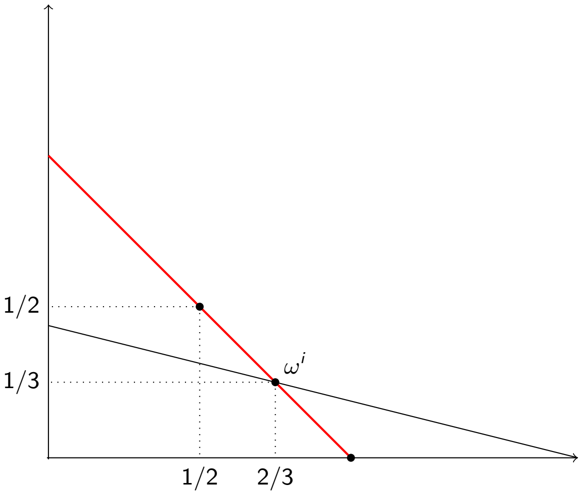
HZ Example



HZ Example



HZ Example



Moreover, . . .

- the first welfare theorem fails.
- There are Pareto ranked Walrasian equilibria.

Economy

An *economy* is a tuple $\Gamma = (I, (Z_i, u_i, \omega_i)_{i \in I})$, where

- I is a finite set of *agents*;
- $Z_i \subseteq \mathbf{R}^L_+$ is i's *consumption space*;
- $u_i : Z_i \to \mathbf{R}$ is i's *utility function*;
- $\omega_i \in Z_i$ is i's *endowment*.

Economy

The *aggregate endowment* is denoted by $\bar{\omega} = \sum_{i \in I} \omega_i$. For every $l \in O$, $\bar{\omega}_l$ is the amount of l in the economy.

A *constrained allocation problem with endowments* is a pair $(\Gamma, \mathcal{C})$ in which Γ is an economy and $\mathcal{C}$ is a set feasible assignments s.t.

1. $\mathcal{C}$ is a polytope;
2. $\omega = (\omega_i)_{i \in I} \in \mathcal{C}$; that is, ω is feasible.

Individual rationality

- A feasible assignment $x \in \mathcal{C}$ is *acceptable* to agent i if $u_i(x_i) \geq u_i(\omega_i)$;
- x is *individually rational* (IR) if it is acceptable to all agents.
- For $\varepsilon > 0$, x is *ε-individually rational* (ε-IR) if $u_i(x_i) \geq u_i(\omega_i) - \varepsilon$ for all $i \in I$.

Equal type

Let $\mathcal{X}_i$ and Ω^* be defined as before.

Two agents i and j are of *equal type* if $\omega_i = \omega_j$, $\mathcal{X}_i = \mathcal{X}_j$, and for all $(a, b) \in \Omega^*$, $a_i = a_j$.

Equilibrium

For any $\alpha \in [0,1]$, we say (x^*, p^*) is an *α-slack equilibrium* if

1. $x_i^* \in \arg\max_{x_i \in \mathcal{X}_i}\{u_i(x_i) : p_i^* \cdot x_i \leq \alpha + (1-\alpha)p_i^* \cdot \omega_i\}$;
2. $x^* \in \mathcal{C}$;
3. For any $c = (a, b) \in \Omega^*$, $\sum_{(i,l)} a_{i,l} x_{i,l}^* < b$ implies that $p_c^* = 0$.

Main result

Assume that for each $c \in \Omega^*$, $\sum_{(i,l)\in supp(c)} \omega_{i,l} > 0$.

Theorem

Suppose u_i is cont., quasi-concave, and st. inc. For any $\alpha \in (0, 1]$:

- *$\exists$ an α-slack eqm. (x^*, p^*), and x^* is weakly $\mathcal{C}$-constrained Pareto efficient.*
- *If agents' utility functions are semi-strictly quasi-concave, $\exists$ an α-slack eqm. assignment x^* that is $\mathcal{C}$-constrained Pareto efficient.*
- *Every α-slack eqm. assignment is equal-type envy-free.*

Individual rationality

Theorem

Suppose u_i are cont., semi-strictly quasi-concave and st. inc. For any $\varepsilon > 0$, $\exists \alpha \in (0, 1]$ and an α-slack equilibrium (x^*, p^*) such that x^* is $\mathcal{C}$-constrained Pareto efficient and

$$\max\{u_i(y) : y \in \mathcal{X}_i \text{ and } p_i^* \cdot y \leq p_i^* \cdot \omega_i\} - u_i(x_i^*) < \varepsilon.$$

In particular, x^* is ε-IR.

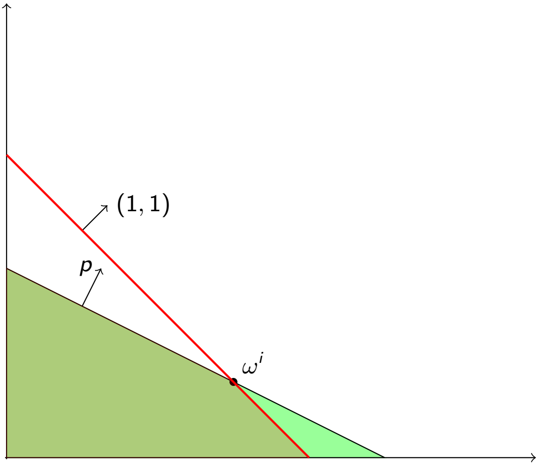
Related Literature

- Mkts. & fairness: Varian (1974), Hylland-Zeckhauser (1979), Budish (2011).
- Allocations with constraints: Ehlers, Hafalir, Yenmez and Yildrim (2014), Kamada and Kojima (2015, 2017).
- Endowments: Mas-Colell (1992), He (2017) , and McLennan (2018).
- Markets and constraints: Kojima, Sun and Yu (2019), Gul, Pesendorfer and Zhang (2019).

More references in the paper. . .

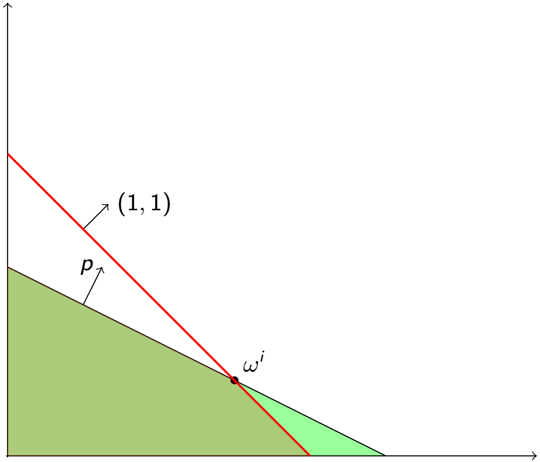
Idea

Classical result relies on Walras Law: $p \cdot z(p) = 0$ for all p. Walras Law does not hold in our model because. . .



Idea

Demand is not responsive to price once boundary is reached.



Idea

Budget constraint:

$$p \cdot x^i \leq \alpha + (1-\alpha)p \cdot \omega^i$$

Idea

Budget constraint:

$$p \cdot (x^i - \omega^i) \leq \alpha(1 - p \cdot \omega^i).$$

This allows prices to matter: large prices imply that the value of excess demand is < 0.

Idea

Consider $\varphi : [0, \bar{p}]^L \to [0, \bar{p}]^L$ defined by

$$\varphi_l(p) = \{\min\{\max\{0, \zeta_l + p_l\}, \bar{p}\} : \zeta \in z(p)\}.$$

where $\bar{p}$ is a large price.

Lemma

φ is upper hemi-continuous, convex- and compact- valued.

(In paper deal with a different φ, which ensures PO.)

Idea

By Kakutani, $\exists$ p^* and $\zeta \in z(p^*)$ s.t

$$p_l^* = \min\{\max\{0, \zeta_l + p_l^*\}, \bar{p}\}.$$

Lemma

$p^* \cdot \zeta \geq 0$.

This is sort of a "weak Walras law."

Pf: $\zeta_l < 0 \Longrightarrow p_l^* = 0$

Idea

Lemma

$p_l^* < \bar{p}$ for all $l \in [L]$

Pf: Suppose $p_l^* = \bar{p}$. $\bar{p}$ is large $\implies 1 - p \cdot \omega^i < 0$; so $p \cdot (x^i - \omega^i) < 0$.
By adding up we get that

$$p \cdot \zeta \leq \alpha(N - p \cdot \bar{\omega}) < 0,$$

in contradiction to prev. lemma.

Idea

Now think about:

$$p_l^* = \min\{\max\{0, \zeta_l + p_l^*\}, \bar{p}\}.$$

when $p_l^* < \bar{p}$.

we have

$$p_l^* = \max\{0, \zeta_l + p_l^*\}.$$

Idea

$$p_l^* = \max\{0, \zeta_l + p_l^*\}.$$

For all l, $\zeta_l = 0$, or $\zeta_l < 0$ and $p_l^* = 0$.

Latter case is not possible.