Complexity: Revealed Preference and Equilibrium

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Three papers:

- *A Revealed Preference Approach to Computational Complexity in Economics,* by Echenique, Golovin & Wierman.
- *Finding a Walrasian equilibrium is easy for a fixed number of agents,* by Echenique & Wierman
- *The Empirical Implications of Rank in Bimatrix Games,* by Barman, Bhaskar, Echenique, & Wierman.

CS and Economics

Recent interest from the theoretical CS literature in economic models. Important new results on our basic models of agents, markets and strategic interactions.

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Many basic results are negative:

- Utility functions are hard to maximize;
- Nash equilibrium is hard to find;
- Walrasian equilibrium is hard to find.

CS critique of positive economics:

Economics is flawed because it assumes agents/society solve hard problems.

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> *"As rational as consumers can possibly be, it is unlikely that they can solve in their minds problems that prove intractable for computer scientists equipped with the latest technology."*
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> – *Gilboa, Schmeidler & Postlewaite*

> *"If an equilibrium is not efficiently computable, much of its credibility as a prediction of the behavior of rational agents is lost"*
>
> – *Christos Papadimitriou*

> *"If your laptop cannot find it, neither can the market"*
>
> – *Kamal Jain*

Theory of the consumer.

"As rational as consumers can possibly be, it is unlikely that they can solve in their minds problems that prove intractable for computer scientists equipped with the latest technology."

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Methodological positivism.

CS (Bded. rationality) critique misunderstands the role of models in positive economics.

Model is a way of thinking about reality, i.e. about *data*.

Economic theory only states that reality behaves as if the theory is true.

Question: What is the empirical content of the hypothesis that consumers are boundedly rational (i.e. that they can't solve hard problems).

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Answer: None.

Our Theorem

Given a consumption data set, the data is either not rationalizable at all, or it is rationalizable by a utility function that is easy to maximize.

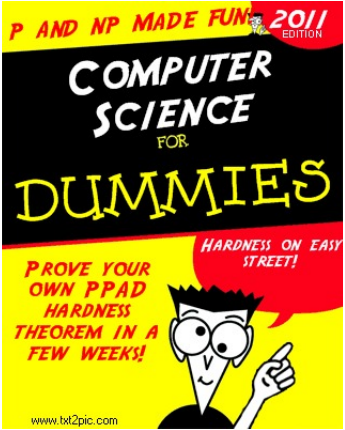
The result is true even if there are indivisible goods.

Digression: complexity for economists.

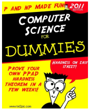
Complexity for dummies.

Economists' reaction to complexity:

- May make sense for computers, not for people/economies.
- Worst case analysis.



Complexity for dumm. . . economists!



A *decision problem* is a problem with a yes/no answer.
Let A be a class of dec. problems.

A dec. problem α is *A-hard* if there is an algorithm that easily transforms any instance of a problem in A into an instance of α, and preserves the answer.

So if you have an algorithm to solve α, you have an algorithm to solve any problem in A. Or, α *is as hard as anything in A.*

Ex: NP-hard problems.

Primitives

$n =$ number of goods
$X \subseteq \mathbf{R}^n_+$ is consumption space
we assume $X \subseteq \mathbf{Z}^n_+$

Data sets

A *consumption data set* D is a collection (x^k, p^k), $k = 1, \dots K$, with $x^k \in X$ and $p^k \in \mathbf{R}^n_{++}$.

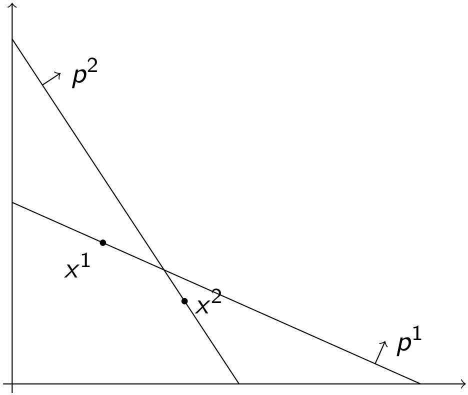
- x^k is the consumption bundle
- purchased at prices p^k.

Rationalization

A utility $u : X \to \mathbf{R}$ *rationalizes* the data if,
for all k and $y \in X$,

$$(p^k \cdot y \leq p^k \cdot x^k \text{ and } y \neq x^k) \Rightarrow u(x^k) > u(y).$$

Are all data sets rationalizable?



p^2
x^1
x^2
p^1

Main result

$u : X \to \mathbf{R}$ is *tractable* if

$$\max \{u(x) : x \in B(p, I)\} .$$

can be solved in polynomial time.

Theorem

In the consumer choice problem with indivisible goods, a dataset is rationalizable iff it is rationalizable via a tractable monotone utility function.

Two approaches in revealed pref. theory

- Construct a utility
- Extend demand.

Constructing a utility does not work.

Theorem (Chambers & Echenique)

In the consumer choice problem with indivisible goods, the following statements are equivalent:

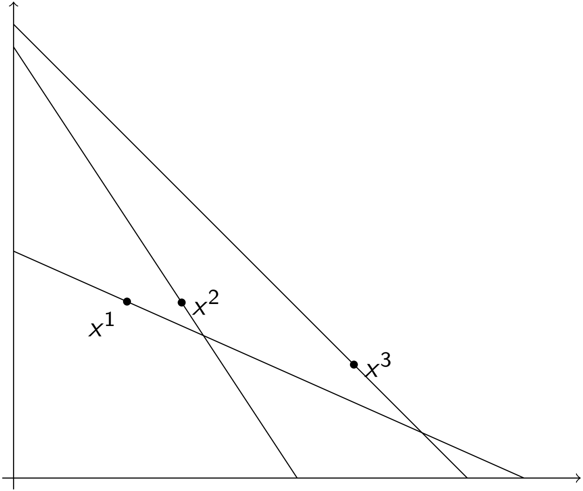
- *The dataset is rationalizable.*
- *The dataset is rationalizable by a supermodular utility function.*
- *The dataset is rationalizable by a submodular utility function.*

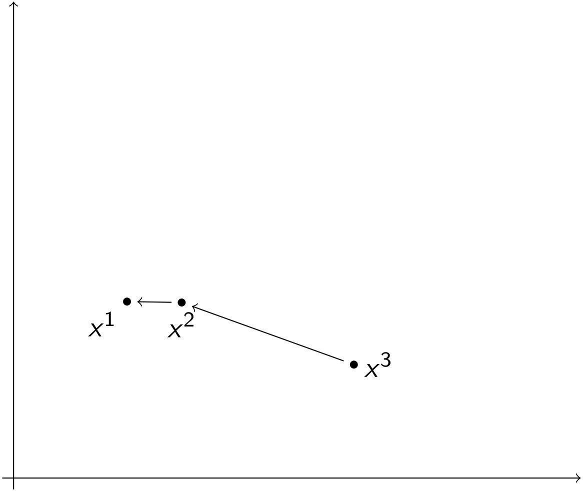
Max. of a super/sub-modular utility subject to a budget constraint is hard.

Revealed preference

x is *revealed preferred* to y if
there is k s.t. $x = x^k$ and $p^k y \leq p^k x^k$

Indicate revealed preference with $\rightarrow$.


x^1
x^2
x^3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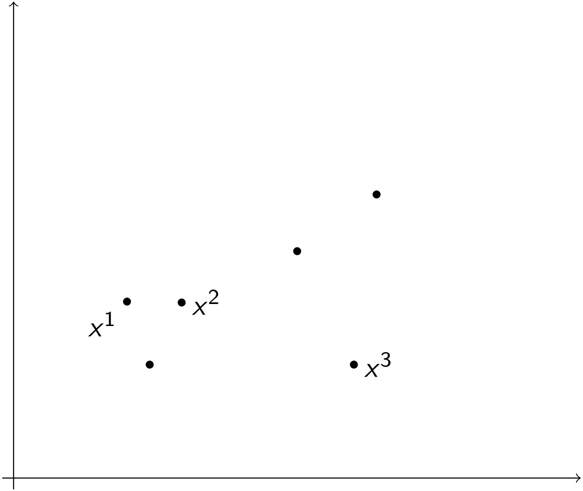
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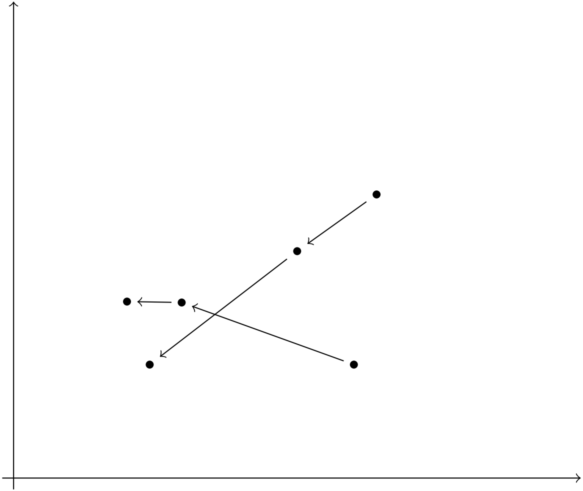
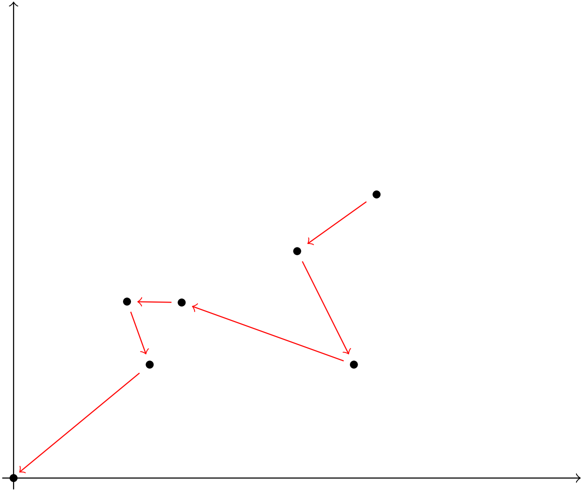
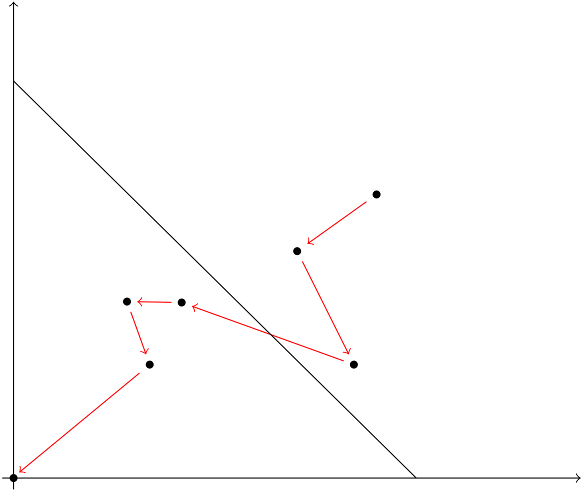
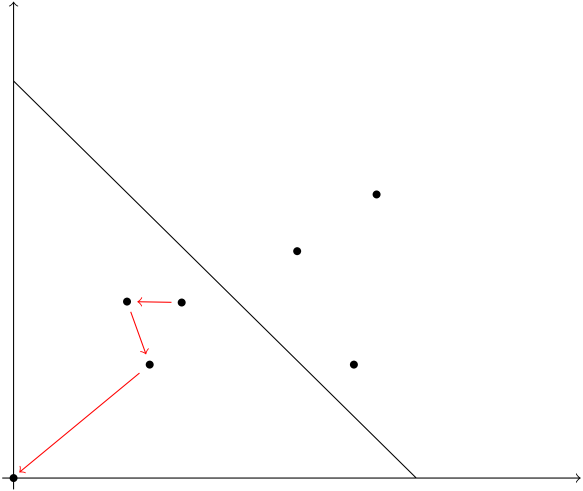
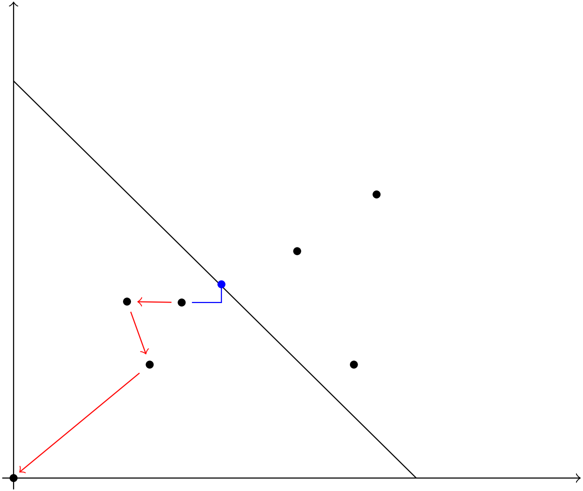








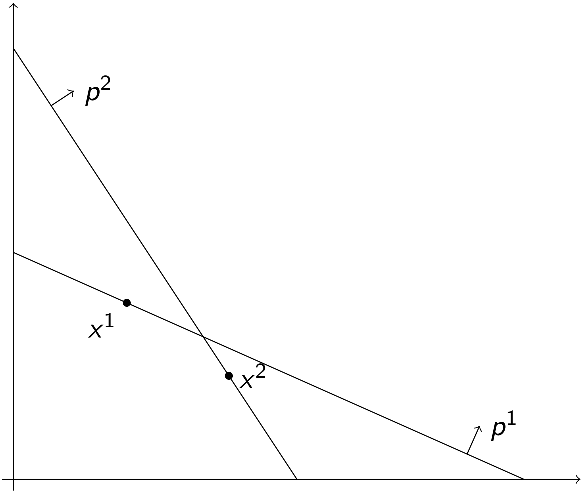


Algorithm:

- Construct a (strict) preference $\succeq$ on data points s.t. $\succeq$ extends the rev. pref.
- Given p and m choose a maximal point in $B(p, m)$ by:
    1. Choose best data point z in $B(p, m)$ for $\succeq$.
    2. Project z into the budget line lexicographically.

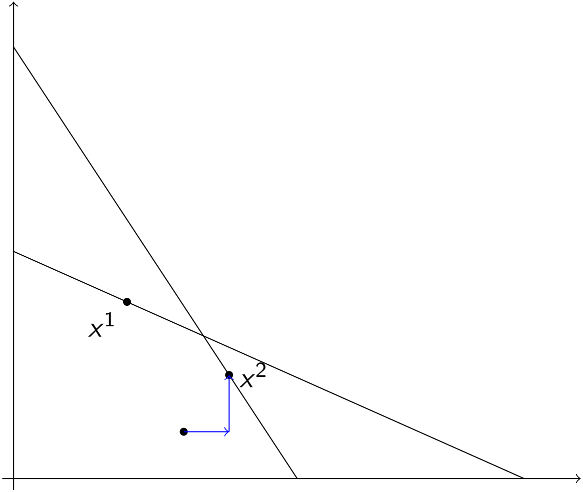
The algorithm defines a demand function $d(p, m)$.
We show that it is a rational demand: it satisfies SARP.

A violation of WARP

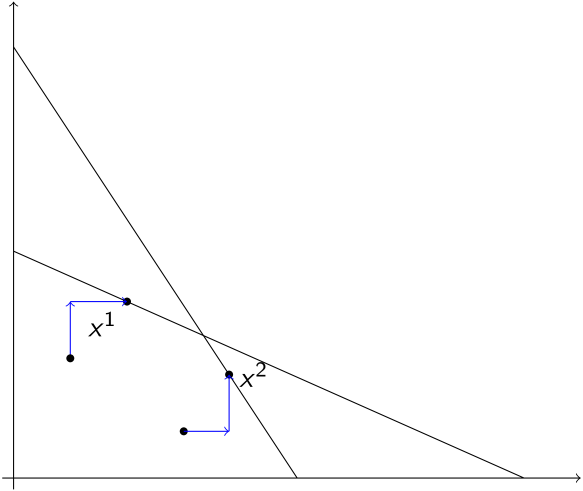


Two possibilities:

- x^1 and x^2 projected from different (data) points;
- x^1 and x^2 projected from same point.



x^1
x^2



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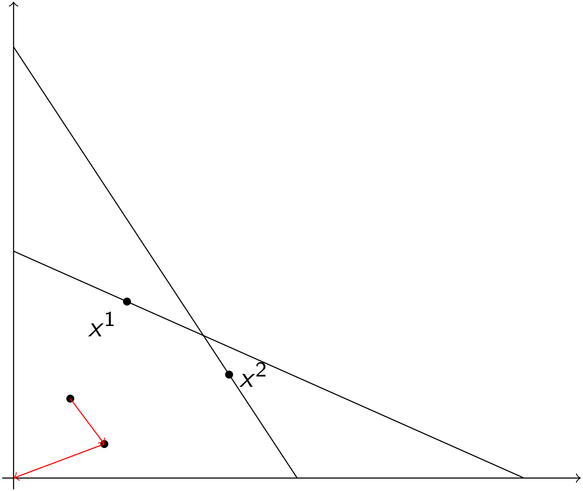

x^1
x^2

Running time of algorithm depends on size of the data set. This turns out to be unavoidable.

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Proposition

Any algorithm that takes as input a data set with n data points, a price vector p, and an income I and outputs $d(p, I)$ for a d which rationalizes the data set requires, in the worst case, $\Omega(n)$ running time on a RAM with word size $\Theta(\log n)$, even when there are only two goods.

Proposition

Any demand function d that rationalizes a data set with n data points requires $\Omega(n \log n)$ bits of space to represent, in the worst case, even when there are only two goods.

Now: general equilibrium theory.

"If your laptop cannot find it, neither can the market"

– Kamal Jain

CS and Economics

For the model of general equilibrium, main CS result is:

Walrasian equilibrium is hard to find.

Hard, even if:

- Utilities are separable over goods and piecewise linear (concave).
- Utilities are Leontief

Our results

Consider exchange economies with std. assumptions on preferences (smooth concave utilities); n agents and l goods.

When n is fixed, it's easy to find a WE.

Exploits the Negishi approach to prove existence of WE.

Why study n fixed?

Macro & finance models $\rightarrow$ many goods, few agents.

- Models w/representative agent.
- Models with n agents and infinitely many goods.

Literally fixed n.

Why study n fixed?

> *The history of all hitherto existing society is the history of class struggles.*
>
> *– Karl Marx*

Many agents but limited heterogeneity: economic “class.”

If preferences are homothetic, and all agents belong to one of a fixed number of endowment classes (e.g. farmers, workers and capitalists), then WE is easy.

Why study n fixed?

Popular model of a large economy: replica of a given economy.

Many classical results on large economies, such as core convergence, hold for replica economies.

Our result implies that WE is easy for (large) replica economies.

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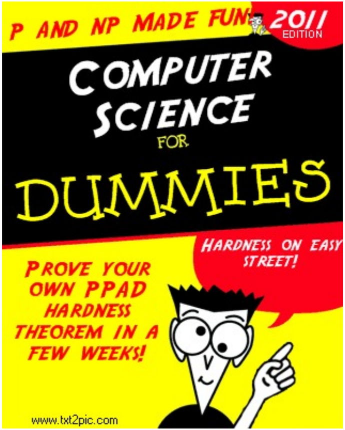

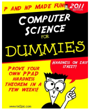

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Ex: NP-hard problems.

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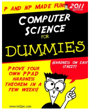
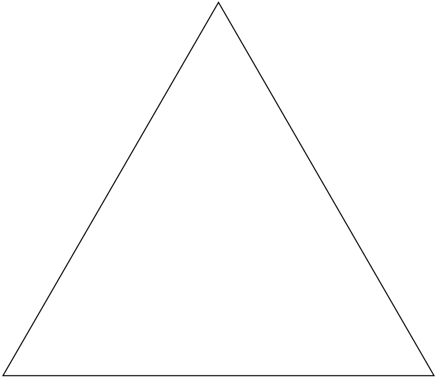
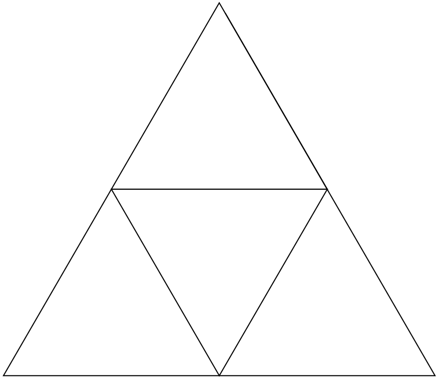


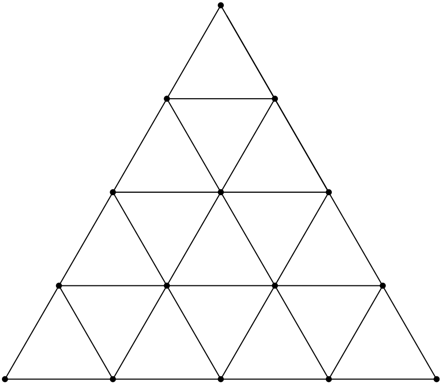
Decision problems are not appropriate for equilibria, because existence is guaranteed.

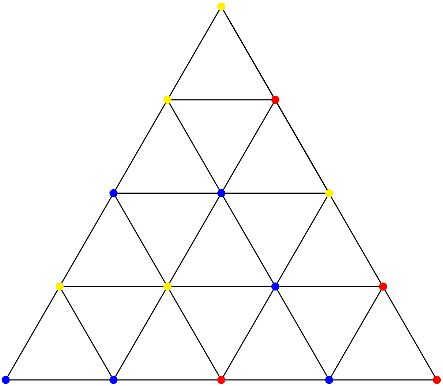
Class of problems based on computing a (total) function: given an input x, compute $f(x)$.

A problem is *PPAD-hard* if it is as hard as END OF THE LINE. Finding Walrasian eq. with Leontief utilities is PPAD-hard.









Exchange economy

An *exchange economy* is a tuple $(\omega_i, u_i)_{i=1}^n$
where $\omega_i \in \mathbf{R}_+^l$ and $u_i : \mathbf{R}_+^l \to \mathbf{R}$.

$l =$ number of goods
$n =$ number of agents
Each agent described an endowments & utiliy fn.

Exchange economy

An *allocation* in $(\omega_i, u_i)_{i=1}^n$ is $x \in \mathbf{R}_+^{nl}$ s.t. $\sum_{i=1}^n x_i = \sum_{i=1}^n \omega_i$.

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A *Walrasian equilibrium* in $(\omega_i, u_i)_{i=1}^n$ is (p, x) s.t.

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1. (p a price vector), $p \in \mathbf{R}_{++}^l$
2. (supply equals demand) $x = (x_i)_{i=1}^n \in \mathbf{R}_+^{nl}$ is an allocation,
3. (agents maximize utility when consuming x_i) and for all i
   $p \cdot \omega_i = p \cdot x_i$ and

$$u_i(y) > u_i(x_i) \Rightarrow p \cdot y > p \cdot x_i$$

Approximate equilibrium

A *Walrasian ε-equilibrium* is (p, x) s.t.

1. $p \in \mathbf{R}^l_+$,
2. x is an allocation,
3. and for all i

$$u_i(y) > u_i(x_i) \Rightarrow p \cdot y > p \cdot x_i$$

and $|p \cdot \omega_i - p \cdot x_i| < \varepsilon$.

E a family of exchange economies.

Each $(u_i, \omega_i)_{i=1}^n \in E$ has n agents (different numb. goods); assume:

1. (all goods exist) $\sum_{i=1}^n \omega_i \in \mathbf{R}^l_{++}$;
2. (regular utilities) u_i is C^1, concave, and strictly monotonic;
3. (boundary condition) If $x \in \mathbf{R}^l_+ \setminus \mathbf{R}^l_{++}$ and $y \in \mathbf{R}^l_{++}$, then $u(x) < u(y)$;
4. (normalization) $\forall x \in \mathbf{R}^{nl}_+$ s.t. $\sum_{i=1}^n \omega_i = \sum_{i=1}^n x_i$, $u_i(x_i) \in [0, 1]$.

Theorem

Let $\varepsilon > 0$. There is an algorithm that, for any economy in E, finds a Walrasian ε-equilibrium in time polynomial in l.

Let $(u_i, \omega_i)_{i=1}^n \in E$.
An allocation x in $(u_i, \omega_i)_{i=1}^n$ is
Pareto optimal iff:

- $\nexists$ allocation y with $\forall i(u_i(y) > u_i(x))$
- iff $\exists \lambda \in \Delta$ s.t. x solves

$$\max \sum_i \lambda_i u_i(\tilde{x}_i)$$
$$\text{s.t.} \tilde{x} \text{ is an allocation}$$

A *Walrasian equilibrium with transfers* is a triple (p, x, T), where:

- $p \in \mathbf{R}^l_+$ (a vector of prices);
- $T \in \mathbf{R}^n$ and $\sum_{i=1}^n T_i = 0$ (a vector of transfers);
- x is an allocation (supply equals demand);
- $\forall i$

$$u_i(y) > u_i(x_i) \Rightarrow p \cdot y > p \cdot \omega_i + T_i$$

and $p \cdot x_i = p \cdot \omega_i + T_i$ (agents are maximizing utility).

Note: a WE is a WET with zero transfers; an approximate WE is a WET with small transfers.

Second Welfare Theorem

Theorem

Let $(u_i, \omega_i)_{i=1}^n \in E$ and x be an interior Pareto optimal allocation. Then $\exists$ p and T s.t. (x, p, T) is Walrasian eq. with transfers.

Negishi's approach

$$\begin{tikzcd} \lambda \in \Delta \arrow[r, mapsto] & x(\lambda) \in \operatorname{argmax} \sum_i \lambda_i u_i \arrow[r, mapsto] & (x(\lambda), p(\lambda), T(\lambda)) \arrow[d, mapsto] \\ & & \lambda' \in \Delta \end{tikzcd}$$

Existence follows by Kakutani's FPT.
Note the fixed-point argument is in the n-dimensional simplex.

Negishi's approach

We:

- Kakutani is non-constructive.
  Instead we use Sperner's lemma.
- Find a zero of $T(\lambda)$.
- Approximation must be *independent* of I.

$$\lambda \in \Delta \longmapsto x(\lambda) \in \operatorname{argmax} \textstyle\sum_i \lambda_i u_i \longmapsto (x(\lambda), p(\lambda), T(\lambda))$$

SWT for undergrads

$$\max \sum_{i=1}^{n} \lambda_i u_i(x_i)$$
$$\text{s.t.} \begin{cases} \sum_{i=1}^{n} x_i \le \sum_{i=1}^{n} \omega_i \\ x_i \ge 0. \end{cases}$$

$$p(\lambda) = \lambda_h D u_i(x_h(\lambda)).$$

$$T_i(\lambda) = p(\lambda) \cdot (x_i(\lambda) - \omega_i).$$

Two little lemmas

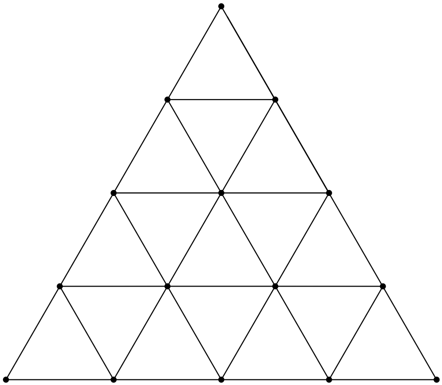
Lemma

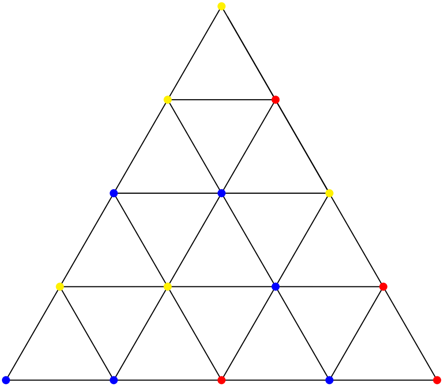
For $\lambda, \lambda' \in \Delta$, $\|T(\lambda) - T(\lambda')\| \leq (n-1)\|\lambda - \lambda'\|$.

Lemma

If $\lambda_i = 0$ *then* $T(\lambda)_i \leq 0$.

$\Rightarrow$ construct simplicial subdivision with mesh $\frac{\varepsilon}{(n-1)^2}$ and color vertexes appropriately.





Sperner's lemma to get $\|T\| < \varepsilon$

- simplicial subdivision with mesh $\frac{\varepsilon}{(n-1)^2}$
- for λ, λ' in same subsimplex, $T(\lambda)$ close to $T(\lambda')$ (Lipschitz lemma)

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- Boundary lemma $\Rightarrow$ proper labeling of subsimplex

Sperner's lemma to get $\|T\| < \varepsilon$

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- color subdivision: vertex λ has color i if $T_i(\lambda) > 0$ (choose largest $T_i(\lambda)$ if more than one).
- Boundary lemma $\Rightarrow$ proper labeling of subsimplex
- polychromatic subsimplex gives $T(\lambda)$ close to each other, for each i one λ with $T_i(\lambda) > 0$.
- Since $\sum_i T_i(\lambda) = 0$ must have $\|T(\lambda)\| < \varepsilon$.

Other notions of approximate equilibria.

Other notions of approximation

An *ε-approximate equilibrium* in an exchange economy $(u_i, \omega_i)_{i=1}^n$ is a pair (p, x) where $p \in \mathbf{R}_+^l$, x is an allocation, and for all i

$$p \cdot y \leq p \cdot \omega_i \Rightarrow u_i(y) \leq u_i(x_i) + \varepsilon,$$

and $|p \cdot \omega_i - p \cdot x_i| < \varepsilon$.

A defn. like the one used in CS.

Other notions of approximation

An *strong ε-approximate equilibrium* in an exchange economy $(u_i, \omega_i)_{i=1}^n$ is a pair (p, x) where $p \in \mathbf{R}_+^l$, $x \in \mathbf{R}_+^{nl}$ with $\| \sum_i x_i - \sum_i \omega_i \| < \varepsilon$, and for all i

$$p \cdot y \leq p \cdot \omega_i \Rightarrow u_i(y) \leq u_i(x_i),$$

and $p \cdot \omega_i = p \cdot x_i$.

A defn. like the one used in GE theory.

Other notions of approximation

Suppose that there is $\Theta > 0$ and $\pi > 0$ such that, for all $(u_i, \omega_i)_{i=1}^n$ in E,

$$\sup_{p \in \Delta} p \cdot \sum_{i=1}^{n} \omega_i \leq \Theta,$$

and if x is an allocation in $(u_i, \omega_i)_{i=1}^n$, then $D_s u_i(x_i) > \pi$.

Other notions of approximation

Theorem

Let $\varepsilon > 0$. There is an algorithm that, for any economy in E, finds an ε-approximate equilibrium, and a strong ε-approximate equilibrium, in time polynomial in l.

Now: game theory.

The Empirical Implications of Rank in Bimatrix Games, by Barman, Bhaskar, Echenique, & Wierman.

Nash equilibrium

A two-player game in normal form is given by a pair of matrices (A, B) of size $n \times n$,
A *Nash equilibrium* is a pair $(i, j) \in [n] \times [n]$ s.t. $\forall\ i' \in [n]$ and $j' \in [n]$,

$$A_{ij} \geq A_{i'j} \text{ and } B_{ij} \geq B_{ij'}.$$

If inequalities are strict, then (i, j) is a *strict Nash equilibrium*.

Rank

Our focus is on games with low rank.
The *rank* of a game (A, B) is the rank of the matrix $C := A + B$.
For a zero-sum game, $C = \mathbf{0}$.

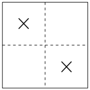
Data

A subgame is denoted by (I, J) where $I, J \subseteq [n]$
A *data set* is a set of triples $((i, j), I, J)$, where (I, J) is a subgame, and $i \in I$ and $j \in J$.

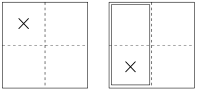
Revealed Preference

A data set T is *rationalizable* if there exist a game (A, B) s.t. (i, j) is a *strict* Nash eq. in the subgame (I, J), $\forall((i, j), I, J) \in T$.

Examples



((a)) Data set that is rationalizable (via a rank one game).



((b)) Data set that is not rationalizable.

Figure: *Examples of a rationalizable data set and a data set that is not rationalizable.*

Crossing number

Two subgames (I, J) and (I', J') *cross* if $(I \times J) \cap (I' \times J') \neq \emptyset$, but $(I \times J) \not\subseteq (I' \times J')$ and $(I' \times J') \not\subseteq (I \times J)$.

The *crossing number of T* is

$$\min \{ |\{i : (i,j) \in \mathcal{O}\}|, \ |\{j : (i,j) \in \mathcal{O}\}| \} .$$

Theorem

For all n, there exists a rationalizable data set T over an $n \times n$ strategy space such that the rank of any bimatrix game that rationalizes T is $\Omega(\sqrt{n})$.

Theorem

Any rationalizable data set T that satisfies the uniqueness property can be rationalized by a bimatrix game of rank at most the crossing number of T.

Despite all the computations
You could just dance to that rock 'n' roll station
And baby it was allright

– Lou Reed