General Revealed Preference Theory

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Revealed Preference

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It means that their *behavior* is as if they maximize some utility.

- behavior $\rightarrow$ observable data.
- utility $\rightarrow$ unobservable.

Revealed Preference

Characterize data for which *there is some* utility function that could rationalize the data.

- Definition is a test
- . . . but it's useless.

The nature of falsifiable theories

Popper's theories:

All swans are white:
$\forall s W(s)$

Falsifiable

There exists a black swan:
$\exists s B(s)$

Not falsifiable

Example: Rev. Pref. problem

Axiomatize R, P (revealed preference relations) for which:

$\exists \succeq$ (satisfying some properties) such that

$\forall x \forall y, x \, R \, y \rightarrow x \succeq y$ and $x \, P \, y \rightarrow x \succ y$

Revealed Preference

A test is an effective "positive axiomatization."

- A universal description of rationalizable data.
- Should not refer to theoretical objects, but only to observables.
- An algorithm should decide in finite time if data passes test.

Our paper:

Gives a sufficient condition for theory to have an effective positive axiomatization.

- Explains classical rev. pref. theory
- new applications to multiple selves (collective dec. making), Nash eq., and barg.

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So:

- Pretend that we can observe theoretical terms.
- Axiomatize the theory using statements about theoretical terms.
- This can be "projected" onto observables as an effective axiomatization.

Example

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Structure:

- $(\mathbf{R}, >^{\mathbf{R}})$,
- $(\mathbf{N}, >^*)$, $n >^* m$ iff $n - m > 5$.

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Structure:

- $(\mathbf{R}, \geq^{\mathbf{R}}, >^{\mathbf{R}})$,
- $(\mathbf{N}, \geq^*, >^*)$, $\geq^* = \geq^{\mathbf{N}}$, $n >^* m$ iff $m - n > 5$.

Game Theory

Nash bargaining: $\mathcal{L} = \langle F, \in \rangle$
Normal-form games: $\mathcal{L} = \langle S_1, \ldots, S_n, C, \in \rangle$

In general:

- A *language* $\mathcal{L}$ is a list of relation symbols.
- A *axiom* is a logical sentence in $\mathcal{L}$.
- *Universal axioms* are those with $\forall$ quantification at the beginning of the sentence.
- A *structure* is a set together with an interpretation of each symbol in $\mathcal{L}$

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Examples of (first order) axioms using symbols $P, R, \succ, \succeq, O$

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- Rationalization: $\forall x \forall y (x \, R \, y) \to (x \succeq y)$

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- Rationalization: $\forall x \forall y (x \, R \, y) \rightarrow (x \succeq y)$
- Optimality: $\forall x O(x) \leftrightarrow \forall y (x \, R \, y)$

Axioms

Universal axioms are those with universal ($\forall$) quantification, coming at the beginning of the sentence:

- $\forall x \forall y (x \succeq y) \vee (y \succeq x)$ is universal.
- $\forall x \exists y (x \succ y)$ is not.
- $\forall x (O(x) \leftrightarrow \forall y (x \, R \, y))$ is not.

What is a theory?

Given a language $\mathcal{L}$. A *theory* is a class of structures of $\mathcal{L}$ that is closed under isomorphism.

Example: Language $\langle \succeq, \succ \rangle$ and theory of utility maximization.

Main Result

Two languages, $\mathcal{L}$ and $\mathcal{F}$, where $\mathcal{F} = (R_1, ..., R_N)$ and $\mathcal{L} = (R_1, ..., R_N, Q_1, ..., Q_K)$.

Symbols R_i or Q_i for k-ary relations.

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Choice example: $\mathcal{F} = (R, P)$ and $\mathcal{L} = (R, P, \succeq, \succ)$. Language $\mathcal{F}$: observables, language $\mathcal{L}$: (observables and unobservables).

Main Result

Recall $\mathcal{F} \subseteq \mathcal{L}$.

Let T be an $\mathcal{L}$-theory.

$F(T)$ is the class of all $\mathcal{F}$-structures $(X, R_1^X, ..., R_N^X)$ for which there exist $Q_1^X, ..., Q_K^X$ s.t.

$$(X, R_1^X, ..., R_N^X, Q_1^X, ..., Q_K^X) \in T.$$

$F(T)$ is a projection of $\mathcal{L}$-theory T onto language $\mathcal{F}$.

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T: $\mathcal{L}$-theory of structures $(X, R^X, P^X, \succeq^X, \succ^X)$ for which

1. $\succeq^X$ is a weak order
2. $\succ^X$ is its strict part
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$F(T)$ is the $\mathcal{F}$-theory of all structures (X, R^X, Q^X) for which there exists $\succ^X$, $\succeq^X$ for which 1-4 is satisfied.

Main result

For languages $\mathcal{F} \subseteq \mathcal{L}$, and $\mathcal{L}$-axioms Σ, the set of $\mathcal{F}$-consequences of Σ is the collection of all logical consequences of Σ involving only symbols from $\mathcal{F}$.

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Theorem

Suppose that T is a universally axiomatizable $\mathcal{L}$-theory, and that $\mathcal{F} \subseteq \mathcal{L}$. Then $F(T)$ is a universally axiomatizable $\mathcal{F}$-theory, and is axiomatized by the set of all universal $\mathcal{F}$-consequences of T.

Example

Let $\mathcal{F} = (R, P)$, and let $\mathcal{L} = (R, P, \succ, \succeq)$.

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Has axiomatization:

1. $\forall x \forall y (\succeq (x, y) \vee \succeq (y, x))$
2. $\forall x \forall y (\succ (x, y) \leftrightarrow (\succeq (x, y) \wedge \neg \succeq (y, x)))$
3. $\forall x \forall y \forall z (\succeq (x, y) \wedge \succeq (y, z)) \rightarrow \succeq (x, z)$
4. $\forall x \forall y (R(x, y) \rightarrow \succeq (x, y))$
5. $\forall x \forall y (P(x, y) \rightarrow \succ (x, y))$

The Strong Axiom of Revealed Preference: For every k,

$$\forall x_1 ... \forall x_k \neg \bigwedge_{i=1}^{k} \left(x_i \; Q_i \; x_{(i+1) \mod k} \right)$$

where for all i, $Q_i \in \{R, P\}$, and for at least one $i \in \{1, ..., k\}$, $Q_i = P$.

Remarks on proof.

Any $\mathcal{F}$-consequence of T is satisfied by $F(T)$, and if an $\mathcal{F}$ axiom is true for $F(T)$, it is true for T (and hence an $\mathcal{F}$ consequence).

Main difficulty is in establishing that $F(T)$ is axiomatizable. Need not necessarily be true.

Remarks on proof.

Proof relies on a result of Tarski, which characterizes universally axiomatizable theories. Also relies on some form of choice (Szpilrajn's theorem is a corollary of our result).

Important: In general, $F(T)$ need not be axiomatizable, even if T is. Universality of T is critical.

Recursive enumerability

A set of axioms is **recursively enumerable** if there is an algorithm for listing them out, one by one.

Corollary

If T is universally and recursively enumerably axiomatizable, then so is $F(T)$.

Fagin's Theorem

$F(T)$ is in class NP if there is a non-deterministic Turing machine which, in poly. time, given any $\mathcal{F}$-structure $\mathcal{M}$, tells us whether $\mathcal{M} \in F(T)$.

Theorem

Suppose that T is a finitely axiomatized $\mathcal{L}$ theory. Then $F(T)$ is in NP.

Applications

- Multiple selves
- Revealed game theory
- Group preferences (Pareto relation, majority rule, etc)
- Choice theory

Multiple selves, or group preferences

Observe relation R. Hypothesize that R is generated by given finite set of agents N and given social choice rule f (satisfying neutrality and IIA).

Agents hypothesized to have “rational” preferences.

This theory is universally and r.e. axiomatizable.

Special case: Pareto extension relation on N agents

Observe $\tilde{P}$.

Hypothesize: $\exists P \exists R_i \exists P_i$ such that:

- $\forall x \forall y, x \, P \, y \leftrightarrow \bigwedge_{i \in N} x \, P_i \, y$
- $\forall x \forall y, x \, \tilde{P} \, y \rightarrow x \, P \, y$
- R_i weak orders, and P_i its strict part.
  - $\forall x \forall y, x \, R_i \, y \vee y \, R_i \, x$
  - $\forall y \forall y \forall z, x \, R_i \, y \wedge y \, R_i \, z \rightarrow x \, R_i \, z$
  - $\forall x \forall y, x \, P_i \, y \leftrightarrow x \, R_i \, y \wedge \neg y \, R_i \, x$

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- $\forall x \forall y, x\, \tilde{P}\, y \rightarrow x\, P\, y$
- R_i weak orders, and P_i its strict part.
  - $\forall x \forall y, x\, R_i\, y \vee y\, R_i\, x$
  - $\forall y \forall y \forall z, x\, R_i\, y \wedge y\, R_i\, z \rightarrow x\, R_i\, z$
  - $\forall x \forall y, x\, P_i\, y \leftrightarrow x\, R_i\, y \wedge \neg y\, R_i\, x$

Axiomatization is known for the case $|N| = 2$.

Related Literature

- Simon (1985) (and other papers by H. Simon)
- Boland
- Mongin
- Chambers-Echenique-Shmaya
- Brown-Kubler (and Brown-Matzkin)