

Response Time and Utility

Federico Echenique and Kota Saito*

Division of Humanities and Social Sciences,
California Institute of Technology[†]

April 19, 2017

Abstract

Response time is the time an agent needs to make a decision. One fundamental finding in psychology and neuroscience is that, in a binary choice, there is a monotonic relationship between the response time and the difference between the utilities of the two options. We consider situations in which utilities are not observed, but rather inferred from revealed preferences: meaning they are inferred from subjects' choices. Given data on subjects' choices, and the time to make those choices, we give conditions on the data that characterize the property that response time is a monotonic function of utility differences.

1 Introduction

Response time refers to the time an agent needs to make a decision. The relationship

*We appreciate Antonio Rangel, Ian Krajbich, and John Clithero for valuable discussion. We also appreciate Yimeng Li for his careful proof reading of the paper, and helpful suggestions. This research is supported by Grant SES1558757 from the National Science Foundation.

[†]1200 E California Blvd, MC 228-77 Pasadena, CA 91125

between utility and response time has been studied extensively in psychology and neuroscience. (See Luce (1986) and Krajbich et al. (2014) for surveys.)

Consider a binary choice experiment. We consider representations where there is a monotonic relation between response time and the utility difference between the two options. The relation can be monotone decreasing, meaning that subjects' response time is shorter as the difference between the utility of the two options becomes larger, or monotone increasing. A monotone decreasing relation is observed in many binary choice experiments in psychology and neuroscience,¹ The monotone decreasing relation can be explained by an agent that is learning the utility differences, so that a large utility difference can be detected more quickly. On the other hand, one can envision a monotone increasing relation when the subject knows the utility difference in advance, but she does not know its sign. The subject may then decide quickly when the difference is small. It simply matters less if she gets the choice wrong.

In either case, decreasing or increasing, there is a monotonic relationship between subjects' response and the difference between the utility of the two options. The purpose of this paper is to provide a formal axiomatic model which describes such monotonic relationships. Our axioms give conceptual foundations for the relation between response time and utility, when utility is not directly observable, and also provide revealed preference tests that can be used on experimental data.

In our model, given is data on agents' choices, and the time taken to make them. We give conditions that describe when data is consistent with the theory that response time is monotonic in utility differences. In a binary choice between x and y , if the agent chooses x over y , then $t(x, y)$ denotes the response time which the agent needs to make

¹This finding is based on having estimates of subjects' values for the different options. In classical economic environments, however, one has no access to such "psychometric" estimates of utility (indeed the traditional position is to discourage economists from using them), and must infer utility from choices.

the choice. We axiomatize the following *response-time representation*:

$$f(t(x, y)) = u(x) - u(y), \tag{1}$$

where f is a monotonic function and $u(x) > u(y)$. We first show an axiomatization for the representation with strictly decreasing f . Then by flipping inequalities in our axiomatization, we obtain an axiomatization of an alternative representation in which f is strictly increasing.

Our representation with strictly decreasing f is consistent with several well-known models in psychology and neuroeconomics. In psychology, *Pieron's law* states that the reaction time t to visual signal is monotonically decreasing in the intensity of the signal:

$$t = r_0 + k \frac{1}{I^\beta},$$

where I is the intensity of the signal. (See figure 2a of Mansfield (1973), figure 3 of Kohfeld (1971), and figure 5 of Kohfeld et al. (1981) for examples of evidence. See Luce (1986) for a survey.) In neuroscience, experimenters have studied economic decision making such as binary choice as in our paper. They use a stochastic model, *Drift Diffusion Model* (DDM) proposed by Ratcliff (1978). DDM predicts that the average response time is decreasing in the utility difference between the two options.

We present three results. First, under a richness assumption of the dataset, we first show that a few simple axioms characterize the representation in (1). Those axioms capture the monotonic relationship between the agent's response time, and the difference between the utility of the two options. For the case of decreasing (increasing) relationship, *Monotonicity* states that if an agent chooses x over y and y over z (i.e., $x \succ y \succ z$), then the response time for him to choose x over z should be shorter (resp. longer) than the response time for him to choose x over y and the response time for him to choose y over z (i.e., $t(x, z) < (resp. >)t(x, y)$, $t(x, z) < (resp. >)t(y, z)$).

In second place we turn to finite data sets. Our richness assumption cannot be satisfied with finite data. We provide an axiomatization that is testable with a finite dataset.

We show that Completeness and a stronger version of Monotonicity, which we call *Compensation*, characterize the response-time representation. (Compensation also implies the standard property of acyclicity commonly used in revealed preference analysis).

In third place, we further generalize the result by dropping Completeness because the axiom may not be satisfied in many experimental designs (it is simply too onerous to present subjects with all pairs of possible options). Then, we show that a stronger version of Compensation alone characterizes the response-time representation. The axiom is easily testable with experimental data.

Ours is the first paper to present an axiomatic study of response time with finite data. Krantz et al. (1971) and Debreu (1958) are the closest precedents within the axiomatic literature, but they assume infinite data and their results do not provide tests for finite datasets. Krantz et al. (1971) provides a general version of our representation. We use their main result to prove a version of our theorem (Theorem 1) under a rich data assumption (see Section 3 for a detailed discussion).

Debreu (1958) studies a model of stochastic choice. His model states that $p(a, b) \geq p(c, d)$ if and only if $u(a) - u(b) \geq u(c) - u(d)$ for some function u , where $p(x, y)$ is the primitive probability that the agent chooses x over y . Given Debreu's result, one might think that one can characterize our representation by imposing Debreu's axioms on $t(x, y)$, instead of $p(x, y)$. However, this transformation does not work because the axiomatization by Debreu (1958) crucially depends on the fact that $p(x, y) + p(y, x) = 1$. This must of course be true for a stochastic choice function p , but it is not true for a response time function t . We define $t(x, y)$ only when the agent chooses x over y , and the time $t(x, y)$ can take any positive real value.

One could get around the problem using an isomorphism between Debreu's and our primitives. Doing so does, however, not yield any fruitful results. For example, one could define Debreu's primitive p from our response time data as follows: Choose T large enough so that $T > t(x, y)$ for any $(x, y) \in \succ$. Then define $p_T(x, y) = t(x, y)/T$ and

$p_T(y, x) = 1 - (t(x, y)/T)$ for any $(x, y) \in \succ$. Notice that the definition depends on T and $p_T(x, y) \neq p_{T'}(x, y)$ whenever $T \neq T'$.

The axiomatization by Debreu (1958) implies the following: There exists a response time representation if there exists a positive number T such that p_T satisfies Debreu's axioms. Such an axiomatization would not be desirable because whether the data satisfy the axioms depends on the choice of T and, hence, it is difficult to interpret the axioms in a meaningful way. For example, one of Debreu's axioms is as follows: $p_T(x, y) \leq p_T(z, w)$ if and only if $p_T(x, z) \leq p_T(y, w)$. Suppose that if the agent chooses x over y and z ; z over w ; and w over y . Then the axiom requires that $t(x, y) \leq t(z, w)$ if and only if $t(x, z) \leq T - t(w, y)$. It is obvious that *whether the requirement is satisfied or not depends on the choice of T* . Moreover, one would need to find a T with which the requirement holds for any x, y, z, w . Finding such a T is a purely computational exercise and conveys little behavioral meaning.

Another precedent to our paper is Scott (1964). Scott considers a finite set X and a quaternary relation D on X . He provides a condition on D under which there exists a real-valued function f on x such that $xyDzw$ if and only if $f(x) - f(y) \geq f(z) - f(w)$. His condition is testable with finite data, although he does not study response time data. Scott's condition does not imply our axiom.

There are several non-axiomatic papers on response times in economics. These are important contributions to the study of response time, but seek different purposes than our paper. The purpose of our axiomatic study is to provide non-parametric tests of response time models, and to provide a basic understanding of the qualitative features behind response time models. The existing literature on response time in economics has a different focus. Wilcox (1993) uses response times to evaluate decision cost in his experiments. Rubinstein (2007) measures response times in his experiments on games, and suggests that choices made instinctively require less response time than choices that require the use of cognitive reasoning. Piovesan and Wengström (2009) experimentally

show that egoistic subjects make faster decisions than subjects with social preferences. In experiments on single-person decision problems, Rubinstein (2013) measures response times to study mistakes and biases such as Allais’s paradox.

Chabris et al. (2009) measure response times in their experiments on intertemporal choices. By assuming linear utility and a specific form of discounting function, they show an inverse relationship between average response times and utility differences. They claim that their evidence is consistent with the models of Gabaix et al. (2006). In the model of Gabaix et al. (2006), an agent uses partially myopic option-value calculations to select their next cognitive operation. More recently, Fudenberg et al. (2015) provide a model of response time. They do not provide an axiomatization of their model, focusing instead on which aspects of optimal sampling are implicitly assumed by the DDM model.

Some recent papers use response time to improve prediction rates. Clithero and Rangel (2013) use response times to predict choices in single-person decision problems. Based on DDM, they made predictions by using response times. Schotter and Trevino (2014) use response times to predict threshold behavior in a simple global game experiment.

Finally, we mention an important limitation of our model. Typically response time data are noisy and look random. Moreover, choices themselves can be random. In this paper, we consider the axiomatization of deterministic choice as a first step. As future research, it would be important to study the case of random choice.

2 Models

Let X be a set and let $\succ \subset X \times X \times \mathbf{R}_+$ be the primitive that describes an agent’s choice behavior. The meaning of $(x, y, t) \in \succ$ is that the agent chooses x over y , and that it takes time t for him to make that decision. We denote this time t by $t(x, y)$. (Notice we do not use the notation $t(y, x)$ unless y is chosen over x .) In this paper, we do not consider indifferences. If we observe a choice of an agent between x and y , we can observe either $t(x, y)$ or $t(y, x)$ but not both.

We assume that for every $x, y \in X$ there is at most one $t \in \mathbf{R}_+$ with $(x, y, t) \in \succ$. So we simply write $x \succ y$ to denote $(x, y, t) \in \succ$. We denote by T the set of $t \in \mathbf{R}_+$ for which there is $x, y \in X$ such that $(x, y, t) \in \succ$.

In the following sections, we provide axiomatizations of the following representation of \succ : for some utility function $u : X \rightarrow \mathbf{R}$ and a monotonic function $f : T \rightarrow \mathbf{R}$ such that

$$f(t(x, y)) = u(x) - u(y). \quad (2)$$

We call the representation *response-time representation*.

3 Axioms for Rich Data

In this section, we consider rich data. We show that two intuitive axioms (Monotonicity and Time-Transitivity) and weak order characterize the response-time representation, under a “richness” condition. Moreover, we show that the response-time representation has a strong uniqueness property. In this section, the set of outcomes, X , is infinite. In the next section, we provide an axiomatization for finite data.

The first two axioms are standard.

Axiom 1. (*Completeness*): For any distinct $x, y \in X$, either $x \succ y$ or $y \succ x$.

Note that the completeness of the strict preference means that we do not allow indifference.

Axiom 2. (*Transitivity*): For any $x, y, z \in X$, if $x \succ y$ and $y \succ z$, then $x \succ z$.

Our next axiom captures the monotonic relationship between response times and the difference in the values of the two options. Consider three options x, y, z such that $x \succ y \succ z$. Then, given a utility representation, and a decreasing relation between utility and response time, the difference $u(x) - u(z)$ must be larger than the differences $u(x) - u(y)$ and $u(y) - u(z)$. So the monotonically decreasing relationship implies that the response time $t(x, z)$ must be smaller than $t(x, y)$ and $t(y, z)$. Formally,

Axiom 3. (*Decreasing Monotonicity*): If $x \succ y, y \succ z$, and $x \succ z$ then $t(x, z) < t(x, y)$ and $t(x, z) < t(y, z)$.

In the same way, the monotonically increasing relationship implies that response time $t(x, z)$ must be larger than $t(x, y)$ and $t(y, z)$. Formally,

Axiom 4. (*Increasing Monotonicity*): If $x \succ y, y \succ z$, and $x \succ z$ then $t(x, z) > t(x, y)$ and $t(x, z) > t(y, z)$.

Notice that the above monotonicity axiom implies antisymmetry.²

The representation has additional implications, beyond Monotonicity. These originate from the difference structure. Given any x, y , and z ; and x', y' , and z' , if the response time $t(x, y)$ is smaller than $t(x', y')$; and $t(y, z)$ is smaller than $t(y', z')$, then by the monotonic decreasing (increasing) relationship between response time and utility differences, the utility difference $u(x) - u(y)$ must be larger (resp. smaller) than $u(x') - u(y')$; and $u(y) - u(z)$ must be larger (resp. smaller) than $u(y') - u(z')$. Therefore, by adding the two differences, we have that the utility difference $u(x) - u(z)$ is larger (resp. smaller) than $u(x') - u(z')$. Hence, by the monotonic relationship again, the response time $t(x, z)$ must be smaller than $t(x', z')$. Our reasoning suggests the following axiom.

Axiom 5. (*Time Transitivity*): If $t(x, y) \leq t(x', y')$ and $t(y, z) \leq t(y', z')$, then $t(x, z) \leq t(x', z')$.

Notice that $t(x, z)$ is well defined because of transitivity; the fact that $t(x, y)$ and $t(y, z)$ are well defined imply $x \succ y$ and $y \succ z$, so we have $x \succ z$.

The next two axioms require that the data are rich enough. To explain the first axiom, consider the case of a monotonic decreasing relation between response times and utility differences. Suppose that $t(x, y) < t(x', y')$. Then the axiom says that by making y (as z) close enough to x or by making x (as w) close enough to y , we can make the response time larger and make it exactly equal to $t(x', y')$.

²If $x \succ x$, then the monotonicity implies $t(x, x) > t(x, x)$, which is a contradiction.

Axiom 6. (*Richness*): If $t(x, y) < t(x', y')$, then there exist $z, w \in X$ such that $t(x, z) = t(x', y') = t(w, y)$.

The last axiom has a flavor of an Archimedean axiom. The axiom says that no matter how small a utility difference $u(x) - u(y)$ is, by “multiplying” the difference we can make it larger than a utility difference $u(z) - u(w)$, no matter how large is it. Since we do not observe the utility itself, instead of multiplying, we consider a sequence x_1, \dots, x_n such that $u(x_{i+1}) - u(x_i)$ is equal to $u(x) - u(y)$ for all i . Hence, the total utility difference $u(x_n) - u(x_1)$ is a multiplication of $u(x) - u(y)$. The axiom says that by considering such a sequence long enough we can make $u(x_n) - u(x_1)$ larger than $u(z) - u(w)$.

By using the monotonic relation between response time and utility differences, we have that $u(x_{i+1}) - u(x_i) = u(x) - u(y)$ if and only if $t(x_{i+1}, x_i) = t(x, y)$. Moreover, if there is monotonic decreasing (increasing) relationship, $u(x_n) - u(x_1) > u(z) - u(w)$ if and only if $t(z, w) > (\text{resp. } <) t(x_n, x_1)$. We can formalize the axiom for each case as follows:

Axiom 7. (*Decreasing Archimedean*): For any $x, y, z, w \in X$ such that $x \succ y$ and $z \succ w$, there exist $x_1, \dots, x_n \in X$ such that $x_{i+1} \succ x_i$, $t(x_{i+1}, x_i) = t(x, y)$ for all $i \leq n - 1$, and $t(z, w) > t(x_n, x_1)$.

Axiom 8. (*Increasing Archimedean*): For any $x, y, z, w \in X$ such that $x \succ y$ and $z \succ w$, there exist $x_1, \dots, x_n \in X$ such that $x_{i+1} \succ x_i$, $t(x_{i+1}, x_i) = t(x, y)$ for all $i \leq n - 1$, and $t(z, w) < t(x_n, x_1)$.

The two axioms above are different only in that $t(z, w) > t(x_n, x_1)$ in the first axiom and $t(z, w) < t(x_n, x_1)$ in the second axiom.

Theorem 1. *Under Richness and Decreasing (Increasing) Archimedean, \succ satisfies Completeness, Transitivity, Decreasing (resp. Increasing) Monotonicity, and Time-Transitivity, if and only if there exists a function $u : X \rightarrow \mathbf{R}$ and a strictly decreasing (resp. increasing) function $f : T \rightarrow \mathbf{R}$ such that*

$$x \succ y \Leftrightarrow u(x) > u(y),$$

and

$$u(x) - u(y) = f(t(x, y)).$$

Moreover, if \succ has two different representations (u, f) and (v, g) , then there exists a positive number α and a real number β such that $u = \alpha v + \beta$ and $f(t) = \alpha g(t)$ for all $t \in T$.

Proof of Theorem 1: The necessity of the axioms is obvious. So we provide the proof for sufficiency. We provide the proof for the case of strictly decreasing f .

For all $x, y, z, w \in X$ such that $x \succ y$ and $z \succ w$, define

$$(x, y) \succsim' (z, w) \text{ if and only if } t(x, y) \leq t(z, w). \quad (3)$$

By definition \succsim' is complete and transitive relation. We will show the following properties of \succsim' .

Claim:

- (i) if $x \succ y, y \succ z$, and $x \succ z$, then $(x, z) \succsim' (x, y)$ and $(x, z) \succsim' (y, z)$;
- (ii) if $(x, y) \succsim' (x', y')$ and $(y, z) \succsim' (y', z')$, then $(x, z) \succsim' (x', z')$;
- (iii) if $(x, y) \succ' (z, w)$, then there exist $w', w'' \in X$ such that $(x, w') \sim' (z, w) \sim' (w'', y)$;
- (iv) for any sequence $\{x_i\} \subset X$, if

- (a) $x_{i+1} \succ x_i$ for all i ,
- (b) $(x_{i+1}, x_i) \sim' (x_2, x_1)$ for all i ,
- (c) there exists $y, z \in X$ such that $y \succ z$ and $(y, z) \succ' (x_i, x_1)$ for all i ,

then the sequence $\{x_i\}$ is finite.

Proof. Note that (i) follows from Decreasing Monotonicity. (ii) follows from Time-Transitivity. (iii) follows from Richness. To see that (iv) follows from Decreasing Archimedean,

suppose (iv) does not hold. Then, there exists an infinite sequence $\{x_i\}_{i=1}^{\infty} \subset X$ and $(z, w) \in \succ$ such that $x_{i+1} \succ x_i$, $t(x_{i+1}, x_i) = t(x_2, x_1)$ for all i , and $t(z, w) < t(x_i, x_1)$ for all i .

By Decreasing Archimedean, on the other hand, there exists a sequence $\{y_i\}_{i=1}^n \subset X$ such that $y_{i+1} \succ y_i$, $t(y_{i+1}, y_i) = t(x_2, x_1)$ for all i , and $t(z, w) > t(y_n, y_1)$. Since $t(y_{i+1}, y_i) = t(x_2, x_1) = t(x_{i+1}, x_i)$ for all i , by using Time-Transitivity repeatedly, we obtain $t(x_n, x_1) = t(y_n, y_1)$. Hence, $t(z, w) < t(x_n, x_1) = t(y_n, y_1) < t(z, w)$, which is a contradiction. \square

By Claim, Theorem 4 (Krantz et al. (1971), Theorem 1, p147) shows that there exists a function $\phi : X \rightarrow \mathbf{R}$ such that $(x, y) \succsim' (z, w)$ if and only if $\phi(x) - \phi(y) \geq \phi(z) - \phi(w)$. Moreover, for any $x, y \in X$ such that $x \succ y$, we have $\phi(x) > \phi(y)$. Furthermore, if there exists another ϕ' satisfies the condition, then there exists a positive number α and a real number β such that $\phi' = \alpha\phi + \beta$. (See Appendix B for the complete statement of the theorem by Krantz et al. (1971).)

For all $x \in X$, define

$$u(x) = \phi(x).$$

For any $x, y \in X$ such that $x \succ y$, we have $\phi(x) > \phi(y)$, so that $u(x) > u(y)$. By Completeness, we have $x \succ y \Leftrightarrow u(x) > u(y)$.

For all $x, y \in X$ such that $x \succ y$, we can define a function $f : T \rightarrow \mathbf{R}$ as follows:

$$f(t(x, y)) = \phi(x) - \phi(y).$$

Then we have $f(t(x, y)) = u(x) - u(y)$. To see that f is well defined notice that

$$t(x, y) = t(z, w) \Leftrightarrow (x, y) \sim' (z, w) \Leftrightarrow \phi(x) - \phi(y) = \phi(z) - \phi(w) = f(t(x, y)) = f(t(z, w)),$$

where $(x, y) \sim' (z, w)$ means $(x, y) \succsim' (z, w)$ and $(z, w) \succsim' (x, y)$.

To show that f is strictly decreasing, choose any $s, s' \in T$. Then, there exist $x, y, x', y' \in$

X such that $s = t(x, y)$ and $s' = t(x', y')$. Then,

$$\begin{aligned}
s \geq s' &\Leftrightarrow t(x, y) \geq t(x', y') \\
&\Leftrightarrow (x', y') \succsim' (x, y) \\
&\Leftrightarrow \phi(x') - \phi(y') \geq \phi(x) - \phi(y) \\
&\Leftrightarrow f(t(x', y')) \geq f(t(x, y)) \\
&\Leftrightarrow f(s') \geq f(s).
\end{aligned}$$

So f is strictly decreasing. The uniqueness result immediately follows from Theorem 4 (Krantz et al. (1971), Theorem 1, p147).

The proof for the axiomatization of the representation in which f is strictly increasing is almost identical. In the definition (3), we need to flip the inequality as follows:

$$(x, y) \succsim' (z, w) \text{ if and only if } t(x, y) \geq t(z, w).$$

Then by flipping inequality consequently, we can obtain the proof for the case of strictly increasing f . ■

4 Axioms for Finite Data

In the previous section we worked with infinite data. Axioms 6 and 7 or 8 require that the data must be infinite. Since typical experimental data are finite, we provide an axiomatization for finite data in this section.

In Section 4.1, we maintain Completeness. We show that Completeness and a stronger version of acyclicity, which we call Decreasing (Increasing) Compensation, characterize the response-time representation.

In Section 4.2, we drop Completeness. We show that a stronger version of Decreasing (Increasing) Compensation characterize the response-time representation.

4.1 Axioms for Finite and Complete Data

In this section, we assume Completeness as follows:

Axiom 9. (*Completeness*): For any two distinct $x, y \in X$, $x \succ y$ or $y \succ x$.

We need one more axiom, which is a stronger version of acyclicity. To state the axiom, we need some preliminary definitions.

Definition 1. A sequence $(x_i)_{i=1}^n$ is called a cycle if $x_1 = x_n$ and for all i either $x_i \succ x_{i+1}$ or $x_{i+1} \succ x_i$ for all $i = 1, \dots, n-1$.

Definition 2. Given a cycle $(x_i)_{i=1}^n$, a decreasing (increasing) compensation is a one-to-one function π that maps any pair (x_i, x_{i+1}) with $x_i \succ x_{i+1}$ into some pair $(x_{i'}, x_{i'+1}) \equiv \pi(x_i, x_{i+1})$ with $x_{i'+1} \succ x_{i'}$ and $t(x_{i'+1}, x_{i'}) \leq$ (resp. \geq) $t(x_i, x_{i+1})$.

Remark. If there is no pair (x_i, x_{i+1}) such that $x_i \succ x_{i+1}$, we regard the empty function (i.e. the function that is the empty set in the set-theoretic notion of a function) as a decreasing (and increasing) compensation.

We call a pair (x_i, x_{i+1}) in a cycle an *improvement* if $x_{i+1} \succ x_i$, and a *worsening* if $x_i \succ x_{i+1}$. We say that an improvement $(x_{i'}, x_{i'+1})$ *compensates* a worsening (x_i, x_{i+1}) if $t(x_{i'+1}, x_{i'}) \leq t(x_i, x_{i+1})$. The idea in compensation is that the utility loss when we go from x_i to x_{i+1} is compensated by the utility gain when we go from $x_{i'}$ to $x_{i'+1}$. To see this interpretation, notice that for the case of a monotonically decreasing relation between time and utility differences, $t(x_{i'+1}, x_{i'}) \leq t(x_i, x_{i+1})$ implies that the utility loss when moving from x_i to x_{i+1} is weakly smaller than the utility gain when moving from $x_{i'}$ to $x_{i'+1}$. For the case of a monotonically increasing relation, $t(x_{i'+1}, x_{i'}) \geq t(x_i, x_{i+1})$ implies that the utility loss by moving from x_i to x_{i+1} is weakly smaller than the utility gain by moving from $x_{i'}$ to $x_{i'+1}$.

Definition 3. Given a cycle $(x_i)_{i=1}^n$, a decreasing (increasing) compensation π is said to be a decreasing (resp. increasing) overcompensation if (i) there exists i such that $t(\pi(x_i, x_{i+1})) <$ (resp. $>$) $t(x_i, x_{i+1})$ or (ii) there exists i' with $x_{i'+1} \succ x_{i'}$ while $(x_{i'}, x_{i'+1}) \notin \text{range } \pi$.

If π is an overcompensation, then (i) for some i , the utility loss by moving from x_i to x_{i+1} is strictly smaller than the utility gain by moving from $x_{i'}$ to $x_{i'+1}$ or (ii) there is some utility gain by moving from $x_{i'}$ to $x_{i'+1}$, which is not used to compensate utility losses. Given a cycle $(x_i)_{i=1}^n$, we must have

$$0 = \sum_{i=1}^n u(x_i) - u(x_{i+1}) = \left(\sum_{i:x_i \succ x_{i+1}} u(x_i) - u(x_{i+1}) \right) - \left(\sum_{i:x_{i+1} \succ x_i} u(x_{i+1}) - u(x_i) \right).$$

In the case of a decreasing (increasing) relation between response time and utility differences, if a decreasing (resp. increasing) compensation exists, then the second sum is larger than the first sum, so it cannot add up to zero. Therefore, there should not exist an overcompensation. This means that the following axioms are necessary for each case.

Axiom 10. (*Decreasing Compensation*) *There is no cycle with a decreasing overcompensation.*

Axiom 11. (*Increasing Compensation*) *There is no cycle with an increasing overcompensation.*

Remark. Observe that these axioms imply that one cannot have a cycle (x_i) with $x_{i+1} \succ x_i$ for all $i = 1, \dots, n-1$. The reason is that such a cycle admits the empty function as a compensation.

The main result of this section is to show that the axioms are also sufficient.

Theorem 2. *A binary relation \succ satisfies Completeness and Decreasing (Increasing) Compensation if and only if there exists a function $u : X \rightarrow \mathbf{R}$ and a strictly decreasing (resp. increasing) function $f : T \rightarrow \mathbf{R}$ such that*

$$x \succ y \Leftrightarrow u(x) > u(y), \tag{4}$$

and moreover that

$$u(x) - u(y) = f(t(x, y)). \tag{5}$$

Remark. Notice that if we do not require that f is decreasing (or increasing), then the response-time representation only requires the existence of utility representation (4) and the condition that $t(x, y) = t(z, y) \implies u(x - u(y)) = u(z) - u(y)$. Given these conditions, once we have u satisfying (4), we can define f by (5).

Proof of Theorem 2: The necessity is obvious. The proof of the sufficiency depends on the proof of Theorem 3 in the next section. In Theorem 3, we will show that Strong Decreasing (resp. Increasing) Compensation is equivalent to a response-time representation. The representation is slightly weaker than the representation in Theorem 2 in that Theorem 3 requires only $x \succ y \implies u(x) > u(y)$. However, under Completeness, this implies that $x \succ y \Leftrightarrow u(x) > u(y)$. It suffices to show that under Completeness, Decreasing (Increasing) Compensation implies Strong Decreasing (resp. Increasing) Compensation.

To show the claim, we show that if there exists a collection $\{(x_i^k)_{i=1}^{n_k} : k = 1, \dots, K\}$ of cycles with an overcompensation π , then there exists a cycle $(x_i)_{i=1}^n$ with an overcompensation $\hat{\pi}$.

We will construct a cycle $(x_i)_{i=1}^n$ as follows. Define

$$x_1 = x_1^1, \dots, x_{n_1} = x_{n_1}^1,$$

$$x_{n_1+1} = x_1^2, \dots, x_{n_1+n_2} = x_{n_2}^2, x_{n_1+n_2+1} = x_1^1.$$

Similarly, for all $k = 2, 3, \dots, K - 1$, define

$$x_{n_1+\dots+n_k+k} = x_1^{k+1}, \dots, x_{n_1+\dots+n_k+n_{k+1}+k-1} = x_{n_{k+1}}^{k+1}, x_{n_1+\dots+n_k+n_{k+1}+k} = x_1^1.$$

By Completeness, we obtain the cycle $(x_i)_{i=1}^{n_1+\dots+n_K+K-1}$.

Now we will define an overcompensation $\hat{\pi}$ by extending π . Notice that the cycle $(x_i)_{i=1}^{n_1+\dots+n_K+K-1}$ contains more edges than the collection $\{(x_i^k)_{i=1}^{n_k} : k = 1, \dots, K\}$ of cycles. Define $\hat{\pi} = \pi$ on the all pairs of worsening in the collection $\{(x_i^k)_{i=1}^{n_k} : k = 1, \dots, K\}$ of cycles. In the cycle $(x_i)_{i=1}^{n_1+\dots+n_K+K-1}$, we have new edges from x_1^1 to x_1^k and x_1^k to x_1^1 for all $k = 2, \dots, K$. By Completeness, we have either $x_1^1 \succ x_1^k$ or $x_1^k \succ x_1^1$ for

each $k = 2, \dots, K$. If $x_1^1 \succ x_1^k$, we define $\hat{\pi}(x_1^1, x_1^k) = (x_1^k, x_1^1)$; if $x_1^k \succ x_1^1$, we define $\hat{\pi}(x_1^k, x_1^1) = (x_1^1, x_1^k)$. Notice that with this definition, the conditions on the relation on the response times are satisfied with equality. Therefore, since π is an overcompensation for the original collection of cycles, then $\hat{\pi}$ is an overcompensation for the cycle. ■

Before we study incomplete (and finite) data in the next section, we mention the relationship between Decreasing (Increasing) Compensation axiom and the two main axioms, Decreasing (resp. Increasing) Monotonicity and Time-Transitivity, in the previous sections.

Remark: *Decreasing (Increasing) Compensation implies Decreasing (resp. Increasing) Monotonicity and Time-Transitivity.*

Here, we will show that Decreasing Compensation implies Decreasing Monotonicity. In the appendix, we show that Decreasing Compensation implies Time-Transitivity. Assume that Decreasing Compensation holds. To show Decreasing Monotonicity, suppose by contradiction that $x \succ y, y \succ z, x \succ z$ and $t(x, z) \geq \min\{t(x, y), t(y, z)\}$. Consider the case, $t(x, z) \geq t(x, y)$. Consider a cycle (z, y, x, z) . Since $z \prec y \prec x \succ z$, there is only one worsening (x, z) . (Here, we are abusing notation \prec . For any $x, y \in X$, by $x \prec y$, we mean $y \succ x$. We will abuse the notation in the following.) So we can define a one-to-one function π by $\pi(x, z) = (x, y)$. Since $t(x, z) \geq t(x, y)$, π is a compensation function and not onto, which contradicts Decreasing Compensation. The case of $t(x, z) \geq t(y, z)$ can be proved in the same way.³

³Consider a cycle (y, x, z, y) . Since $y \prec x \succ z \prec y$, in the cycle, there is only one worsening (x, z) . So we can define a one-to-one function π by $\pi(x, z) = (z, y)$. Since $t(x, z) \geq t(y, z)$, π is a compensation function and not onto, which contradicts Decreasing Compensation.

4.2 Axioms for Finite and Incomplete Data

In this section, we do not assume Completeness. We will show that a stronger version of Decreasing (Increasing) Compensation characterize the response-time representation. The axiom is stronger only because we consider a collection of cycles instead of one cycle.

Definition 4. A set

$$\{(x_i^k)_{i=1}^{n_k} : k = 1, \dots, K\}$$

in which $x_1^k = x_{n_k}^k$ and for all $i = 1, \dots, n_k - 1$ either $x_i^k \succ x_{i+1}^k$ or $x_{i+1}^k \succ x_i^k$ is called a collection of cycles ⁴.

We need to modify the concept of compensation accordingly:

Definition 5. Given a collection $\{(x_i^k)_{i=1}^{n_k} : k = 1, \dots, K\}$ of cycles, a decreasing (increasing) compensation is a one-to-one function π that maps any pair (x_i^k, x_{i+1}^k) with $x_i^k \succ x_{i+1}^k$ into some pair $(x_{i'}^{k'}, x_{i'+1}^{k'}) = \pi(x_i^k, x_{i+1}^k)$ with $x_{i'}^{k'} \succ x_{i'+1}^{k'}$ and $t(x_{i'+1}^{k'}, x_{i'}^{k'}) \leq$ (resp. \geq) $t(x_i^k, x_{i+1}^k)$.

Notice that the above definition is different from the one in the previous section only because we do not require $k = k'$. In other words, we allow that a pair of worsening in a cycle can be compensated by a pair of improvement in another cycle.

Definition 6. Given a collection $\{(x_i^k)_{i=1}^{n_k} : k = 1, \dots, K\}$ of cycles, a decreasing (increasing) compensation π is said to be a decreasing (resp. increasing) overcompensation if (i) there exist k and i such that $t(\pi(x_i^k, x_{i+1}^k)) <$ (resp. $>$) $t(x_i^k, x_{i+1}^k)$ or (ii) there exists k' and i' with $x_{i'+1}^{k'} \succ x_{i'}^{k'}$ and $(x_{i'}^{k'}, x_{i'+1}^{k'}) \notin \text{range } \pi$.

Given a collection $\{(x_i^k)_{i=1}^{n_k} : k = 1, \dots, K\}$ of cycles, if we sum up the utility differences for each k , it must add up to zero (i.e., $\sum_{i=1}^{n_k} u(x_i^k) - u(x_{i+1}^k) = 0$, where $n_k + 1 = 1$). Hence, if we sum up the sum over k , it must add up to zero (i.e., $\sum_k \sum_{i=1}^{n_k} u(x_i^k) - u(x_{i+1}^k) = 0$,

⁴Sums are mod n_k .

where $n^k + 1 = 1$ for each k). Therefore, by a similar argument in the previous section, there should not exist an overcompensation for the collection of cycles. This means that the following axioms are necessary for each case.

Axiom 12. (*Strong Decreasing Compensation*) *There exists no collection of cycles with a decreasing overcompensation.*

Axiom 13. (*Strong Increasing Compensation*) *There exists no collection of cycles with an increasing overcompensation.*

The main result in this section shows that the above axioms are also sufficient.

Theorem 3. *\succ satisfies Strong Decreasing (Increasing) Compensation if and only if there exists a function $u : X \rightarrow \mathbf{R}$ and a strictly decreasing (resp. increasing) function $f : T \rightarrow \mathbf{R}$ such that*

$$x \succ y \implies u(x) > u(y), \tag{6}$$

and moreover that

$$u(x) - u(y) = f(t(x, y)). \tag{7}$$

Remark. Since typical experimental data are finite and often incomplete, the results in the sections are useful to test such data. Moreover, Strong Decreasing Compensation and Strong Increasing Compensation can be checked computationally: in fact there is an efficient algorithm to decide whether a dataset satisfies Strong Decreasing Compensation and Strong Increasing Compensation. This fact is easy to see from the proof of the theorem.

Remark. In the following, we prove the case of decreasing monotonic relationship. Strong Decreasing Compensation holds if and only if there exists a solution to a system of inequalities, and there are efficient algorithms to decide whether a system of linear inequalities has a solution.

In the definitions (8) and (9) in the proof below, we define matrices A, B, E from the data and then we show that a response-time representation exists if and only if there is a solution u of the following system of equations and linear inequalities

$$A \cdot u = 0, \quad (B, E) \cdot u \gg 0.$$

The main part of the proof is to show the existence of a solution is equivalent to Strong Decreasing Compensation.

Proof of Theorem 3: We show the proof for case of strictly decreasing f . Remember that T is the set of $t \in \mathbf{R}_+$ for which there is $x, y \in X$ such that $(x, y, t) \in \succ$. So we write $t(x, y)$ to mean the $t \in \mathbf{R}$ such that $(x, y, t) \in \succ$.

Define first a matrix A which has $|X| + T$ columns in $|\succ|$ rows. We arrange T columns in a way that the largest t appears at the last column. (This arrangement is made to capture the implication $x \succ y \implies u(x) > u(y)$. This will be clear later.) For each triple $(x, y, t) \in \succ$ there is a row. In the row corresponding to (x, y, t) there are zeros in every entry except for a 1 in the row for x , a -1 in the row for y , and a -1 in the row for t . A looks as follows

$$(x,y,t) \in \succ \begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \quad (8)$$

Consider the system $A \cdot u = 0$. If there are numbers solving the right hand side of equation (7), then these define a solution $u \in \mathbf{R}^{|X|+|T|}$. If, on the other hand, there is a solution $u \in \mathbf{R}^{|X|+|T|}$ to the system $A \cdot u = 0$, then the vector u defines a solution to (7).

Define the matrix B as a matrix with $|X| + |T|$ columns and one row for every pair (t, t') with $t < t'$. In the row corresponding to (t, t') with $t < t'$ there are zeroes in every entry except for a 1 in the column for t and a -1 in the column for t' . In symbols, the

row corresponding to (t, t') is $1_t - 1_{t'}$. B looks as follows

$$\begin{array}{c}
\vdots \\
t < t' \\
\vdots
\end{array}
\left[
\begin{array}{cccccc|cccccc}
\cdots & x & \cdots & y & \cdots & & \min T & \cdots & t & \cdots & t' & \cdots & \max T \\
\vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 & \cdots & -1 & \cdots & 0 \\
\vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots
\end{array}
\right] \quad (9)$$

Then, the system $B \cdot u \gg 0$ captures the requirement that f be strictly decreasing.

In third place, we have a matrix E that captures the requirement that $f(\max T) > 0$. The matrix E has a single row and $|X| + T$ columns. It has zeroes everywhere except for 1 in the last column. Note that since f is strictly decreasing and $f(\max T) > 0$, it follows that $f > 0$. Therefore by equation (7), we have $x \succ y \implies u(x) > u(y)$.

To sum up, there is a solution to system (7) and (6) if and only if there is a vector $u \in \mathbf{R}^{|X|+|T|}$ that solves the following system of equations and linear inequalities

$$(S1) : A \cdot u = 0, (B, E) \cdot u \gg 0.$$

The entries of A , B , and E are either 0, 1 or -1 . By Lemma 3, then, there is such a solution u to $S1$ if and only if there is no rational vector (θ, η, λ) that solves the system of equations and linear inequalities as follows:⁵

$$(S2) : \theta \cdot A + \eta \cdot B + \lambda \cdot E = 0, (\eta, \lambda) > 0,$$

where $(\eta, \lambda) > 0$ means that $\eta > 0, \lambda \geq 0, \eta \geq 0, \lambda > 0$, or $\eta > 0, \lambda > 0$.

Suppose that there is no solution u of $S1$. Then, there exists a rational vector (θ, η, λ) that solves the above system of equations and linear inequalities. By multiplying (θ, η, λ) by any positive integer we obtain new vectors that solve $S2$, so we can take (θ, η, λ) to be integer vectors. We transform the matrices A using θ and η . Define a matrix A^* from A by letting A^* have the same number of columns as A and including: (i) θ_r copies of the r th row when $\theta_r > 0$; (ii) omitting row r when $\theta_r = 0$; and (iii) $|\theta_r|$ copies of the r th row

⁵We apply Lemma 3 by considering B in the lemma as an empty matrix.

multiplied by -1 when $\theta_r < 0$. We refer to rows that are copies of some r with $\theta_r > 0$ as *original* rows, and to those that are copies of some r with $\theta_r < 0$ as *reversed* rows.

Similarly, we define the matrix B^* from B by including the same columns as B and η_r copies of each row (and thus omitting row r when $\eta_r = 0$; recall that $\eta_r \geq 0$ for all r).

Henceforth, we use the following notational convention: For a matrix D with $|X| + |T|$ columns, write D_1 for the submatrix of D corresponding to the first $|X|$ columns; let D_2 be the last $|T|$ column. Thus, $D = [D_1 | D_2]$.

Then, $1 \cdot A_i^* + 1 \cdot B_i^* + \lambda E_i = 0$ for each $i = 1, 2$. Moreover, $B_1^* = 0 = E_1$. Hence, we obtain $A_1^* = 0$ and $1 \cdot A_2^* + 1 \cdot B_2^* = -\lambda E_2 \leq 0$. The rows of B^* are vectors in $\{-1, 0, 1\}^{|X|+|T|}$ that we can write as $1_t - 1_{t'}$ with $t < t'$.

We can assume without loss of generality that if there is a row associated with (t, t') , then there is no row associated with (t'', t) (i.e. $t'' < t$) or with (t', t'') (i.e., $t' < t''$). The reason that this is without loss is the following. Say that there are two rows associated with (t, t') and (t', t'') . Then the sum of the rows will give

$$(1_t - 1_{t'}) + (1_{t'} - 1_{t''}) = 1_t - 1_{t''}.$$

Then the matrix B^* can be replaced with a matrix which omits the rows for (t, t') and (t', t'') and includes instead a row for (t, t'') while preserving the property that $1 \cdot A_2^* + 1 \cdot B_2^* = -\lambda E_2 \leq 0$.

The matrix A_1^* defines a graph (X, F) in which F is a multiset. The matrix A^* is the incidence matrix of the graph (X, F) ; note that there is an edge (x, y) in this graph only if $x \succ y$ or $y \succ x$. By the Poincarè-Veblen-Alexandre theorem, since the sum of the rows of A^* is zero, the graph can be decomposed into a collection of cycles C^1, \dots, C^K .

Each cycle C^k is a sequence $(x_i^k)_{i=1}^{n_k}$ such that (i) $x_{n_k}^k = x_1^k$, (ii) each pair (x_l^k, x_{l+1}^k) corresponds to a row $r(x_l^k, x_{l+1}^k)$ of A^* , and (iii) in the row $r(x_l^k, x_{l+1}^k)$, we have -1 at the column for x_l^k and 1 at the column for x_{l+1}^k . Notice that the function r is one-to-one and onto between the edges in F and the rows of A^* . Also, if the row $r(x_l^k, x_{l+1}^k)$ is a reversed row, then $x_l^k \succ x_{l+1}^k$ because in A , we have 1 at the column for x_l^k and -1 at the column

for x_{l+1}^k . Similarly, if $r(x_l^k, x_{l+1}^k)$ is an original row, then $x_{l+1}^k \succ x_l^k$ because in A , we have -1 at the column for x_l^k and 1 at the column for x_{l+1}^k .

The rows of A are identified with a triple $(x, y, t) \in \succ$. For each (x, y) there is at most one row associated to $(x, y, t(x, y))$. So we identify each row r with the corresponding pair (x, y) , and write $t(r)$ for $t(x, y)$.

Suppose first that there are no reversed rows in A^* . Then, it is a violation of Strong Decreasing Compensation because there exists a cycle (x_i) with $x_{i+1} \succ x_i$ for all i (with addition mod n).

Lemma 1. *For each $k \in K$, for each reversed row $r(x_l^k, x_{l+1}^k)$ of C^k , there is an original row $\pi(r(x_l^k, x_{l+1}^k))$ of A^* with $t(\pi(r(x_l^k, x_{l+1}^k))) \leq t(r(x_l^k, x_{l+1}^k))$. The function π is one-to-one.*

Proof. If the row $r(x_l^k, x_{l+1}^k)$ in A^* is reversed, then in the column for $t^* = t(x_l^k, x_{l+1}^k)$ there is a 1 in row $r(x_l^k, x_{l+1}^k)$. Then, $1 \cdot A_2^* + 1 \cdot B_2^* \leq 0$ means that in the column for t^* there is some row ρ of A^* or of B^* in which the entry in the column for t^* is -1 . There are two cases to consider.

Case 1: Firstly, if row ρ is in A^* , then ρ must be an original row because its entry in column t^* is -1 . Let $\pi(r(x_l^k, x_{l+1}^k))$ be equal to ρ . We have

$$t(\pi(r(x_l^k, x_{l+1}^k))) = t^* = t(r(x_l^k, x_{l+1}^k)).$$

Case 2: Secondly, if row ρ is in B^* then ρ corresponds to some pair (t, t^*) with $t < t^*$. The reason is that there is a -1 in row ρ and column t^* , and hence $\rho = 1_t - 1_{t^*}$ for some $t < t^*$. Moreover, by the assumption, there is no row that corresponds to some pair (t', t) with $t' < t$. Therefore there is a 1 in the column for t and row ρ in B^* , and no -1 in the column for t in B^* . Then, $A_2^* + B_2^* < 0$ implies that there is a row $\hat{\rho}$ in A^* for which the column for t has a -1 . The row $\hat{\rho}$ must then be original. Let $\pi(r(x_l^k, x_{l+1}^k)) = \hat{\rho}$. Note that $t = t(\hat{\rho}) < t^*$. We have

$$t(\pi(r(x_l^k, x_{l+1}^k))) < t^* = t(r(x_l^k, x_{l+1}^k)). \quad (10)$$

In this way, we have assigned $\pi(r(x_l^k, x_{l+1}^k))$ to $r(x_l^k, x_{l+1}^k)$. In order to proceed, we define $A^*(1)$ and $B^*(1)$ by deleting the rows used above from A^* and B^* . In particular, in Case 1, we delete $r(x_l^k, x_{l+1}^k)$ and $\pi(r(x_l^k, x_{l+1}^k))$ from A^* to define $A^*(1)$. We define $B^*(1) = B^*$. In Case 2, we delete $r(x_l^k, x_{l+1}^k)$ and $\pi(r(x_l^k, x_{l+1}^k))$ from A^* in order to define $A^*(1)$. From B^* , we delete the row ρ which corresponds to the pair (t, t^*) in order to define $B^*(1)$. Note that in either case, the sum of deleted rows in the last $|T|$ columns is zero. Since $1 \cdot A_2^* + 1 \cdot B_2^* \leq 0$, it follows that $1 \cdot A_2^*(1) + 1 \cdot B_2^*(1) \leq 0$.

Now with $A^*(1)$ and $B^*(1)$, we proceed to define the function π : we choose a reversed row from $A^*(1)$ and assign an original row exactly in the same way above. After deleting the corresponding rows, we obtain matrices $A^*(2)$ and $B^*(2)$ by the same procedure that produced $A^*(1)$ and $B^*(1)$. We keep this construction until we exhaust all rows in A^* . This ensures that the function π is one-to-one because when a row is selected to be the image of π it is deleted from the corresponding matrix, and unavailable for assignment in the rest of the process. The process must end after a finite number of steps, say n , because the number of rows in A^* is finite. \square

Suppose that the process of the deletion at the end of the proof of Lemma 1 ends after n steps. Now consider matrices $A^*(n)$ and $B^*(n)$. To finish the proof, it suffices to show the following lemma because it will exhibit an overcompensation.

Lemma 2. (i) π is not onto; or (ii) there exists a reversed row $r(x_l^k, x_{l+1}^k)$ such that $t(\pi(r(x_l^k, x_{l+1}^k))) < t(r(x_l^k, x_{l+1}^k))$.

Proof. There are two cases to consider. The first case is when $\lambda > 0$; the second case is when $\lambda = 0$.

Case 1: Consider the case $\lambda > 0$. Since $1 \cdot A_2^*(n) + 1 \cdot B_2^*(n) + \lambda E_2 = 0$ and the last column of E_2 is 1, there must be a row ρ in $A_2^*(n)$ or in $B_2^*(n)$ which has -1 at the last column.

Subcase 1.1: ρ is in $A_2^*(n)$. Then the row is an original row. Since the row is in $A_2^*(n)$ (and not deleted), $\rho \notin \text{range } \pi$. This completes the proof.

Subcase 1.2: ρ is in $B_2^*(n)$. Then ρ corresponds to some pair $(t, \max T)$ with $t < \max T$. The reason is that there is a -1 in row ρ and column $\max T$, and hence $\rho = 1_t - 1_{\max T}$ for some $t < \max T$. Moreover, by the assumption, there is no row that corresponds to some pair (t', t) with $t' < t$. Therefore, there is a 1 in the column for t and row ρ in $B^*(n)$, and no -1 in the column for t in $B^*(n)$. Then, $A_2^*(n) + B_2^*(n) < 0$ implies that there is a row $\hat{\rho}$ in $A^*(n)$ for which the column for t has a -1 . The row $\hat{\rho}$ must then be original. Since the row is in $A_2^*(n)$ (and not deleted), $\hat{\rho} \notin \text{range } \pi$. So, π is not onto. This completes the proof for the first case.

Case 2: Consider the case $\lambda = 0$. Since $(\eta, \lambda) > 0$, then $\eta > 0$. So B^* is not zero. there must be a row in B_2^* in which there exists 1 in a column and there is no -1 at the column because the assumption on B . Since $1 \cdot A_2^* + 1 \cdot B_2^* \leq 0$, the 1 in B_2^* must be canceled out some -1 in A^* . Therefore, Case 2 must exist in Lemma 1. Therefore, (10) holds. So there exists a reversed row $r(x_l^k, x_{l+1}^k)$ such that $t(\pi(r(x_l^k, x_{l+1}^k))) < t(r(x_l^k, x_{l+1}^k))$. \square

To finish the proof of Theorem 3, in the collection $\{(x_i^k)_{i=1}^{n_k}\}_{k=1}^K$ of cycles, remember that if the row $r(x_l^k, x_{l+1}^k)$ is a reversed row, then $x_l^k \succ x_{l+1}^k$; and if $r(x_l^k, x_{l+1}^k)$ is an original row, then $x_{l+1}^k \succ x_l^k$. So the function π is a function from a pair of worsening to a pair of improvement, as desired. By Lemma 1, π is a compensation function. Moreover, Lemma 2 shows that π is an overcompensation. Thus we obtain a violation of Strong Decreasing Compensation.

The proof for the case of increasing f is almost identical. We need to change the definition (9) of B and the definition of E . In B , we need to flip 1 and -1 , which makes f strictly increasing. In E , we put 1 in the column of $\min T$ instead of $\max T$, which gives us $f(\min T) > 0$. Together with strictly increasingness of f , we have $f > 0$.

In Lemma 1, we need to flip the inequality as $t(\pi(r(x_l^k, x_{l+1}^k))) \geq t(r(x_l^k, x_{l+1}^k))$. In Lemma 2, we need to flip the inequality as $t(\pi(r(x_l^k, x_{l+1}^k))) > t(r(x_l^k, x_{l+1}^k))$. \blacksquare

References

- CHABRIS, C. F., C. L. MORRIS, D. TAUBINSKY, D. LAIBSON, AND J. P. SCHULDT (2009): “The allocation of time in decision-making,” *Journal of the European Economic Association*, 7, 628–637.
- CHAMBERS, C. P. AND F. ECHENIQUE (2011): “On the consistency of data with bargaining theories,” Forthcoming, *Theoretical Economics*.
- CLITHERO, J. A. AND A. RANGEL (2013): “Combining Response Times and Choice Data Using a Neuroeconomic Model of the Decision Process Improves Out-of-Sample Predictions,” *unpublished, California Institute of Technology*.
- DEBREU, G. (1958): “Stochastic choice and cardinal utility,” *Econometrica: Journal of the Econometric Society*, 440–444.
- FUDENBERG, D., P. STRACK, AND T. STRZALECKI (2015): “Stochastic choice and optimal sequential sampling,” *working paper*.
- GABAIX, X., D. LAIBSON, G. MOLOCHE, AND S. WEINBERG (2006): “Costly information acquisition: Experimental analysis of a boundedly rational model,” *The American Economic Review*, 1043–1068.
- KOHFELD, D. L. (1971): “Simple reaction time as a function of stimulus intensity in decibels of light and sound.” *Journal of experimental psychology*, 88, 251.
- KOHFELD, D. L., J. L. SANTEE, AND N. D. WALLACE (1981): “Loudness and reaction time: I,” *Attention, Perception, & Psychophysics*, 29, 535–549.
- KRAJBICH, I., B. OUD, AND E. FEHR (2014): “Benefits of Neuroeconomic Modeling: New Policy Interventions and Predictors of Preference,” *The American Economic Review*, 104, 501–506.

- KRANTZ, D., D. LUCE, P. SUPPES, AND A. TVERSKY (1971): *Foundations of measurement, Vol. I: Additive and polynomial representations*, Academic press.
- LUCE, R. D. (1986): *Response times*, 8, Oxford University Press.
- MANSFIELD, R. (1973): “Latency functions in human vision,” *Vision research*, 13, 2219–2234.
- PIOVESAN, M. AND E. WENGSTRÖM (2009): “Fast or fair? A study of response times,” *Economics Letters*, 105, 193–196.
- RATCLIFF, R. (1978): “A theory of memory retrieval.” *Psychological review*, 85, 59.
- RUBINSTEIN, A. (2007): “Instinctive and cognitive reasoning: A study of response times,” *The Economic Journal*, 117, 1243–1259.
- (2013): “Response time and decision making: An experimental study,” *Judgment and Decision Making*, 8, 540–551.
- SCHOTTER, A. AND I. TREVINO (2014): “Is response time predictive of choice? An experimental study of threshold strategies,” .
- SCOTT, D. (1964): “Measurement structures and linear inequalities,” *Journal of mathematical psychology*, 1, 233–247.
- WILCOX, N. T. (1993): “Lottery choice: Incentives, complexity and decision time,” *The Economic Journal*, 1397–1417.

A Appendix: Theorem of the Alternative

We make use of the following version of the theorem of the alternative. See Lemma 12 in Chambers and Echenique (2011). It has solutions over the real field in the primal system, and over the rational field in the dual.

Lemma 3. *Let A be an $m \times n$ matrix, B be an $l \times n$ matrix, and E be an $r \times n$ matrix. Suppose that the entries of the matrices A , B , and E are rational numbers. Exactly one of the following alternatives is true.*

1. *There is $u \in \mathbf{R}^n$ such that $A \cdot u = 0$, $B \cdot u \geq 0$, and $E \cdot u \gg 0$.*
2. *There is $\theta \in \mathbf{Q}^m$, $\eta \in \mathbf{Q}^l$, and $\pi \in \mathbf{Q}^r$ such that $\theta \cdot A + \eta \cdot B + \pi \cdot E = 0$; $\pi > 0$ and $\eta \geq 0$,*

where $\pi > 0$ means that all of the coordinates are nonnegative and some coordinates are positive.

B Appendix: Theorem 1 in Krantz et al. (1971) (p.147)

Theorem 4. *Let A be a nonempty set, A^* be a nonempty subset of $A \times A$, and \succsim be a binary relation on A^* . Suppose that*

- *\succsim is complete and transitive,*
- *If $ab, bc \in A^*$, then $ac \in A^*$,*
- *If $ab, bc \in A^*$, then $ac \succ ab, ac \succ bc$,*
- *If $ab, bc, a'b', b'c' \in A^*$ $ab \succsim a'b'$, and $bc \succsim b'c'$, then $ac \succsim a'c'$,*
- *If $ab, cd \in A^*$ and $ab \succ cd$, there exist $d', d'' \in A$ such that $ad' \sim cd \sim d''b$,*
- *For any sequence $\{a_i\}$, if $a_{i+1}a_i \sim a_2a_1$ for all i and there exist $d'd'' \succ a_i a_1$ for all i , then the sequence is finite.*

Then, there exists $\psi : A^ \rightarrow \mathbf{R}_{++}$ such that for all $a, b, c, d \in A$*

(i) if $ab, cd \in A^$, then*

$$ab \succsim cd \Leftrightarrow \psi(ab) \geq \psi(cd).$$

(ii) if $ab, bc \in A^*$, then

$$\psi(ac) = \psi(ab) + \psi(bc).$$

If ψ' also has these properties, then there exists a positive number α such that $\psi = \alpha\psi'$. If, in addition, for all $a, b \in A$, $a \neq b$, either $ab \in A^*$ or $ba \in A^*$, then there exists $\phi : A \rightarrow \mathbf{R}$ such that for all $ab \in A^*$, $\psi(ab) = \phi(a) - \phi(b)$. If ϕ' satisfies the same property, then there exists a real number β such that $\phi = \phi' + \beta$.

C Appendix: Compensation implies Time Transitivity

Assume Decreasing Compensation to show Time-Transitivity. Suppose by contradiction that $t(x, y) \leq t(x', y')$, $t(y, z) \leq t(y', z')$, and $t(x, z) > t(x', z')$. We have $x \succ z \prec y \prec x$ and $x' \succ y' \prec z' \prec x'$. There are two cases to consider $x \succ x'$ and $x' \succ x$. First we consider the case where $x \succ x'$. Consider a cycle:

$$x' \succ y' \succ z' \prec x' \prec x \succ z \prec y \prec x \succ x'.$$

We can define a one to one function π as follows: $\pi(x', y') = (y, x)$, $\pi(y', z') = (z, y)$, $\pi(x, z) = (z', x')$, and $\pi(x, x') = (x', x)$. This function is a compensation function. Since $t(x, y) \leq t(x', y')$, $t(y, z) \leq t(y', z')$, and $t(x, z) > t(x', z')$, this contradicts Decreasing Compensation. The other case can be proved in the same way. \blacksquare

In the same way, Increasing Compensation implies Time Transitivity.

⁶Consider a cycle $z \prec y \prec x \prec x' \succ y' \succ z' \prec x' \succ x \succ z$. We can define a one to one function π as follows: $\pi(x', y') = (y, x)$, $\pi(y', z') = (z, y)$, $\pi(x, z) = (z', x')$, and $\pi(x', x) = (x, x')$. This function is a compensation function. Since $t(x, y) \leq t(x', y')$, $t(y, z) \leq t(y', z')$, and $t(x, z) > t(x', z')$, which is a contradiction to Decreasing Compensation.