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A Solution to Matching with Preferences over Colleagues [1]

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Abstract

We study many-to-one matchings, such as the assignment of students to colleges, where the students have preferences over the other students who would attend the same college. It is well known that the core of this model may be empty, without strong assumptions on agents' preferences. We introduce a method that finds all core matchings, if any exist. The method requires no assumptions on preferences. Our method also finds certain partial solutions that may be useful when the core is empty.
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1 Introduction

The many-to-one matching model is a commonly-used model of how workers are assigned to firms, or how students are assigned to schools. The model assumes that students do not care who the other students matched to the same school are. This assumption seems problematic for two reasons. First, it is crucial to obtaining the results in the literature: it is widely recognized that the results break down without it. Second, while crucial, the assumption is unlikely to hold in some important applications. Indeed, in many labor markets (such as the academic market) the set of colleagues is an important consideration in choosing whom to work for. In school choice, it seems that students, and their parents, care primordially about colleagues.

In this paper, we study the matching model when students do care about who else goes to the same school. Our approach is not (mainly) to obtain a general structure on preferences that will guarantee existence of some solution to the model. Instead, we propose an algorithm that will find the solutions *if* they exist.

Our approach is motivated by a certain pessimism. It seems that general conditions for nonemptiness of the core are difficult to obtain, and that the few that are known are very strong. We choose then to be agnostic about the emptiness of the core; we present an algorithm that works without any structure on preferences and that finds the core when it exists.

A second motivation is that, in practical problems, where one needs to devise a centralized matching procedure, it is often difficult to verify that agents' preferences satisfy this or that property. We believe our algorithm will then be useful, as it is guaranteed to work for any preferences.

Our main results hold without any structure on agents' preferences, but we study the behavior of our algorithm under some restrictions on preferences that will ensure a nonempty core. Under these restrictions the algorithm is efficient. The algorithm also identifies certain partial solutions which may be useful when the core is empty. In a partial solution that we call "simple matchings," the agents who are matched are matched in a stable way, and blocks can only involve agents who are single. Simple matchings were introduced by Marilda Sotomayor in a sequence of recent papers (Sotomayor, 2005a,b,c).

In the rest of the Introduction, we relate this paper to the existing literature.

Nearly all publications on the many-to-one model rule out that a student may care about who her colleagues are. This is true of the seminal papers (e.g. Gale and Shapley (1962), Kelso and Crawford (1982), Roth (1982), Blair (1988)) as well as of the exposition of the theory in Roth and Sotomayor (1990).

The only exceptions are the papers by Dutta and Massó (1997) and Revilla (2004); they present some strong conditions under which the core will be nonempty. Dutta and Massó essentially study lexicographic preferences: the students either first care about the college, then about their colleagues, in which case the core is nonempty; or they care first about their colleagues and then about the college, in which case they need additional assumptions for the core to be nonempty. Revilla generalizes Dutta and Massó's results for couples (see below) to more general preferences over colleagues. He proves that the core is nonempty under certain hypotheses that include a weakened version of lexicographic preferences.

Our paper is also related to the literature on matching with couples (Roth (1984), (Roth and Sotomayor, 1990, page 140), Klaus and Klijn (2005)), and Dutta and Massó (1997). This literature is mainly motivated by the role of married couples in medical-interns matching. In Roth (1984) and Klaus and Klijn (2005), each member in a couple cares about the school choice of his/her partner. But the model is different from ours because couples make a joint decision, and care about the school choice of a partner, even among colleges that they do not both attend. Dutta and Massó's (1997) model of couples is in the spirit of our model of preferences regarding colleagues. In Section 9 we present an extension of our model to the model with couples in Dutta and Massó; our algorithm can thus be used to find all the core matchings in that model.

For our model of matching with couples, we give a solution to Open Problem 4 in Roth and Sotomayor (1990): when does the pairwise stable set coincide with the core.

Ours is essentially a model of hedonic coalition formation (see e.g. Greenberg (1994), Banerjee et al. (2001) or Bogomolnaia and Jackson (2002)). We are able to adapt some preference restrictions from the coalition-formation literature and use them in our approach. We also note that the method presented here should be easily applicable to the study of stability in more general coalition-formation models.

Finally, we should mention the literature on finding all core matchings- see Gusfield and Irving (1989) for an exposition. The recent paper by Martínez et al. (2004) presents an algorithm for the many-to-many case. These papers assume an absence of preferences over colleagues.

We present our model in Section 2 and give a statement of the problem and outline of our solution. We translate finding the core into a fixed-point problem in Sections 3 and 4. In Section 5 we present the algorithm and discuss partial solutions in Section 7. In Section 8 we restrict preferences to obtain the existence of core matchings. In Section 9 we develop a model with couples.

2 Statement of the Problem

We state the problem by first specifying a model of matchings with preferences over colleagues and defining the notion of the core. We then outline the difficulties created by preferences over colleagues, and sketch our main contributions.

2.1 The Model

There are two disjoint sets of agents, the set of n colleges, C, and the set of m students, S. Each college c has a strict, transitive, and complete preference $P(c)$ over 2^S. Each student s has a strict, transitive, and complete preference $P(s)$ over $C \times S_s \cup \{(\emptyset, \emptyset)\}$; where S_s is the set of subsets of S which contain s. A preference profile is a collection of preference relations for all colleges and students- that is, an $(n+m)$-tuple $P = (P(c_1), ..., P(c_n), P(s_1), ..., P(s_m))$. A college admissions problem is a 3-tuple $\langle C, S, P \rangle$.

A *matching* μ is a mapping defined on the set $C \cup S$ which satisfies for all $c \in C$ and $s \in S$:

(1) $\mu(s) \in C \times S_s \cup \{(\emptyset, \emptyset)\}$.
(2) $\mu(c) \in 2^S$.
(3) If $s \in \mu(c)$ then $\mu(s) = (c, \mu(c))$.
(4) If $\mu(s) = (c, S')$ for some college c then $\mu(c) = S'$.

Here, $\mu(s) = (\emptyset, \emptyset)$ means that s is not matched to any college. Similarly, if $\mu(c) = \emptyset$ then there are no students matched to college c.

Notation. Given a preference relation of a college c, $P(c)$, and a group of students S', let $Ch(S', P(c))$ denote the choice set of S' according to $P(c)$; that is, for every $A \subseteq S'$ we have $Ch(S', P(c))R(c)A$. Since $P(c)$ is strict, $Ch(S', P(c))$ is well-defined.

A matching μ is *individually rational* if $\mu(s)R(s)(\emptyset, \emptyset)$ for all students s and $\mu(c) = Ch(\mu(c), P(c))$ for all colleges c.

A triple $\langle C', S', \mu' \rangle$, where $C' \subseteq C$, $S' \subseteq S$ and μ' is a matching, is a *block* of μ if the following hold:

(1) $C' \cup S' \neq \emptyset$; at least one agent is involved.
(2) For all $c \in C'$ and $s \in S'$, $\mu'(c) \in 2^{S'}$ and $\mu'(s) \in C' \times S'_s \cup \{(\emptyset, \emptyset)\}$; the agents in $C' \cup S'$ can implement μ' without outside help.

(3) For all $f \in C' \cup S'$, $\mu'(f)R(f)\mu(f)$; all agents in $C' \cup S'$ are weakly better of.
(4) There exists $f \in C' \cup S'$ such that $\mu'(f)P(f)\mu(f)$; at least one agent is strictly better off.

The *core* is the set of matchings for which there is no block, denoted by $C_W(P)$.

Note that this definition of the core is usually called the weak core (Roth and Sotomayor, 1990). In the appendix we discuss the relation of the core to other solution concepts, such as the pairwise-stable set.

2.2 Problem created by preferences over colleagues.

It is well-known among specialists in matching that preferences over colleagues creates problems for core existence. We illustrate the problems by an example. The example has an empty core, and nothing obviously pathological—for instance, colleges' preferences satisfy "substitutability," the structure on preferences that is known to guarantee a non-empty core in standard many-to-one matching problems (Kelso and Crawford (1982) first proved this in a matching model with wages). See Section 3.3 for a more substantive explanation of the source of problems.

Example 2.1 *Consider two colleges c_1, c_2 and three students s_1, s_2, s_3 with the following preferences:*

$$P(c_1) : \{s_1, s_2\}, \{s_1, s_3\}, \{s_1\}, \{s_2\}, \{s_3\}$$
$$P(c_2) : \{s_2, s_3\}, \{s_3\}, \{s_2\}$$
$$P(s_1) : (c_1, \{s_1, s_2\}), (c_1, \{s_1, s_3\}), (c_1, \{s_1\})$$
$$P(s_2) : (c_2, \{s_2, s_3\}), (c_1, \{s_1, s_2\}), (c_1, \{s_2\}), (c_2, \{s_2\})$$
$$P(s_3) : (c_1, \{s_1, s_3\}), (c_2, \{s_2, s_3\}), (c_2, \{s_3\}).$$

This notation means that c_1 prefers $\{s_1, s_2\}$ to $\{s_1, s_3\}$, $\{s_1, s_3\}$ to $\{s_1\}$, and so on. The potential groups of students not listed are worse for c_1 than being single.

It is easy to check that, in an individually-rational matching, every student is matched to a college. There are three such matchings:

$$\mu_1: \begin{array}{cc} c_1 & c_2 \\ s_1 s_2 & s_3 \end{array} \qquad \mu_2: \begin{array}{cc} c_1 & c_2 \\ s_1 s_3 & s_2 \end{array} \qquad \mu_3: \begin{array}{cc} c_1 & c_2 \\ s_1 & s_2 s_3 \end{array}$$

Now, μ_1 is blocked by $\langle\{c_2\},\{s_2,s_3\},\mu_3\rangle$, μ_2 is blocked by $\langle\{c_1\},\{s_1,s_2\},\mu_1\rangle$, and μ_3 is blocked by $\langle\{c_1\},\{s_1,s_3\},\mu_2\rangle$.

2.3 Outline of our solution.

We show that the core matchings coincide with the fixed points of a certain function T. Motivated by the discussion above, we do not then impose a structure on preferences that will let us prove the existence of fixed points. Instead, we present an algorithm that finds fixed points of T^2, the composition of T with itself. The fixed points of T, and thus the core matchings, are also fixed points of T^2. Our algorithm may not find all the fixed points of T^2, but it will find all the fixed points of T—or report that the core is empty if that is the case. Hence we have an algorithm that finds all the matchings in the core, when it is nonempty.

When the core is empty, our algorithm identifies matchings where a subset of the agents are matched in a stable way—their assignments will not be blocked. Other agents are left single in these matchings, and they may block their assignments.

We present some structure on preferences that will guarantee that the core is nonempty, and that our algorithm will find the core quickly.

3 The Core as a set of fixed points

We present a construction that allows us to characterize the core as the fixed points of a certain function. This type of construction has been used in the matching literature before, see Adachi (2000), Echenique and Oviedo (2004), Fleiner (2003), Echenique and Oviedo (2006), Hatfield and Milgrom (2005), and Ostrovsky (2005).

A *prematching* is a mapping ν, defined on the set $C \cup S$, which satisfies, for all $c \in C$ and $s \in S$,

(1) $\nu(s) \in C \times S_s \cup \{(\emptyset,\emptyset)\}$.
(2) $\nu(c) \in 2^S$.

Let Φ denote the set of prematchings ν.

Remark 1 *A prematching ν is a matching if and only if the following hold: (a) If $s \in \nu(c)$ then $\nu(s) = (c,\nu(c))$. (b) If $\nu(s) = (c,S')$ then $\nu(c) = S'$. Note that (a) and (b) correspond to (3) and (4) in the definition of matching.*

We now proceed to define a function $T : \Phi \to \Phi$. Let ν be a prematching. We need the following constructions:

$$\begin{aligned} U(c,\nu) &= \{S' \subseteq S : \forall s \in S', (c,S')R(s)\nu(s)\} \\ V(s,\nu) &= \{(c,S') \in C \times S_s : \forall s' \in S' \backslash \{s\}, (c,S')R(s')\nu(s') \\ &\quad \text{and } \ S'R(c)\nu(c)\} \cup \{(\emptyset,\emptyset)\} \end{aligned}$$

That is, $U(c,\nu)$ is the collection of sets of students S' so that (c,S') is better than their matches in ν, for each one of them. $V(s,\nu)$ is the set of (c,S') so that, for each student in $S'\backslash\{s\}$, and for the college c, the matching in which c is matched to S' is better than their matches in ν.

Now, define $T : \Phi \to \Phi$ by $(T\nu)(f) = max_{P(f)}U(f,\nu)$ if $f \in C$ and $(T\nu)(f) = max_{P(f)}V(f,\nu)$ if $f \in S$. The function T takes each college to its optimal set of students, out of those who are willing to attend that college as a group, and each student to its optimal college-group of students pair, out of those willing to accept him/her.

Let $\mathcal{E}(T) = \{\nu \in \Phi : \nu = T\nu\}$.

The main result of this section is

Theorem 3.1 *$\mathcal{E}(T) = C_W(P)$.*

The proof of Theorem 3.1 is in Section 3.2.

3.1 An intermediate notion of stability.

We first introduce a notion of stability that is instrumental in obtaining our results. A pair $(B,c) \in 2^S \times C$ *blocks** a matching μ if $B \cap \mu(c) = \emptyset$ and there exists $A \subseteq \mu(c)$ so that for every $s' \in A \cup B, (c, A \cup B)P(s')\mu(s')$ and $A \cup BP(c)\mu(c)$. A matching is *stable** if it is individually rational and there does not exist a student-group-college pair that blocks* μ. Denote the set of stable* matchings by $S^*(P)$.

Lemma 3.2 *$S^*(P) = C_W(P)$.*

The proof of Lemma 3.2 is in the appendix.

We isolate part of the proof of Theorem 3.1 as Lemma 3.3, as it will be useful in other results.

Lemma 3.3 *Let μ be a matching and $\nu = T\mu$.*

(1) If $\nu(c) \neq \mu(c)$ then $(c, \nu(c))$ blocks μ. If $\nu(s) \neq \mu(s)$ then $\nu(s)$ blocks* μ.*
(2) If $\nu(c) = \mu(c)$ then there is no block (c, D) of μ, for any $D \subseteq S$. If $\nu(s) = \mu(s)$ then there is no block* (c', D) of μ, for any $c' \in C$ and $D \subseteq S$ with $D \ni s$.*

Proof We first prove (1). Let $\nu(c) \neq \mu(c)$. That μ is a matching implies $\mu(c) \in U(c, \mu)$; so $\nu(c)P(c)\mu(c)$. That $\nu(c) \in U(c, \mu)$ implies for every $s \in \nu(c)$, $(c, \nu(c))R(s)\mu(s)$. But μ is a matching, so $\nu(c) \neq \mu(c)$ implies that for all $s \in \nu(c)$, $(c, \nu(c)) \neq \mu(s)$. Hence, for all $s \in \nu(c)$, $(c, \nu(c))P(s)\mu(s))$.

The proof that, if $\nu(s) \neq \mu(s)$, then $\nu(s)$ blocks* μ, is analogous.

We now prove (2). Let $c \in C$ with $\nu(c) = \mu(c)$. Let $D \subseteq S$ be such that for every $s \in D$, $(c, D)R(s)\mu(s)$, then $D \in U(c, \mu)$. But $\mu(c) = \nu(c)$ implies that $\mu(c)R(c)D$. So (c, D) is not a block* of μ. Now let $s \in S$ with $\nu(s) = \mu(s)$. If (c', D), with $s \in D$ is such that $DR(c')\mu(c')$ and for all $s' \in D \backslash \{s\}$, $(c, D)R(s')\mu(s')$, then $(c', D) \in V(s, \mu)$. But $\mu(s) = \nu(s)$ then gives $\mu(s)R(s)(c', D)$, so (c', D) is not a block* of μ. □

3.2 Proof of Theorem 3.1.

By Lemma 3.2, it is enough to prove that $S^*(P) = \mathcal{E}(T)$.

We need to show that for every $\nu \in \mathcal{E}(T)$, ν is a matching and that it is stable* and also if μ is a stable* matching then μ is a fixed point of T.

Now suppose that $\nu \in \mathcal{E}(T)$. We first show that it is a matching.

Since we already know that ν is a prematching we only need to show the following: (a) If $s \in \nu(c)$ then $\nu(s) = (c, \nu(c))$. (b) If $\nu(s) = (c, S')$ then $\nu(c) = S'$ by Remark 1.

(a) $s \in \nu(c) = (T\nu)(c) = max_{P(c)}\{U(c, \nu)\}$. Therefore, $(c, \nu(c))R(\hat{s})\nu(\hat{s})$ for all $\hat{s} \in \nu(c)$ and in particular

$$(c, \nu(c))R(s)\nu(s). \tag{1}$$

Thus, we have $(c, \nu(c)) \in V(s, \nu)$. But now $\nu(s) = (T\nu)(s) = max_{P(s)}\{V(s, \nu)\}$. Therefore,

$$\nu(s)R(s)(c, \nu(c)). \tag{2}$$

Since $P(s)$ is strict (1) and (2) imply that $\nu(s) = (c, \nu(c))$.

(b) $\nu(s) = (c, S')$. Now, $\nu(s) = (T\nu)(s) = max_{P(s)}\{V(s, \nu)\}$. Thus, $\nu(s) \in$

$V(s,\nu)$. Therefore, we have

$$S' R(c) \nu(c) \tag{3}$$

and also that for all $s' \in S' - \{s\}$, $(c, S')R(s')\nu(s')$. This, along with $\nu(s) = (c, S')$ implies that $S' \in U(c,\nu)$. But $\nu(c) = (T\nu)(c) = max_{P(c)}\{U(c,\nu)\}$. So we get

$$\nu(c) R(c) S' \tag{4}$$

Since $P(c)$ is strict (3) and (4) imply that $\nu(c) = S'$.

Next, we show that ν is a stable* matching. Assume that (B, c) blocks* ν. Then, there exists $A \in \nu(c)$ such that $(c, A \cup B)P(s)\nu(s)$ for all $s \in A \cup B$ and $A \cup BP(c)\nu(c)$. This implies that $A \cup B \in U(c,\nu)$. Therefore, $\nu(c) = (T\nu)(c) = max_{P(c)}\{U(c,\nu)\}$ gives us that $\nu(c)R(c)A \cup B$. This is a contradiction to $A \cup BP(c)\nu(c)$.

To finish the proof, we need to show that for every $\mu \in S^*(P)$ we have $\mu = T\mu$. This is a direct consequence of Lemma 3.3. Let $\nu = T\mu$. Since μ is stable*, there are no blocking coalitions, which implies that $\nu(c) = \mu(c)$ for every college c and $\nu(s) = \mu(s)$ for every student s. Thus, $\mu = \nu = T\mu$. $\square$

3.3 Discussion

The matching literature that uses constructions like the T function usually proceeds by ordering prematchings and then showing that T is monotone increasing. By application of Tarski's fixed-point theorem (Theorem 4.4), then, one proves that $\mathcal{E}(T)$, and thus the core, is nonempty. It may be interesting to see where that approach would fail if applied to our model.

The order on prematchings always involves saying that a prematching ν' is larger than another prematching ν if all agents on one side of the market prefer ν' to ν, while the other side of the market prefers ν (see Echenique and Oviedo (2006) for a discussion of the two main orders used). Now, if one compares $T\nu$ with $T\nu'$ one should get that $T\nu'$ is larger than $T\nu$. In the present model, that is a problem because students are choosing their best match out of sets ($V(s,\nu)$ and $V(s,\nu')$) that include agents from both sides of the market. So the set out of which students choose *does not depend in a systematic way* on the prematching involved. Without preferences over colleagues, since colleges are better off in ν', the set from which students choose shrinks, and thus students prefer $T\nu$ to $T\nu'$.

4 The fixed points of T^2.

We have seen that the core can be empty; thus T may not have any fixed points. However, we can prove that T^2, i.e. the composition of T with itself, must have fixed points. These may not be matchings, let alone core matchings. But if the core is nonempty, the core matchings must be fixed points of T^2 (and the fixed points of T^2 that are matchings must be a "partial" solution, see Section 7).

The importance of T^2 becomes clear in Section 5, where we present an algorithm for finding fixed points of T^2; an algorithm that will find all the fixed points of T.

Consider the following partial order on prematchings.

Definition 4.1 *Let $\nu, \nu' \in \Phi$; $\nu \succ \nu'$ if and only if $\nu(f)R(f)\nu'(f)$ for all agents $f \in C \cup S$ and $\nu(f)P(f)\nu'(f)$ for some agent f.*

Since the preferences are strict, the weak partial order $\succeq$ associated with $\succ$ can be defined as follows: $\nu \succeq \nu'$ if and only if $\nu = \nu'$ or $\nu \succ \nu'$.

We now define what it means for T to be monotone with respect to $\succeq$. Note that the same definition can be used for any partial order on prematchings instead of $\succeq$.

Definition 4.2 *T is monotone increasing with respect to $\succeq$ if $\nu \succeq \nu'$ implies $T\nu \succeq T\nu'$; T is monotone decreasing if $\nu \succeq \nu'$ implies $T\nu' \succeq T\nu$.*

Lemma 4.3 *T is monotone decreasing with respect to $\succeq$.*

Proof Let $\nu \succeq \nu'$. We are going to show that $T\nu' \succeq T\nu$, that is $T\nu'(f) \succeq T\nu(f)$ for all agents f. We split this into two cases according to whether f is a student or a college:

Let $f \in C$. Let $S' \in U(f, \nu)$. Then, for all $s \in S'$, $(f, S')R(s)\nu(s)$. Since $\nu \succeq \nu'$, we have $\nu(s)R(s)\nu'(s)$. Now, by transitivity we get $(f, S')R(s)\nu'(s)$ for all $s \in S'$, which implies that $S' \in U(f, \nu')$. Thus, $U(f, \nu') \supseteq U(f, \nu)$, which in turn implies

$$T\nu'(f) = max_{P(f)}U(f, \nu')R(f)max_{P(f)}U(f, \nu) = T\nu(f).$$

Hence, $T\nu'(f)R(f)T\nu(f)$. The proof of $T\nu'(f)R(f)T\nu(f)$ when $f \in S$ is analogous. $\square$

Let Φ' be the set of individually-rational prematchings. That is,

$$\Phi' = \{\nu \in \Phi : \forall s \in S, \nu(s)R(s)(\emptyset, \emptyset) \text{ and } \forall c \in C, \nu(c)R(c)\emptyset\}.$$

When endowed with the partial order $\succeq$, Φ' is a lattice because it is a product set endowed with a product order (Echenique and Oviedo, 2004).

Tarski's fixed-point theorem is crucial to our results. We include a statement to keep our paper self-contained. See, for example, Topkis (1998) for a proof.

Theorem 4.4 ***(Tarski's Fixed-point Theorem)*** *Let X, endowed with a partial order $\geq$, be a lattice. If $f : X \to X$ is monotone increasing, then the set of fixed points of f, endowed with $\geq$, is a non-empty lattice.*

Note that for all $\nu \in \Phi$, $T\nu \in \Phi'$, so we can regard T as mapping Φ' into Φ'.

Let $\mathcal{E}(T^2) = \{\nu \in \Phi : \nu = T^2\nu\}$.

Theorem 4.5 *$\mathcal{E}(T^2)$ is a non-empty lattice.*

Proof Consider the partial order $\succeq$. We have shown in Lemma 4.3 that T is monotone decreasing with respect to this partial order. Thus, if $\nu \succeq \nu'$ then $T\nu' \succeq T\nu$. Now, apply the same lemma to $T\nu'$ and $T\nu$ to get $T^2\nu \succeq T^2\nu'$. We have that T^2 on Φ' is monotone increasing, and also that $(\Phi', \succeq)$ is a lattice. Tarski's fixed point theorem (Theorem 4.4) implies that $(\mathcal{E}(T^2), \succeq)$ is a non-empty lattice. $\square$

Proposition 4.6 *No two fixed points of T are ordered by $\succeq$.*

Proof Assume the contrary: There exist $\mu, \mu' \in \mathcal{E}(T)$, such that $\mu \succeq \mu'$ and $\mu \neq \mu'$. Now, by applying Lemma 4.3 to this inequality we get $T\mu' \succeq T\mu$; that is, $\mu' \succeq \mu$. Since $\succeq$ is a partial order, we must have $\mu = \mu'$, which is a contradiction. $\square$

Proposition 4.7 *There exist two prematchings $\overline{\nu}, \underline{\nu} \in \mathcal{E}(T^2)$ such that for all $\nu \in \mathcal{E}(T)$ $\overline{\nu} \succeq \nu \succeq \underline{\nu}$. Moreover, if one of these two prematchings is also a fixed point of T, then T has a unique fixed point.*

Proof The existence of $\overline{\nu}$ and $\underline{\nu}$ follows from Theorem 4.5, as a finite lattice must have a smallest and a largest element.

Now assume that $\overline{\nu}$ is also a fixed point of T. If there was another fixed point of T, it would also be a fixed point of T^2. But by the first part we know that $\overline{\nu}$ is better than this fixed point, which contradicts Proposition 4.6. The case where $\underline{\nu}$ is a fixed point of T is similar. $\square$

Proposition 4.8 *The iterations of T^2 starting at the largest prematching in Φ' will reach $\overline{\nu}$ in a finite number of steps. Similarly, if we start from the smallest prematching in Φ', we will reach $\underline{\nu}$.*

Proof Let ν_0 be the largest prematching in Φ'. Define $\nu_n = T^2\nu_{n-1}$ inductively for $n \geq 1$.

First we prove by induction that $\nu_n \succeq \overline{\nu}$. Base assumption holds since ν_0 is the largest prematching. Now, suppose $\nu_{n-1} \succeq \overline{\nu}$. As we have shown in the proof of theorem 4.5, T^2 preserves the order in $\succeq$. Therefore, $\nu_n = T^2\nu_{n-1} \succeq T^2\overline{\nu} = \overline{\nu}$ which completes the induction.

Second, for some finite n, $\nu_{n-1} = \nu_n$. Assume this does not hold for any n. Then, $\{\nu_n\}$ is an infinite sequence of distinct prematchings in Φ'. Since there exists a finite number of agents, number of prematchings is also finite. This gives a contradiction.

We've shown that $\nu_{n-1} = \nu_n$ holds for some n, that means ν_n is a fixed point of T^2. Now, by the first part of the proof $\nu_n \preceq \overline{\nu}$. Since $\overline{\nu}$ is the largest fixed point of T^2, we get that $\nu_n = \overline{\nu}$.

The proof of the second statement in the proposition is exactly the same. □

5 An Algorithm

We describe an algorithm and prove that it finds all the core matchings.

5.1 Description.

Let $\{1, 2, \ldots, n+m\}$ be an enumeration of the elements of $C \cup S$. Given a college-admissions problem $\langle C, S, P\rangle$, let $\langle F_1, F_2, \ldots, F_{m+n}\rangle$ denote the problem with the same sets of agents, in which each agent's preference list is restricted to those with F_f being the top choice for agent f. So, in $\langle F_1, F_2, \ldots, F_{m+n}\rangle$, agent f finds unacceptable the partners that were originally better than F_f.

For every agent $f \in C \cup S$ and for every prematching ν, let $i(f, \nu)$ denote the best choice of f that is worse than $\nu(f)$.

Algorithm 5.1 *Find the smallest $\underline{\nu}$ and largest $\overline{\nu}$ fixed points by applying T^2 repeatedly to the largest and smallest prematchings in Φ', as suggested in Proposition 4.8, until it finds a fixed point. If $T\overline{\nu} = \overline{\nu}$ then let $\hat{\mathcal{E}} = \{\overline{\nu}\}$ and*

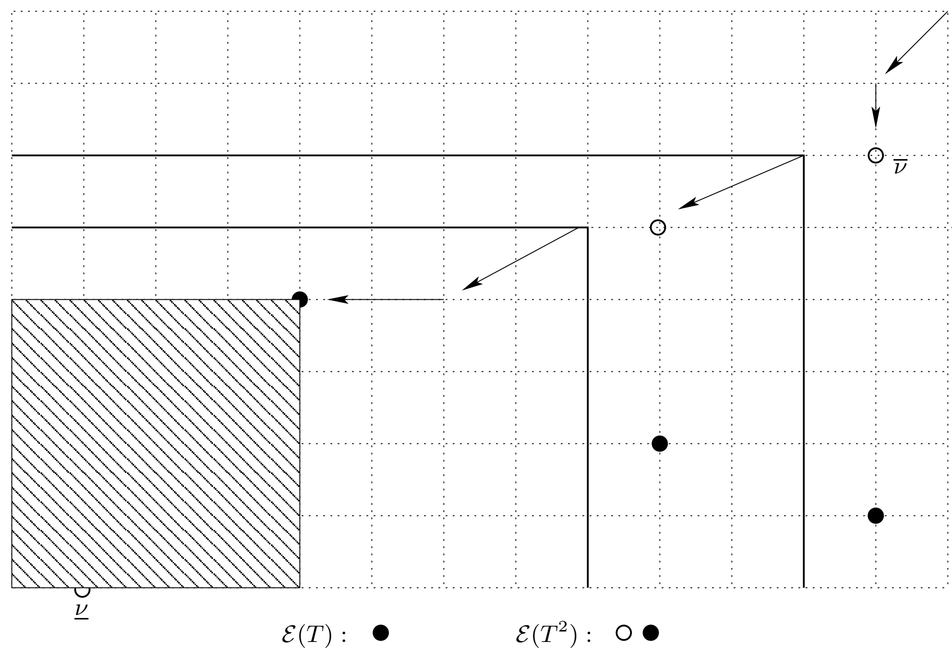

Fig. 1. An illustration of the algorithm

the algorithm is finished. Otherwise proceed as follows: let $\hat{\mathcal{E}} = \emptyset$. The possible states of the algorithm are the sets of individually-rational prematchings, and the initial state is $\boldsymbol{Q} = \{\overline{\nu}\}$. While $\boldsymbol{Q} \neq \emptyset$ do the following subroutine.

*SUBROUTINE: Set **Q'** $= \emptyset$. For all $\nu \in \boldsymbol{Q}$ and for all f such that $i(f,\nu)R(f)\underline{\nu}(f)$ do steps 1-2 to get a new state **Q'**. Then set $\boldsymbol{Q} =$ **Q'**.*

STEP 1. Find the largest fixed point of T^2 for the problem $\langle \nu(1), \ldots, \nu(f-1), i(f,\nu), \nu(f+1) \ldots, \nu(m+n) \rangle$, call it $\overline{\nu}_f$.

*STEP 2. If $\overline{\nu}_f = T\overline{\nu}_f$ then add $\overline{\nu}_f$ to $\hat{\mathcal{E}}$; otherwise if $\overline{\nu}_f \succ \underline{\nu}$ then add $\overline{\nu}_f$ to **Q'**.*

The algorithm is easy to explain using a picture; see Figure 1. The set of prematchings is a product set, and $\succeq$ is a product order. We can represent it as the grid on Figure 1. Note how the core matchings, the matchings in $\mathcal{E}(T)$, are unordered, and the matchings in $\mathcal{E}(T^2)$ form a lattice (the smallest element is hidden by the shaded area).

First iterate T^2 from the largest prematching—represented by the upper right corner. By monotonicity of T^2 one obtains a monotone decreasing sequence, which has to stop at a fixed point $\overline{\nu}$. Again by monotonicity of T^2, $\overline{\nu}$ must be the largest fixed point of T^2. Now the algorithm iterates T^2 in a restricted problem, the problem obtained by setting $i(f,v)$ as the best partner for one agent, and $\overline{\nu}(f)$ for everyone else. This restriction is represented by thick lines in Figure 1, to the left and down of $\overline{\nu}$. By iterating T^2 we find the largest fixed point in the restricted problem.

The procedure of restricting and finding is repeated. Note that when a core matching is found, we know by Proposition 4.6 that there cannot be any more core matchings down and to the left. This is illustrated by a shaded area in Figure 1.

Each restriction changes how T operates, not just the domain of T. The successive restrictions can make us lose fixed points of T^2, but, it turns out, not of T (Proposition 5.2). For example, often $\underline{\nu} = T\overline{\nu}$ so that once $\overline{\nu}$ is eliminated, $\underline{\nu}$ is no longer a fixed point of T^2. The algorithm is based on Echenique's (2003) algorithm for non-cooperative games. Echenique's algorithm searches and finds all the fixed points of a monotone function. On the other hand, our algorithm searches for the fixed points of T^2 but will in general miss some; it is only guaranteed to find all the fixed points of T.

5.2 Results.

The algorithm proceeds by restricting agents' preferences. Our first result is that one does not lose fixed points with these restrictions.

Proposition 5.2 *If $\mu \in \mathcal{E}(T)$ and $F_f R(f)\mu(f)$ for all $f \in C \cup S$, then μ is also a fixed point of T for the problem $\langle F_1, F_2, \ldots, F_{m+n}\rangle$.*

Proof Let $\tilde{T}$, $\tilde{U}$ and $\tilde{V}$ be the corresponding T, U, and V for the restricted problem $\langle F_1, F_2, \ldots, F_{m+n}\rangle$. Now, it is clear that $U(c,\mu) \supseteq \tilde{U}(c,\mu)$ for all $c \in C$ and similarly $V(s,\mu) \supseteq \tilde{V}(s,\mu)$ for all $s \in S$. Therefore, $(T\mu)(c)R(c)(\tilde{T}\mu)(c)$ for all $c \in C$ and $(T\mu)(s)R(s)(\tilde{T}\mu)(s)$ for all $s \in S$.

Since μ is a fixed point of T we have $(T\mu)(c) = \mu(c)$ and $(T\mu)(s) = \mu(s)$.

Now, that μ is a matching and that $F_f R(f)\mu(f)$ for all $f \in C \cup S$ imply $\mu(c) \in \tilde{U}(c,\mu)$ and $\mu(s) \in \tilde{U}(s,\mu)$. Therefore, $(\tilde{T}\mu)(c)R(c)\mu(c)$ and $(\tilde{T}\mu)(s)R(s)\mu(s)$.

To complete the proof we need to put together the inequalities we got: $\mu(c) = (T\mu)(c)R(c)(\tilde{T}\mu)(c)R(c)\mu(c)$ and similarly $\mu(s) = (T\mu)(s)R(s)\,(\tilde{T}\mu)(s)R(s)\mu(s)$. Since $R(c)$ and $R(s)$ are linear orders we get that $(\tilde{T}\mu)(s) = \mu(s)$ and $(\tilde{T}\mu)(c) = \mu(c)$. □

Theorem 5.3 $\hat{\mathcal{E}} = \mathcal{E}(T)$*, that is, the set $\hat{\mathcal{E}}$ produced by the algorithm above coincides with the fixed points of T which are the core matchings.*

Proof We first show that the algorithm stops after a finite number of steps when $\mathbf{Q} = \emptyset$. Then we establish $\hat{\mathcal{E}} \subseteq \mathcal{E}(T)$ and $\hat{\mathcal{E}} \supseteq \mathcal{E}(T)$ to complete the proof.

Since Φ' is a product set, we can identify Φ' with a grid. Let the distance between each consecutive point in the grid be one unit, and use the resulting Euclidean distance between prematchings. Let $d(\mathbf{Q})$ be the maximum of distances between each prematching in $\mathbf{Q}$ and $\underline{\nu}$ (if $\mathbf{Q}$ is empty let $d(\mathbf{Q}) = 0$). Let $\mathbf{Q}$ and **Q'** be successive states in the algorithm. It is clear from the definition that $d(\mathbf{Q}) > d(\mathbf{Q'})$. Note that since Φ' is a finite set d takes only a finite number of values. This shows that, after a finite number of steps, we must get $\mathbf{Q} = \emptyset$ which means that the algorithm stops after a finite number of steps.

Now, let us show $\hat{\mathcal{E}} \subseteq \mathcal{E}(T)$. Let $\mu \in \hat{\mathcal{E}}$. This means that $\mu = T\mu$ by Step 2. Therefore, $\mu \in \mathcal{E}(T)$ which proves $\hat{\mathcal{E}} \subseteq \mathcal{E}(T)$.

To complete the proof we have to show that $\hat{\mathcal{E}} \supseteq \mathcal{E}(T)$. Let $\mu \in \mathcal{E}(T)$. We prove by induction that at every stage $\mathbf{Q}$ of the algorithm, either $\mu \in \hat{\mathcal{E}}$ or there exists $\nu \in \mathbf{Q}$ such that $\nu \succeq \mu$. The beginning state is $\mathbf{Q} = \{\overline{\nu}\}$ and $\hat{\mathcal{E}} = \emptyset$. By the first statement in Proposition 4.7 we get $\overline{\nu} \succeq \mu$ thus the initial condition is satisfied. Now, let $\mathbf{Q}$ be an intermediate state, from applying the subroutine on a previous state $\mathbf{Q}_0$. Let $\hat{\mathcal{E}}$ and $\hat{\mathcal{E}}_0$ be the associated sets of fixed points. If $\mu \in \hat{\mathcal{E}}_0$ then $\mu \in \hat{\mathcal{E}}$ since $\hat{\mathcal{E}} \supseteq \hat{\mathcal{E}}_0$. If not then, by the inductive hypothesis, there exists $\nu \in \mathbf{Q}_0$ so that $\nu \succeq \mu$. Now, if $\nu = \mu$ then μ should have already been in $\hat{\mathcal{E}}_0$ since at the previous stage in Step 2 it checks for this. Therefore $\nu \succ \mu$. Which implies, by Proposition 5.2, that there exists f so that μ is a fixed point of the restricted problem $\langle \nu(1), \ldots, \nu(f-1), i(f,\nu), \nu(f+1) \ldots, \nu(m+n) \rangle$. Now, let ν' be the greatest fixed point of T^2 for the restricted problem. By Proposition 4.6, either $\nu' = \mu$ or ν' is not a fixed point of T. If $\nu' = \mu$ then $\mu \in \hat{\mathcal{E}}$, otherwise $\nu' \in \mathbf{Q}$ and $\nu' \succeq \mu$ completing the induction. Now, we have shown in the previous paragraph that the algorithm ends when $\mathbf{Q} = \emptyset$. Hence, the inductive hypothesis implies that $\mu \in \hat{\mathcal{E}}$. □

5.3 Algorithm performance.

We now discuss the performance of the algorithm. In Section 8.1 we prove, under a certain structure on agents' preferences, a bound on the number of steps the algorithm takes to find the core.

Without assumptions on preferences one cannot guarantee that the algorithm does not visit all possible matchings, and thus performs very slowly. But such "worst case" performance seems of little use in evaluating the algorithm. Ideally, we could calculate the average performance of the algorithm in some class of relevant problems. Unfortunately we do not have such results. Indeed average-time results are difficult to find in computer science; for example there are no results on average time for the Gale-Shapley algorithm in many-to-one matchings, or for the numerous algorithms for finding Nash equilibria (see the surveys by McKelvey and McLennan (1996) and von Stengel (2002)). There are also very few results on worst-case performance for these algorithms. [2]

Beyond our result in Section 8.1, we think the algorithm is likely to be fast in most applications. The reason can be inferred from Figure 1. Each iteration of the algorithm from the upper-right corner reduces the size of subsequent problems in which T must iterate. So if T-iterations take large steps in the picture, the algorithm will be fast because the steps eliminate cases the algorithm would have to consider. In a different context, Echenique (2003) simulates non-cooperative games where a similar idea is used to find all Nash equilibria. The effect of "large steps" is also present in his application to non-cooperative games, and it makes the algorithm run very fast in his simulations. For example it finds all equilibria of games where each player has 60,000 strategies, and there are 500 equilibria, in less than a second.

The only alternative to our algorithm is to perform an exhaustive search of all possible matchings, and test whether each of them is in the core. We shall argue that this alternative is, in practice, not feasible because the number of matchings one needs to test quickly becomes too large. We calculate the number of matchings in a problem with n colleges and m students: Pick k of the n colleges to be non-single; this can be done in $\binom{n}{k}$ ways. Partition the k colleges into k nonempty sets. Each partition then generates $k!$ different matchings, as there are $k!$ ways of assigning the elements of the partition to the k colleges. The number of partitions of m elements in k sets is expressed by the Stirling number of the second kind (see e.g. Comtet (1974)). Thus there are

$$\sum_{k=1}^{n} \binom{n}{k} S_k^m k!$$

different matchings. For example, one can assign 1200 students to 9 colleges in 1.233×10^{1145} different ways.

[2] Some results on the worst-case performance of the Gale-Shapley algorithm are in Segal (2003). His results, as well as the remarks above, are relative to input size, which tends to be very large (Echenique, 2006). In actual implementations, one would have to avoid this problem, for example by bounding the number of acceptable partners. In actual implementations of the Gale-Shapley algorithm, this is accomplished by setting a maximum length on submitted rank-order lists (e.g. the National Resident Matching Program or the matching of elementary-school children to schools in the New York public school system).

6 Examples

We now illustrate how the algorithm operates on two examples. The first example has two core matchings and the second has no core matchings. We note that an actual implementation of the algorithm would operate differently from the examples, optimizing in several points. For example, if the algorithm reaches a $\overline{\nu}$ for which $T\overline{\nu}$ has already been calculated, it would stop iterating. And it would avoid computing the sets $V(s,\nu)$ and $U(c,\nu)$, as it is enough to run down an agents' preference ordering, picking the best available partner at ν.

Example 6.1 *Let $S = \{s_1, s_2\}$ and $C = \{c_1, c_2\}$. Suppose that agents' preferences are:*

$$P(c_1) : S$$

$$P(c_2) : S$$

$$P(s_1) : (c_1, S), (c_2, S)$$

$$P(s_2) : (c_2, S), (c_1, S).$$

First we need to calculate $\overline{\nu}$ and $\underline{\nu}$. Since T maps the smallest prematching to the largest prematching, we can calculate both $\overline{\nu}$ and $\underline{\nu}$ in one iteration starting from the smallest prematching.

	s_1	s_2	c_1	c_2
ν_0	$(\emptyset,\emptyset)$	$(\emptyset,\emptyset)$	$\emptyset$	$\emptyset$
$V(s,\nu_0)/U(c,\nu_0)$	$(c_1,S),(c_2,S)$	$(c_1,S),(c_2,S)$	S	S
$\nu_1 = T\nu_0$	(c_1,S)	(c_2,S)	S	S
$V(s,\nu_1)/U(c,\nu_1)$	(c_2,S)	(c_1,S)	$\emptyset$	$\emptyset$
$\nu_2 = T\nu_1$	(c_2,S)	(c_1,S)	$\emptyset$	$\emptyset$
$V(s,\nu_2)/U(c,\nu_2)$	$(c_1,S),(c_2,S)$	$(c_1,S),(c_2,S)$	S	S
$\nu_3 = T\nu_2$	(c_1,S)	(c_2,S)	S	S
$V(s,\nu_3)/U(c,\nu_3)$	(c_2,S)	(c_1,S)	$\emptyset$	$\emptyset$
$\nu_4 = T^2\nu_2$	(c_2,S)	(c_1,S)	$\emptyset$	$\emptyset$

Hence, $\overline{\nu} = v_3$ and $\underline{\nu} = v_4$. As we can see, $\overline{\nu}$ is not a fixed point of T. Therefore, $\hat{\mathcal{E}} = \emptyset$ and $\boldsymbol{Q} = \{\overline{\nu}\}$.

Secondly, we have to run the subroutine for each agent and for each prematch-

ing in $\boldsymbol{Q}$*. We have to find the largest fixed points of problems* $\langle (c_2, S), (c_2, S), S, S\rangle$, $\langle (c_1, S), (c_1, S), S, S\rangle$, $\langle (c_1, S), (c_2, S), \emptyset, S\rangle$ *and* $\langle (c_1, S), (c_2, S), S, \emptyset\rangle$.

The iteration for problem $\langle (c_2, S), (c_2, S), S, S\rangle$*:*

	s_1	s_2	c_1	c_2
ν_0	(c_2, S)	(c_2, S)	S	S
$V(s,\nu_0)/U(c,\nu_0)$	(c_2, S)	(c_2, S)	$\emptyset$	S
$\nu_1 = T\nu_0$	(c_2, S)	(c_2, S)	$\emptyset$	S
$V(s,\nu_1)/U(c,\nu_1)$	(c_2, S)	(c_2, S)	$\emptyset$	S
$\nu_2 = T\nu_1$	(c_2, S)	(c_2, S)	$\emptyset$	S

Here, ν_2 *is the largest fixed point of the subproblem and also a fixed point of* T*. Let* $\nu_2 = \mathcal{E}_1$*, then by the second step of the subroutine* $\hat{\mathcal{E}} = \{\mathcal{E}_1\}$ *and* $\boldsymbol{Q}$*'* $= \emptyset$*.*

The iteration for problem $\langle (c_1, S), (c_1, S), S, S\rangle$*:*

	s_1	s_2	c_1	c_2
ν_0	(c_1, S)	(c_1, S)	S	S
$V(s,\nu_0)/U(c,\nu_0)$	(c_1, S)	(c_1, S)	S	$\emptyset$
$\nu_1 = T\nu_0$	(c_1, S)	(c_1, S)	S	$\emptyset$
$V(s,\nu_1)/U(c,\nu_1)$	(c_1, S)	(c_1, S)	S	$\emptyset$
$\nu_2 = T\nu_1$	(c_1, S)	(c_1, S)	S	$\emptyset$

As before, ν_2 *is the largest fixed point of the subproblem and also a fixed point of* T*. Let* $\nu_2 = \mathcal{E}_2$*, then by the second step of the subroutine* $\hat{\mathcal{E}} = \{\mathcal{E}_1, \mathcal{E}_2\}$ *and* $\boldsymbol{Q}$*'* $= \emptyset$*.*

The iteration for problem $\langle (c_1, S), (c_2, S), \emptyset, S\rangle$*:*

	s_1	s_2	c_1	c_2
ν_0	(c_1, S)	(c_2, S)	$\emptyset$	S
$V(s,\nu_0)/U(c,\nu_0)$	(c_2, S)	(c_2, S)	$\emptyset$	S
$\nu_1 = T\nu_0$	(c_2, S)	(c_2, S)	$\emptyset$	S
$V(s,\nu_1)/U(c,\nu_1)$	(c_2, S)	(c_2, S)	$\emptyset$	S
$\nu_2 = T\nu_1$	(c_2, S)	(c_2, S)	$\emptyset$	S

Here $\nu_2 = \mathcal{E}_1$ *is the largest fixed point of the subproblem which is also a fixed* *point of T. Thus, $\hat{\mathcal{E}} = \{\mathcal{E}_1, \mathcal{E}_2\}$ and **Q'** $= \emptyset$.*

The iteration for problem $\langle (c_1, S), (c_2, S), S, (\emptyset, \emptyset)\rangle$:

	s_1	s_2	c_1	c_2
ν_0	(c_1, S)	(c_2, S)	S	$\emptyset$
$V(s, \nu_0)/U(c, \nu_0)$	(c_1, S)	(c_1, S)	S	$\emptyset$
$\nu_1 = T\nu_0$	(c_1, S)	(c_1, S)	S	$\emptyset$
$V(s, \nu_1)/U(c, \nu_1)$	(c_1, S)	(c_1, S)	S	$\emptyset$
$\nu_2 = T\nu_1$	(c_1, S)	(c_1, S)	S	$\emptyset$

*In this case $\nu_2 = \mathcal{E}_2$ is the largest fixed point of the subproblem which is also a fixed point of T. Thus, $\hat{\mathcal{E}} = \{\mathcal{E}_1, \mathcal{E}_2\}$ and **Q'** $= \emptyset$.*

*Thus our new state is **Q** $=$ **Q'** $= \emptyset$. The algorithm stops here and gives us $\hat{\mathcal{E}} = \{\mathcal{E}_1, \mathcal{E}_2\}$ as the core.*

Example 6.2 *Let $S = \{s_1, s_2, s_3\}$ and $C = \{c_1, c_2\}$. Suppose that agents' preferences are:*

$$P(c_1) : \{s_1, s_2\}, \{s_2, s_3\}$$
$$P(c_2) : \{s_1, s_3\}$$
$$P(s_1) : (c_2, \{s_1, s_3\}), (c_1, \{s_1, s_2\})$$
$$P(s_2) : (c_1, \{s_1, s_2\}), (c_1, \{s_2, s_3\})$$
$$P(s_3) : (c_1, \{s_2, s_3\}), (c_2, \{s_1, s_3\}).$$

In this example, we will not write down the sets $V(s, \nu)$ and $U(c, \nu)$ when we are iterating T, as we only care about the best option for each set and we hope that it is clear how they are calculated from the previous example (indeed a computer implementation of the algorithm would not compute the U and V sets).

We start by calculating the largest and smallest fixed points.

	s_1	s_2	s_3	c_1	c_2
ν_0	$(\emptyset,\emptyset)$	$(\emptyset,\emptyset)$	$(\emptyset,\emptyset)$	$\emptyset$	$\emptyset$
$\nu_1 = T\nu_0$	$(c_2,\{s_1,s_3\})$	$(c_1,\{s_1,s_2\})$	$(c_1,\{s_2,s_3\})$	$\{s_1,s_2\}$	$\{s_1,s_3\}$
$\nu_2 = T\nu_1$	$(c_1,\{s_1,s_2\})$	$(\emptyset,\emptyset)$	$(c_2,\{s_1,s_3\})$	$\emptyset$	$\emptyset$
$\nu_3 = T\nu_2$	$(c_2,\{s_1,s_3\})$	$(c_1,\{s_1,s_2\})$	$(c_1,\{s_2,s_3\})$	$\{s_1,s_2\}$	$\{s_1,s_3\}$
$\nu_4 = T^2\nu_2$	$(c_1,\{s_1,s_2\})$	$(\emptyset,\emptyset)$	$(c_2,\{s_1,s_3\})$	$\emptyset$	$\emptyset$

Hence, $\overline{\nu} = \nu_3$ *and* $\underline{\nu} = \nu_4$. *As we can see,* $\overline{\nu}$ *is not a fixed point of* T. *Therefore,* $\hat{\mathcal{E}} = \emptyset$ *and* $\boldsymbol{Q} = \{\overline{\nu}\}$.

Now, we have to run the subroutine for each agent and for each prematching in $\boldsymbol{Q}$. *We have to find the largest fixed points of problems*

$$\begin{gathered}\langle(c_1,\{s_1,s_2\}),(c_1,\{s_1,s_2\}),(c_1,\{s_2,s_3\}),\{s_1,s_2\},\{s_1,s_3\}\rangle,\\ \langle(c_2,\{s_1,s_3\}),(c_1,\{s_2,s_3\}),(c_1,\{s_2,s_3\}),\{s_1,s_2\},\{s_1,s_3\}\rangle,\\ \langle(c_2,\{s_1,s_3\}),(c_1,\{s_1,s_2\}),(c_2,\{s_1,s_3\}),\{s_1,s_2\},\{s_1,s_3\}\rangle,\\ \langle(c_2,\{s_1,s_3\}),(c_1,\{s_1,s_2\}),(c_1,\{s_2,s_3\}),\{s_2,s_3\},\{s_1,s_3\}\rangle,\\ \langle(c_2,\{s_1,s_3\}),(c_1,\{s_1,s_2\}),(c_1,\{s_2,s_3\}),\{s_1,s_2\},\{\emptyset,\emptyset\}\rangle.\end{gathered}$$

The iteration for problem $\langle(c_1,\{s_1,s_2\}),(c_1,\{s_1,s_2\}),(c_1,\{s_2,s_3\}),\{s_1,s_2\},\{s_1,s_3\}\rangle$*:*

	s_1	s_2	s_3	c_1	c_2
ν_0	$(c_1,\{s_1,s_2\})$	$(c_1,\{s_1,s_2\})$	$(c_1,\{s_2,s_3\})$	$\{s_1,s_2\}$	$\{s_1,s_3\}$
$\nu_1 = T\nu_0$	$(c_1,\{s_1,s_2\})$	$(c_1,\{s_1,s_2\})$	$(\emptyset,\emptyset)$	$\{s_1,s_2\}$	$\emptyset$
$\nu_2 = T\nu_1$	$(c_1,\{s_1,s_2\})$	$(c_1,\{s_1,s_2\})$	$(\emptyset,\emptyset)$	$\{s_1,s_2\}$	$\emptyset$

v_2 *is neither a fixed point of* T *nor bigger than* $\underline{\nu}$, *so* ***Q'*** $= \emptyset$ *and* $\hat{\mathcal{E}}$ *remains unchanged.*

The iteration for problem $\langle (c_2, \{s_1, s_3\}), (c_1, \{s_2, s_3\}), (c_1, \{s_2, s_3\}), \{s_1, s_2\}, \{s_1, s_3\} \rangle$*:*

	s_1	s_2	s_3	c_1	c_2
ν_0	$(c_2, \{s_1, s_3\})$	$(c_1, \{s_2, s_3\})$	$(c_1, \{s_2, s_3\})$	$\{s_1, s_2\}$	$\{s_1, s_3\}$
$\nu_1 = T\nu_0$	$(\emptyset, \emptyset)$	$(\emptyset, \emptyset)$	$(c_2, \{s_1, s_3\})$	$\{s_2, s_3\}$	$\emptyset$
$\nu_2 = T\nu_1$	$(c_2, \{s_1, s_3\})$	$(c_1, \{s_2, s_3\})$	$(c_1, \{s_2, s_3\})$	$\{s_2, s_3\}$	$\{s_1, s_3\}$
$\nu_3 = T\nu_2$	$(\emptyset, \emptyset)$	$(c_1, \{s_2, s_3\})$	$(c_1, \{s_2, s_3\})$	$\{s_2, s_3\}$	$\emptyset$
$\nu_4 = T\nu_3$	$(\emptyset, \emptyset)$	$(c_1, \{s_2, s_3\})$	$(c_1, \{s_2, s_3\})$	$\{s_2, s_3\}$	$\emptyset$

v_4 *is neither a fixed point of* T *nor bigger than* $\underline{\nu}$*, so* ***Q'*** $= \hat{\mathcal{E}} = \emptyset$*.*

The iteration for problem $\langle (c_2, \{s_1, s_3\}), (c_1, \{s_1, s_2\}), (c_2, \{s_1, s_3\}), \{s_1, s_2\}, \{s_1, s_3\} \rangle$*:*

	s_1	s_2	s_3	c_1	c_2
ν_0	$(c_2, \{s_1, s_3\})$	$(c_1, \{s_1, s_2\})$	$(c_2, \{s_1, s_3\})$	$\{s_1, s_2\}$	$\{s_1, s_3\}$
$\nu_1 = T\nu_0$	$(c_2, \{s_1, s_3\})$	$(\emptyset, \emptyset)$	$(c_2, \{s_1, s_3\})$	$\emptyset$	$\{s_1, s_3\}$
$\nu_2 = T\nu_1$	$(c_2, \{s_1, s_3\})$	$(\emptyset, \emptyset)$	$(c_2, \{s_1, s_3\})$	$\emptyset$	$\{s_1, s_3\}$

Let $q_1 = v_2$*.* q_1 *is not a fixed point of* T *but it's bigger than* $\underline{\nu}$*. Thus,* ***Q'*** $= \{q_1\}$ *and* $\hat{\mathcal{E}} = \emptyset$*.*

The iteration for problem $\langle (c_2, \{s_1, s_3\}), (c_1, \{s_1, s_2\}), (c_1, \{s_2, s_3\}), \{s_2, s_3\}, \{s_1, s_3\} \rangle$*:*

	s_1	s_2	s_3	c_1	c_2
ν_0	$(c_2, \{s_1, s_3\})$	$(c_1, \{s_1, s_2\})$	$(c_1, \{s_2, s_3\})$	$\{s_2, s_3\}$	$\{s_1, s_3\}$
$\nu_1 = T\nu_0$	$(\emptyset, \emptyset)$	$(c_1, \{s_2, s_3\})$	$(c_2, \{s_1, s_3\})$	$\emptyset$	$\emptyset$
$\nu_2 = T\nu_1$	$(c_2, \{s_1, s_3\})$	$(c_1, \{s_2, s_3\})$	$(c_1, \{s_2, s_3\})$	$\{s_2, s_3\}$	$\{s_1, s_3\}$
$\nu_3 = T\nu_2$	$(\emptyset, \emptyset)$	$(c_1, \{s_2, s_3\})$	$(c_1, \{s_2, s_3\})$	$\{s_2, s_3\}$	$\emptyset$
$\nu_4 = T\nu_3$	$(\emptyset, \emptyset)$	$(c_1, \{s_2, s_3\})$	$(c_1, \{s_2, s_3\})$	$\{s_2, s_3\}$	$\emptyset$

ν_4 *is neither a fixed point of* T *nor bigger then* $\underline{\nu}$*. Therefore,* ***Q'*** *and* $\hat{\mathcal{E}}$ *remain unchanged.*

The iteration for problem $\langle (c_2, \{s_1, s_3\}), (c_1, \{s_1, s_2\}), (c_1, \{s_2, s_3\}), \{s_1, s_2\}, \{\emptyset, \emptyset\}\rangle$*:*

	s_1	s_2	s_3	c_1	c_2
ν_0	$(c_2, \{s_1, s_3\})$	$(c_1, \{s_1, s_2\})$	$(c_1, \{s_2, s_3\})$	$\{s_1, s_2\}$	$\emptyset$
$\nu_1 = T\nu_0$	$(c_1, \{s_1, s_2\})$	$(\emptyset, \emptyset)$	$(\emptyset, \emptyset)$	$\emptyset$	$\emptyset$
$\nu_2 = T\nu_1$	$(c_1, \{s_1, s_2\})$	$(c_1, \{s_1, s_2\})$	$(c_1, \{s_2, s_3\})$	$\{s_1, s_2\}$	$\emptyset$
$\nu_3 = T\nu_2$	$(c_1, \{s_1, s_2\})$	$(c_1, \{s_1, s_2\})$	$(\emptyset, \emptyset)$	$\{s_1, s_2\}$	$\emptyset$
$\nu_4 = T\nu_3$	$(c_1, \{s_1, s_2\})$	$(c_1, \{s_1, s_2\})$	$(\emptyset, \emptyset)$	$\{s_1, s_2\}$	$\emptyset$

ν_4 *is neither a fixed point of* T *nor bigger than* $\underline{\nu}$*. Therefore, the first loop of the algorithm terminates at* ***Q'*** $= \{q_1\}$ *and* $\hat{\mathcal{E}} = \emptyset$*.*

For the second loop of the algorithm, we need to consider only two subproblems:

$$\langle (c_1, \{s_1, s_2\}), (\emptyset, \emptyset), (c_2, \{s_1, s_3\}), \emptyset, \{s_1, s_3\}\rangle \text{ and}$$
$$\langle (c_2, \{s_1, s_3\}), (\emptyset, \emptyset), (c_2, \{s_1, s_3\}), \emptyset, \emptyset\rangle.$$

For both problems the largest fixed point is the prematching in which each agent is single. This prematching is not a fixed point of T *nor bigger than* $\underline{\nu}$*. Thus, after this loop* ***Q'*** $= \hat{\mathcal{E}} = \emptyset$ *and the algorithm terminates. Therefore, the core is empty.*

7 Partial solutions

What will the algorithm deliver when the core is empty? It turns out that the algorithm can solve the problem partially. It can identify a subset of agents that are matched in a way that will not be blocked, while the rest of the agents block their assignments. Following Sotomayor (2005a,b,c), we call the resulting matchings simple matchings.

Definition 7.1 *A matching* μ *is a simple matching if, for any block** (c, D) *of* μ*,* $\mu(c) = \emptyset$ *and* $\mu(s) = (\emptyset, \emptyset)$ *for all* $s \in D$*.*

Theorem 7.2 *Let* μ *be a matching. If* $\mu \in \mathcal{E}(T^2)$*, then* μ *is a simple matching.*

It is worth emphasizing that the algorithm does not confuse simple matchings with matchings in the core. It identifies the fixed points of T, and reports those as the core. But it also finds fixed points of T^2 that are not fixed points of T, and when those are in addition matchings, by Theorem 7.2, they must simple matchings.

Proof Let $\nu = T\mu$. We shall first prove that $\mu(c) \neq \emptyset$ implies that $\mu(c) = \nu(c)$, and that $\mu(s) \neq (\emptyset, \emptyset)$ implies that $\mu(s) = \nu(s)$.

Let $c \in C$ be such that $\mu(c) \neq \emptyset$. Since $\mu = T^2\mu$, we know that $\mu(c) \in U(c, \nu)$ so $(c, \mu(c))R(s)\nu(s)$ for all $s \in \mu(c)$. But that μ is a matching means that $(c, \mu(c)) = \mu(s)$; so we have shown that $\mu(s)R(s)\nu(s)$.

On the other hand, that μ is a matching implies that $\mu(s) \in V(s, \mu)$. This follows from the definition of $V(s, \mu)$, and that $\mu(s) = (c, \mu(c))$ for all $s \in \mu(c)$. Now, $\nu(s) = (T\mu)(s)$ and $\mu(s) \in V(s, \mu)$ gives $\nu(s)R(s)\mu(s)$. But we proved that $\mu(s)R(s)\nu(s)$, so $\mu(s) = \nu(s)$ follows because $P(s)$ is strict.

Similarly, $\mu = T^2\mu$ implies that $\mu(s) \in V(s, \nu)$ for any $s \in \mu(c)$. By the definition of $V(s, \mu)$, then, $\mu(c)R(c)\nu(c)$. But that μ is a matching implies that $\mu(c) \in U(c, \mu)$; so $\nu(c) = (T\mu)(c)$ implies $\nu(c)R(c)\mu(c)$. Hence $\nu(c) = \mu(c)$.

Let (c, D) be a block* of μ. Item (2) of Lemma 3.3 implies that $\mu(c) \neq \nu(c)$ and that for every $s \in D$ $\mu(s) \neq \nu(s)$. $\square$

Corollary 7.3 *Let μ be a matching in which no agent is single. Then μ is a core matching if and only if $\mu \in \mathcal{E}(T^2)$.*

Let ν be a prematching. Denote by $C_\nu \subseteq C$ the set of colleges c such that $(c, \nu(c)) = \nu(s)$ for all $s \in \nu(c)$. Let $S_\nu = \cup_{c \in C_\nu} \nu(c)$. Thus the restriction of ν to $C_\nu \cup S_\nu$ is a matching.

Proposition 7.4 *Let $\nu \in \mathcal{E}(T^2)$. Then the restriction of ν to $C_\nu \cup S_\nu$ is a core matching of $\langle C_\nu, S_\nu, P|_{C_\nu \cup S_\nu}\rangle$.*

Proposition 7.5 *Let μ be a simple matching, and let C' and S' denote the agents who are single in μ. If μ' is a simple matching for $\langle C', S', P|_{C' \cup S'}\rangle$, then the matching (μ, μ'), which matches C' and S' according to μ', and $C \backslash C'$ and $S \backslash S'$ according to μ, is a simple matching for $\langle C, S, P\rangle$.*

Proof Denote the matching (μ, μ') by $\hat{\mu}$. Suppose, by way of contradiction, that there is a block* (c^*, S^*) of $\hat{\mu}$ such that the agents in (c^*, S^*) are not single under $\hat{\mu}$.

First, suppose that $c^* \in C \backslash C'$. Then $S^* \nsubseteq S \backslash S'$, so there is $s \in S' \cap S^*$. Thus $S^* P(c)\mu(c)$, as $\hat{\mu}(c) = \mu(c)$ and $P(c)$ is strict. If we prove that $S^* \in U(c, \mu)$ we have reached a contradiction, since $\mu(c) = (T\mu)(c)$ (see proof of Theorem 7.2). Now, if $s \in S' \cap S^*$, $\mu(s) = (\emptyset, \emptyset)$. Since $(c^*, S^*)R(s)\hat{\mu}(s)R(s)(\emptyset, \emptyset)$, we have $(c^*, S^*)R(s)\mu(s)$. On the other hand, if $s \in S^* \backslash S'$, then $\hat{\mu}(s) = \mu(s)$ so $(c^*, S^*)R(s)\mu(s)$, as (c^*, S^*) is a block of $\hat{\mu}$.

Second, suppose that $c^* \in C'$. Then $S^* \nsubseteq S'$, as (c^*, S^*) cannot be a block of μ'. Let $s \in S' \cap S^*$; then $(c^*, S^*) \in V(s, \mu)$. Now we have a contradiction, as before, between $\mu'(s) = (T\mu')(s)$ and (c^*, S^*) being a block. □

Proposition 7.5 suggests a recursive procedure for finding a core matching: run T^2 to find a simple matching; put the non-single agents aside; run T^2 in the reduced market $\langle C', S', P|_{C' \cup S'} \rangle$. This procedure will, in some cases, be very fast.

8 Restrictions on preferences

8.1 The top coalition property

Banerjee et al. (2001) study coalition-formation games, of which our model is a special case. They introduce the so-called top-coalition property, and prove that it is sufficient for the core to be nonempty and unique. We prove that the top-coalition property is also sufficient to bound the number of iterations of the algorithm (Theorem 8.2).

We take the following notational liberty: Let $\mathcal{F}$ be the set of subsets of $C \cup S$ with at most one element from C. Let $F = \{c\} \cup \hat{S} \in \mathcal{F}$ and $F' = \{c'\} \cup S' \in \mathcal{F}$. If $c = c'$ we say that $FP(c)F'$ if $\hat{S}P(c)S'$. If $s \in \hat{S} \cap S'$ we say that $FP(s)F'$ if $(c, \hat{S})P(s)(c', S')$. If $F \subseteq S$, substitute $(\emptyset, \emptyset)$ for (c, S) in the statement above. For all $F \in \mathcal{F}$ with $f \notin F$, say that $\{f\}\, P(f)F$.

Definition 8.1 *A college-admissions problem satisfies the weak top-coalition property if there exists a partition $(F_1, F_2, \ldots, F_k)$ of all the agents, where $F_i \in \mathcal{F}$ for all i, with the following property: For all $f \in F_1$, F_1 is the top choice in $\mathcal{F}$ for $P(f)$, and for all $f \in F_i$, F_i is the top choice for $P(f)$ over the sets $F \in \mathcal{F}$ with*

$$F \subseteq (C \cup S) \setminus \cup_{j=1}^{i-1} F_j, i = 2, \ldots, k.$$

Theorem 8.2 *If a college-admissions problem satisfies the weak top-coalition property, then it has a unique core matching μ. Moreover, μ is the largest fixed point of T^2, and if k is the cardinality of the partition in Definition 8.1, then the algorithm finds μ in at most k steps.*

The first statement in Theorem 8.2 follows from Banerjee et al. (2001). We provide an independent proof to illustrate how our fixed-point method can be used and because we need it to prove the second part of the theorem.

Proof Let $(F_1, F_2, \ldots, F_k)$ be a partition of $C \cup S$ which satisfies the weak top-coalition property. Let μ be the matching which matches every agent in one partition to the agents in that partition. We first prove that μ is a stable* matching.

First note that μ is individually rational since for any agent f, $\mu(f)R(f)\emptyset$ if $f \in C$ or $\mu(f)R(f)(\emptyset, \emptyset)$ if $f \in S$ since $\{f\}$ is also an admissible coalition. Now, μ is stable* since no agents in F_1 want to block* since F_1 is their best choice, no agent in F_2 wants to block* without the agents in F_1 since F_2 is their best choice among $(C \cup S) \backslash F_1, \ldots$, no agent in F_k wants to block* without the agents in $F_1 \cup F_2 ... \cup F_{k-1}$.

Uniqueness of μ follows from Proposition 4.7 and by the next part of this theorem that μ is the largest fixed point of T^2.

Now, let ν_0 be the largest prematching in Φ'. Define $\nu_k = T\nu_{k-1}$ inductively. Now, ν_0 matches each agent in F_1 to F_1 since F_1 is their best choice. ν_1 might not match each agent in F_2 to F_2 since F_2 might not be their best overall choice but it still keeps agents in F_1 matched to F_1. However, ν_2 does match each agent in $f \in F_2$ to F_2 since each agent appearing in $\nu_1(f)$ is also an element of $C \cup S - F_1$ and $F_2 = Ch(C \cup S - F_1, P(f))$. It is easy to see with an inductive argument that $\nu_{2(i-1)}$ matches each agent in $F_1 \cup F_2 \ldots \cup F_i$ to its corresponding coalition for $i = 1, 2, \ldots, k$. Thus $\nu_{2(k-1)} = \mu$. Since μ is a stable* matching $\nu_{2k-1} = T\mu = \mu = \nu_{2(k-1)}$. Thus, we'll be able to get μ in at most in $2k-1$ iterations using T or equivalently in at most k iterations using T^2. $\square$

Example 8.3 shows that the weak top-coalition property is not necessary for the result in Theorem 8.2.

Example 8.3 *Let $S = \{s_1, s_2\}$ and $C = \{c_1, c_2\}$. Suppose that agents' preferences are:*

$$P(c_1) : S, \{s_1\}$$
$$P(c_2) : S, \{s_2\}$$
$$P(s_1) : (c_1, S), (c_2, S), (c_1, \{s_1\})$$
$$P(s_2) : (c_2, S), (c_2, \{s_2\}).$$

The following array shows the iterations of the algorithm.

	s_1	s_2	c_1	c_2
ν_0	$(\emptyset,\emptyset)$	$(\emptyset,\emptyset)$	$\emptyset$	$\emptyset$
$V(s,\nu_0)/U(c,\nu_0)$	$(c_2,S),(c_1,\{s_1\})$	$(c_2,S),(c_2,\{s_2\})$	$\{s_1\}$	$S,\{s_2\}$
$\nu_1 = T\nu_0$	(c_2,S)	(c_2,S)	$\{s_1\}$	S
$V(s,\nu_1)/U(c,\nu_1)$	(c_2,S)	(c_2,S)	$\emptyset$	S
$T^2\nu_0$	(c_2,S)	(c_2,S)	$\emptyset$	S

By Proposition 4.7, $T^2\nu_0$ is the unique core matching. The preferences in this example do not satisfy the weak top-coalition property.

8.2 Respecting preferences

We introduce a second restriction on preferences. The assumption is that the "projection" of any agent's preferences to either the set of colleges, or the sets of students, be the same. Agents can thus only differ in how they trade off different colleges and students. Under this restriction, the problem turns out to have the weak top-coalition property.

Definition 8.4 *A preference profile P is called respecting if there exist a preference relation P_S over 2^S and a preference relation P_C over $C \cup \emptyset$ with the following properties:*

(1) For all $s \in S$, $(c,\hat{S})P(s)(c,S')$ if and only if $\hat{S}P_S S'$.
(2) For all $s \in S$, $(c,\hat{S})P(s)(c',\hat{S})$ if and only if $cP_C c'$.
(3) For all $c \in C$, $\hat{S}P(c)S'$ if and only if $\hat{S}P_S S'$.
(4) For all $s \in S$, if $\emptyset P_C c$ then $(\emptyset,\emptyset)P(s)(c,\hat{S})$ for all $\hat{S} \subseteq S$.

Proposition 8.5 *If P is respecting then it satisfies the weak top-coalition property.*

Proof Let F_1 be the union of the top college in P_C with the top group of students in P_S. Clearly, every agent in F_1 prefers F_1 to any other coalition. Now, let F_2 be the union of the top college in P_C among the remaining colleges with the top group of students in P_S among the remaining group of students. Continue similarly until we exhaust all the colleges c such that $cP_C\emptyset$ or all the admissible groups of students S' such that $S'P_S\emptyset$. Then let each remaining agent be a coalition on its own. Assume that we have formed k coalitions. It is clear that $(F_1, F_2, ..., F_k)$ satisfies Definition 8.1. □

In view of Proposition 8.5, respecting preferences is sufficient for a unique core matching, and for the algorithm to find this core matching in relatively few iterations.

8.3 Monotonicity of T.

Now order Φ by $\nu' \trianglerighteq \nu$ if, for all c and s, $\nu'(c)R(c)\nu(c)$ and $\nu(s)R(s)\nu'(s)$. This is the order normally used in two-sided matching problems (see e.g. Adachi (2000); Fleiner (2003); Echenique and Oviedo (2004)).

The restriction on preferences we shall consider now is that preferences are such that T is monotone increasing, when Φ is ordered by $\nu' \trianglerighteq \nu$.

Proposition 8.6 *If T is monotone increasing with respect to $\trianglerighteq$, then $\mathcal{E}(T)$ is a non-empty lattice. In particular, $\mathcal{E}(T)$ has a smallest (in $\trianglerighteq$) element $\underline{\mu}$, and a largest element $\overline{\mu}$. These satisfy, for all c and s,*

$$\overline{\mu}(c)R(c)\overline{\nu}(c)R(c)\underline{\nu}(c)R(c)\underline{\mu}(c)$$
$$\underline{\mu}(s)R(s)\overline{\nu}(s)R(s)\underline{\nu}(s)R(s)\overline{\mu}(s),$$

where $\underline{\nu}$ and $\overline{\nu}$ were defined in Proposition 4.7.

Proof That $\mathcal{E}(T)$ is a non-empty lattice follows from Tarski's fixed-point theorem (Theorem 4.4).

There are smallest and largest prematchings, ν'_0 and ν'_1, in the order $\trianglerighteq$. So $\nu'_1 \trianglerighteq \underline{\nu} \trianglerighteq \nu'_0$ and $\nu'_1 \trianglerighteq \overline{\nu} \trianglerighteq \nu'_0$. The monotonicity of T implies that T^2 is monotone increasing in the order $\trianglerighteq$. Then, for any iteration k of T^2, we have

$$T^{2k}(\nu'_1) \trianglerighteq \underline{\nu} \trianglerighteq T^{2k}(\nu'_0)$$

and

$$T^{2k}(\nu'_1) \trianglerighteq \overline{\nu} \trianglerighteq T^{2k}(\nu'_0).$$

The result follows (similarly to the proof of Proposition 4.8) because there is an iteration k such that $T^{2k}(\nu'_1) = \overline{\mu}$ and $T^{2k}(\nu'_0) = \underline{\mu}$. □

As a consequence of Proposition 8.6, if T is monotone increasing, and there is a unique core matching μ, we have $\underline{\mu} = \overline{\mu} = \mu$. So $\underline{\nu} = \overline{\nu} = \mu$, and our algorithm finds the unique core matching in fewer steps than the algorithm of iterating T (called the T-algorithm in Echenique and Oviedo (2006))

8.4 Preference cycles.

We show that a type of preference cycle must be present every time a fixed point of T^2 is not a core matching. So absence of cycles is a useful restriction on preferences. There is nothing pathological about preference cycles, though.

Definition 8.7 *A matching problem $\langle C, S, P\rangle$ exhibits a preference cycle if there is a sequence $((c_1, S_1), (c_2, S_2), \dots (c_K, S_K))$ such that $(c_1, S_1) = (c_K, S_K)$ and, for all $k = 1, \dots K-1$, either $c_k = c_{k+1}$ and $S_{k+1}P(c_k)S_k$ or there is $s \in S_k \cap S_{k+1}$ such that $(c_{k+1}, S_{k+1})P(s)(c_k, S_k)$.*

Theorem 8.8 *Let μ be a matching. If $\mu = T^2\mu$ but $\mu \neq T\mu$, then $\langle C, S, P\rangle$ exhibits a preference cycle $((c_1, S_1), (c_2, S_2), \dots (c_K, S_K))$. Moreover, each (c_k, S_k) blocks* μ, $\mu(c_k) = \emptyset$ for all k and $\mu(s) = (\emptyset, \emptyset)$ for all $s \in \cup_k S_k$.*

Proof Let $\nu = T\mu$.

STEP 1 Let $\mu(c) \neq \nu(c)$. We shall prove that there is $s \in \nu(c)$ such that $\nu(s)P(s)(c, \nu(c))$, and that $\nu(s)$ blocks* μ. First, note that $\mu(c) \in U(c, \mu)$, as μ is a matching; this and $\nu = T\mu$ implies that $\nu(c)P(c)\mu(c)$. Now, $\mu(c) = (T\nu)(c)$, since $\mu = T^2\mu$. Then $\nu(c)P(c)\mu(c)$ implies $\nu(c) \notin U(c, \nu)$. By definition of $U(c, \nu)$ there must be $s \in \nu(c)$ such that $\nu(s)P(s)(c, \nu(c))$.

Further, $\nu(c) \in U(c, \mu)$ implies that $(c, \nu(c))R(s)\mu(s)$. So

$$\nu(s)P(s)(c, \nu(c))R(s)\mu(s).$$

Hence $\nu(s) \neq \mu(s)$. By Lemma 3.3, $\nu(s)$ blocks* μ.

STEP 2 Let $\mu(s) \neq \nu(s)$. We shall prove that there is a block* (c', S') of μ such that either (a) $c' = \nu(s)$ and $S'P(s)\nu(s)$ or (b) there is $\tilde{s} \in \nu(s)$ such that $\nu(\tilde{s})P(\tilde{s})\nu(s)$. First, that $\nu(s) \notin V(s, \nu)$ follows analogously to $(\mu(c) \neq \nu(c) \Rightarrow \nu(c) \notin U(c, \nu))$ above. The definition of $V(s, \nu)$ implies that either $\nu(\nu(s))P(\nu(s))\nu(s)$ or there is $\tilde{s} \in \nu(s)$ such that $\nu(\tilde{s})P(\tilde{s})\nu(s)$. Setting $(c', S') = (\nu(s), \nu(\nu(s)))$ in the first case, and $(c', S') = \nu(\tilde{s})$ in the second, proves the claim. That (c', S') is a block* follows applying Lemma 3.3 as in Step 1.

STEP 3 We shall construct a cycle. If there is c with $\mu(c) \neq \nu(c)$, let $(c_1, S_1) = (c, \nu(c))$. If there is s with $\mu(s) \neq \nu(s)$, let $(c_1, S_1) = \nu(s)$. Let $((c_1, S_1), (c_2, S_2), \dots (c_k, S_k))$ be a sequence that would be a preference cycle if $(c_1, S_1) = (c_k, S_k)$, and such that either (a) $(c_k, S_k) = (c_k, \nu(c_k))$ or (b) $(c_k, S_k) = \nu(s)$ for some $s \in S_k$.

In case (a), by Step 1, there is $s \in \nu(c_k)$ such that $\nu(s)P(s)(c_k, \nu(c_k))$. Let $(c_{k+1}, S_{k+1}) = \nu(s)$. Then $(c_{k+1}, S_{k+1}) \neq \mu(s)$, and (c_{k+1}, S_{k+1}) blocks* μ.

In case (b), by Step 2, either $\nu(c_k)P(c_k)S_k$ and $\nu(c_k)$ blocks* μ, or there is $\tilde{s} \in S_k$ such that $\nu(\tilde{s})P(\tilde{s})(c_k, S_k)$, and $\nu(\tilde{s}) \neq \mu(\tilde{s})$ is a block* of μ. Let $(c_{k+1}, S_{k+1}) = \nu(c_k)$, or $(c_{k+1}, S_{k+1}) = \nu(\tilde{s})$, respectively.

For each element (c_k, S_k) in the range of the resulting sequence, (c_k, S_k) is in the image of ν. There are finitely many elements in the image of ν, so there is some K such that $(c_1, S_1) = (c_K, S_K)$.

□

Corollary 8.9 *Let μ be a matching and $\mu \in \mathcal{E}(T^2)$. If preferences do not exhibit a preference cycle, then μ is a stable* matching.*

Proof If $\mu = T\mu$ then μ is stable* matching by Theorem 3.1. Otherwise, $\mu \neq T\mu$ which implies that $\langle C, S, P\rangle$ exhibits a preference cycle which is a contradiction. □

9 Extension: A model with Couples.

We present an extension of our model to a model with couples. The couples introduce a specific form of preferences over colleagues, but it does not reduce to the one we have discussed so far. We present a fixed-point construction, similar to the one above. One can thus use our algorithm to find the core matchings in the model with couples, if there are any. As a by-product, we obtain a result that may be of independent interest: we extend the classical result in the theory of many-to-one matchings, that under Kelso-Crawford substitutable preferences the core coincides with a less restrictive pair-wise stable solution. [3]

We now assume that there is a subset of students that form couples. So, for each student s in the subset that forms couples, there is one and only one student s' so that s forms a couple with s and s' forms a couple with s.

Split the set of students into two (disjoint) sets, Q and L such that if s forms a couple with s' they cannot both be in Q or both in L. Thus Q and L form a partition of S that splits all couples. Suppose now that we add a copy of the "singlehood" symbol $\emptyset$ to Q and L; in a convenient abuse of notation we shall refer to the different copies by the same label, $\emptyset$. In the sequel, s still denotes a generic element of $S = Q \cup L$, while q and l denote elements of Q and L, respectively.

[3] This is a solution to Open Problem 4 in Roth and Sotomayor (1990)

We extend the preferences $P(c)$ to preferences over $2^{Q\times L}$ by: $AP(c)B$, for $A, B \subseteq Q \times L$ if and only if

$$\{l : \exists q \text{ s.t. } (l,q) \in A\} \cup \{q : \exists l \text{ s.t. } (l,q) \in A\}\, P(c)\, \{l : \exists q \text{ s.t. } (l,q) \in B\} \cup \{q : \exists l \text{ s.t. } (l,q) \in B\}\,.$$

Note that we abuse notation, using $P(c)$ for the extension of c's original preferences.

Students $l \in L$ have preferences $P(l)$ over $C \times Q$, and $q \in Q$ has preferences $P(q)$ over $C \times L$.

A *prematching* is a function μ on $S \cup C$ such that

(1) $\mu(l) \in C \times Q$, if $l \in L$ and $l \neq \emptyset$;
(2) $\mu(q) \in C \times L$, if $q \in Q$ and $q \neq \emptyset$;
(3) $\mu(c) \subseteq L \times Q$, if $c \in C$.

Let $\mathcal{V}$ be the set of all prematchings.

A *matching* is a prematching such that, for all $(c, l, q) \in C \times L \times Q$,

$$\mu(c) \ni (l,q) \Rightarrow (l \neq \emptyset \Rightarrow \mu(l) = (c,q)) \wedge (q \neq \emptyset \Rightarrow \mu(q) = (c,l))$$
$$\mu(l) = (c,q) \Rightarrow (\mu(c) \ni (l,q)) \wedge (q \neq \emptyset \Rightarrow \mu(q) = (c,l))$$
$$\mu(q) = (c,l) \Rightarrow (\mu(c) \ni (l,q)) \wedge (l \neq \emptyset \Rightarrow \mu(l) = (c,q))$$

9.1 Stability.

A matching μ is *individually rational* if, for all c, l, q, $\mu(c) = Ch(\mu(c), P(c))$, $\mu(l)P(l)(\emptyset, \emptyset)$ and $\mu(q)P(q)(\emptyset, \emptyset)$.

Let μ be a matching. A pair $(c, (l, q))$ is a *couples-block* of μ if there is some $A \subseteq \mu(c)$ such that

(1) $A \cup \{(l,q)\}\, P(c)\mu(c)$
(2) $l \neq \emptyset \Rightarrow (c,q)P(l)\mu(l)$
(3) $p \neq \emptyset \Rightarrow (c,l)P(q)\mu(q)$

Note that definition of a couples-block includes the possibility that $(q, \emptyset) \in \mu(c)$ and that $(c, (l, q))$ blocks μ.

A matching is *couples-stable* if it is individually rational and has no couples blocks. Denote the set of all couples-stable matchings by $S^c(P)$.

A pair (D, c) where $D \subseteq Q \times L$ and $c \in C$ is a block* of a matching μ if $DR(c)\mu(c)$, for all $(l,q) \in D$ $(c,q)R(l)\mu(l)$ and $(c,l)R(q)\mu(q)$, and if one of the stated relations holds with P in place of R.

We state here without proof that the core—denoted $C_W(P)$—is the set of matchings for which there is no block*. The proof is very similar to the proof of Lemma 3.2.

9.2 Fixed-point construction.

To ease notation, when $l = \emptyset$ say that $(c,q)P(l)(c',q')$ holds by definition. Similarly for $q = \emptyset$.

$$\begin{aligned} V(l,\nu) &= \{(c,q) : (l,q) \in Ch(\nu(c) \cup \{(l,q)\}, P(c)) \\ &\quad \text{and } (c,l)R(q)\nu(q)\} \cup \{(\emptyset,\emptyset)\} \\ W(q,\nu) &= \{(c,l) : (l,q) \in Ch(\nu(c) \cup \{(l,q)\}, P(c)) \\ &\quad \text{and } (c,q)R(l)\nu(l)\} \cup \{(\emptyset,\emptyset)\} \\ U(c,\nu) &= \{(l,q) : (c,q)R(l)\nu(l)(c,l)R(q)\nu(q)\} \cup \{(\emptyset,\emptyset)\} \end{aligned}$$

Now let $T : \mathcal{V} \to \mathcal{V}$ be defined by letting $T\nu(c) = Ch(U(c,\nu), P(c))$, and $T\nu(s)$ be the maximal element in $V(s,\nu)$ if $s \in L$ and in $W(q,\nu)$ if $q \in Q$.

Denote the set of fixed points of T by $\mathcal{E}(T)$.

9.3 Results.

Lemma 9.1 *If $\mu \in \mathcal{E}(T)$ then μ is individually rational.*

Lemma 9.1 follows immediately from the definition of the map T.

Proposition 9.2 $\mathcal{E}(T) \subseteq C_W(P) \subseteq S^c((P)$

Proof That $C_W(P) \subseteq S^c(P)$ is immediate. Let $\mu \in \mathcal{E}(T)$ and suppose, by way of contradiction, that $\mu \notin C_W(P)$. Let (D,c) be a block* of μ. By definition of a block*, $D \subseteq U(c,\mu)$. So $\mu = T\mu$ implies that $\mu(c)R(c)D$. But $DR(c)\mu(c)$, since (D,c) is a block*.

Now, $DR(c)\mu(c)$ and $\mu(c)R(c)D$ implies that, for all $(l,q) \in D$, $\mu(l)R(l)(c,q)$ and $\mu(q)R(q)(c,l)$. This is a contradiction with (D,c) being a block*. $\square$

Let $c \in C$. Say that $P(c)$ is *substitutable* if, for any $A, B \subseteq Q \times L$, if $(q,l) \in A \subseteq B$, and $(q,l) \in Ch(B, P(c))$ then $(q,l) \in Ch(A, P(c))$. Say that a profile $(P(c))_{c \in C}$ is substitutable if each individual $P(c)$ is substitutable.

Proposition 9.3 *If $(P(c))_{c\in C}$ is substitutable, then $S^c(P) = \mathcal{E}(T)$.*

Proposition 9.3 translates into

Corollary 9.4 *If $(P(c))_{c\in C}$ is substitutable, then $S^c(P) = C_W(P)$.*

Proof [Proof of Proposition 9.3] We need to prove that $S^c(P) \subseteq \mathcal{E}(T)$. Let $\mu \neq T\mu$. We shall prove that $\mu \notin S^c(P)$.

First, suppose there is c such that $T\mu(c) \neq \mu(c)$. Let $D = Ch(U(c,\mu), P(c)) \neq \mu(c)$. Since $\mu(c) \subseteq U(c,\mu)$, because μ is a matching, $DP(c)\mu(c)$. Since μ is individually rational, $\mu(c) = Ch(\mu(c), P(c))$. So we have that $D \nsubseteq \mu(c)$, and hence that there is $(l,q) \in D \backslash \mu(c)$. We shall prove that $(c,(l,q))$ is a couples block of μ. Now,

$$(l,q) \in Ch(U(c,\mu), P(c)) = Ch(U(c,\mu) \cup \{(l,q)\}, P(c)),$$

and substitutability of $P(c)$ implies that $(l,q) \in Ch(\mu(c) \cup \{(l,q)\}, P(c))$, as $\mu(c) \subseteq U(c,\mu)$. Let $A = Ch(\mu(c) \cup \{(l,q)\}, P(c)) \cap \mu(c)$. By definition of A, it satisfies (1) in the definition of a couples block.

We now verify (2) and (3) in the definition of a couples block. If $l \neq \emptyset$ then $\mu(l) = (c',q')$ with $c \neq c'$, as μ is a matching and $(l,q) \notin \mu(c)$. But $(l,q) \in D \subseteq U(c,\mu)$ so $(c,q)R(l)\mu(l) = (c',q')$. Preferences are strict, so $c \neq c'$ implies $(c,q)P(l)\mu(l)$. By the same argument (3) follows.

Second, suppose that there is l such that $T\mu(l) \neq \mu(l)$. Let $(c,q) = T\mu(l)$. We shall prove that $(c,(l,q))$ is a couples block of μ. That μ is a matching implies $\mu(l) \in V(l,\mu)$. So the definition of T gives $(c,q)P(l)\mu(l)$, requirement (2) in the definition of a couples block.

That $(c,q) \in V(l,\mu)$ implies that

$$(l,q) \in Ch(\mu(c) \cup \{(l,q)\}, P(c)) \quad (5)$$

$$(c,l)R(q)\mu(q) \quad (6)$$

Since μ is a matching, $(c,q) \neq \mu(l)$ implies $(l,q) \notin \mu(c)$, so Statement (5) implies $Ch(\mu(c) \cup \{(l,q)\}, P(c))P(c)\mu(c)$. Let $A = Ch(\mu(c) \cup \{(l,q)\}, P(c)) \setminus \{(l,q)\}$; A satisfies (1) in the definition of a couples block.

Finally, Statement (6) and that $(c,q) \neq \mu(l)$ implies (3) in the definition of a couples block. □

A Appendix: Proof of Lemma 3.2 and Weaker notions of stability.

We prove Lemma 3.2 in the text and discuss briefly pairwise stability and $S^*(P)$.

Proof [Proof of Lemma 3.2] First we shall show $C_W(P) \subseteq S^*(P)$ and then $S^*(P) \subseteq C_W(P)$ to complete the proof.

Let $\mu \in C_W(P)$. Since for no $\hat{\mu}$, $\langle \{c\}, \emptyset, \hat{\mu} \rangle$ or $\langle \emptyset, \{s\}, \hat{\mu} \rangle$ blocks μ, μ is individually rational. Moreover, since for no $S' \subseteq S$ and $\hat{\mu}$, $\langle \{c\}, S', \hat{\mu} \rangle$ blocks μ, μ is a stable* matching.

Now we show that $S^*(P) \subseteq C_W(P)$ by contradiction. Assume that there exists a matching μ such that $\mu \in S^*(P)$ and $\mu \notin C_W(P)$. Hence, there exists a coalition $S' \cup C'$ and $\hat{\mu}$ that satisfy the definition of a block. Therefore, there exists $f \in S' \cup C'$ so that $\hat{\mu}(f)P(f)\mu(f)$. We split this into two cases:

Case 1. $f \in C$. Then, for all $s \in \hat{\mu}(f), \hat{\mu}(s)R(s)\mu(s)$ by construction of $\hat{\mu}$. Since $\hat{\mu}(f)P(f)\mu(f)$, $\hat{\mu}(f) \neq \mu(f)$ which implies that $\hat{\mu}(s) \neq \mu(s)$ for all $s \in \hat{\mu}(f)$. Since preferences are strict, we get that for all $s \in \hat{\mu}(f), \hat{\mu}(s)P(s)\mu(s)$. Therefore, if we let $B = \hat{\mu}(f) - \mu(f)$ then (B, f) blocks* μ with $A = \hat{\mu}(f) \cap \mu(f)$, a contradiction to stability* of μ.

Case 2. $f \in S$. Since μ is a stable* matching, it is individually rational. Thus, $\mu(f)R(f)(\emptyset, \emptyset)$ which implies together with strictness of $P(f)$ and $\hat{\mu}(f)P(f)\mu(f)$ that $\hat{\mu}(f)P(f)(\emptyset, \emptyset)$. Hence, $\hat{\mu}(f) = (c, \hat{\mu}(c))$ for some college c. Moreover, $\{c\} \cup \hat{\mu}(c) \in C' \cup S'$ and $\hat{\mu}$ is at least as good as μ for all agents in $C' \cup S'$. Since $\hat{\mu}(f)P(f)\mu(f)$, $\hat{\mu}(f) \neq \mu(f)$. Therefore, $\hat{\mu}(s) \neq \mu(s)$ for all students $s \in \hat{\mu}(c)$ and also $\hat{\mu}(c) \neq \mu(c)$. Now, let $B = \hat{\mu}(c) - \mu(c)$. Hence, the matching $\hat{\mu}$ is better for all students in $\hat{\mu}(c)$ and also for college c. We get that (B, c) blocks* μ with $A = \hat{\mu}(c) \cap \mu(c)$. A contradiction to stability* of μ. □

Pairwise stability has been studied widely in many-to-one matchings without preferences over colleagues. The following is the adaptation of pairwise stability to our model:

A pair $(s, c) \in S \times C$ is a *pairwise-block* of a matching μ if $s \notin \mu(c)$, but $s \in Ch(\{s\} \cup \mu(c), P(c))$ and $\forall s' \in Ch(\{s\} \cup \mu(c), P(c))$,

$$(c, Ch(\{s\} \cup \mu(c), P(c)))P(s')\mu(s').$$

A matching μ is *stable* if it is individually rational and there does not exist a pairwise block of μ.

Notation. Given a preference profile P, we denote the set of stable matchings by $S(P)$.

The following simple proposition and example show that the core is smaller than the set of pairwise stable matchings, and may be strictly smaller. In many-to-one models without preferences over colleagues, the two solutions coincide if colleges' preferences are substitutable (Roth and Sotomayor, 1990).

Proposition A.1 $S^*(P) \subseteq S(P)$

Proof Let $\mu \in S^*(P)$. Assume that $\mu \notin S(P)$. Since μ is individually rational there must be a blocking pair (s, c). Hence, $\exists s \in S$ such that $s \notin \mu(c)$, but $s \in Ch(\{s\} \cup \mu(c), P(c))$ and $\forall s' \in Ch(\{s\} \cup \mu(c), P(c)), (c, Ch(\{s\} \cup \mu(c), P(c)))P(s')\mu(s')$. Therefore, $(\{s\}, c)$ blocks* μ with $A = Ch(\{s\} \cup \mu(c), P(c)) - \{s\}$. Contradiction to stability*. □

In general, $S^*(P)$ can be different from $S(P)$; we make this point through an example.

Example A.2 *Consider three colleges $C = \{c_1, c_2, c_3\}$ and three students $S = \{s_1, s_2, s_3\}$ with the following preferences:*

$$P(c_1) : \{s_2, s_3\}, \{s_2, s_1\}, \{s_1, s_3\}, \{s_1\}, \{s_2\}, \{s_3\}$$
$$P(c_2) : \{s_2\}$$
$$P(c_3) : \{s_3\}$$
$$P(s_1) : (c_1, s_1), (c_1, \{s_1, s_3\}), (c_1, \{s_2, s_1\})$$
$$P(s_2) : (c_1, \{s_2, s_3\}), (c_2, \{s_2\}), (c_1, \{s_2\}), (c_1, \{s_1, s_2\})$$
$$P(s_3) : (c_1, \{s_2, s_3\}), (c_3, \{s_3\}).$$

There is only one stable matching μ_1 which is $\mu_1(c_1) = \{s_2, s_3\}$ and $\mu_1(c_2) = \emptyset$ but there is another matching μ_2 which is stable and given by $\mu_2(c_1) = \{s_1\}$, $\mu_2(c_2) = \{s_2\}$ and $\mu_2(c_3) = \{s_3\}$.*

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