Published: Econometrica, Vol. 77 Issue 4, July 2009, pp 1281–1298

TESTING MODELS WITH MULTIPLE EQUILIBRIA BY QUANTILE METHODS

FEDERICO ECHENIQUE AND IVANA KOMUNJER

Abstract. This paper proposes a method for testing complementarities between explanatory and dependent variables in a large class of economic models. The proposed test is based on the monotone comparative statics (MCS) property of equilibria. Our main result is that MCS produces testable implications on the (small and large) quantiles of the dependent variable, despite the presence of multiple equilibria. The key features of our approach are: (1) we work with a nonparametric structural model of a continuous dependent variable in which the unobservable is allowed to be correlated with the explanatory variable in a reasonably general way; (2) we do not require the structural function to be known or estimable; (3) we remain fairly agnostic on how an equilibrium is selected. We illustrate the usefulness of our result for policy evaluation within Berry, Levinsohn, and Pakes's (AER, 1999) model.

Affiliations and contact information. Echenique: Division of the Humanities and Social Sciences, California Institute of Technology, `fede@caltech.edu`. Komunjer: Department of Economics, University of California at San Diego, `komunjer@ucsd.edu`.

Acknowledgments: Many thanks to Rabah Amir, Kate Antonovics, Andrés Aradillas-López, Patrick Bajari, Peter Bossaerts, Colin Camerer, Chris Chambers, Xiaohong Chen, Valentino Dardanoni, Han Hong, Joel Horowitz, Matt Jackson, Michael Jansson, Chuck Manski, Andrea Mattozzi, Rosa Matzkin, Tom Palfrey, Jim Powell, Andrea Rotnitzky, Paul Ruud, Max Stinchcombe, Andrew Sweeting, Elie Tamer, Xavier Vives, Quang Vuong, Hal White, Bill Zame, as well as participants at presentations given at Northwestern University, UC Berkeley, UC Davis, and the North American ES summer meeting, the EEE2005 conference, and the 2005 ESSET workshop. Echenique thanks the NSF for its support through award SES-0751980. Anonymous referees and a co-editor provided excellent suggestions and comments that greatly improved an earlier version of the paper. All remaining errors are ours.

1. Introduction

In many conventional economic models, equilibrium uniqueness comes at a cost of strong and often untenable assumptions. Consider, for example, general equilibrium models: the uniqueness conditions with some natural economic meaning imply the strong weak axiom, which in turn cannot be expected to hold beyond single-agent economies (Arrow and Hahn, 1971). Therefore it is not surprising to find equilibrium multiplicity present in a variety of contexts, ranging from general equilibrium models in microeconomics, oligopoly models and network externalities in industrial organization, to non-convex growth models in macroeconomics or models of statistical discrimination in labor economics.

Performing comparative statics with multiple equilibria is a challenge. How changes in explanatory variables affect dependent variables depends on the way a particular equilibrium is selected. Unfortunately, the theoretical literature offers little guidance on equilibrium selection.[1] As a consequence, policy analysis seems impossible as policy effects may well vary across different equilibria. More to the point, without equilibrium selection, it is hard to identify the structure underlying economic models when multiple equilibria are present. And with no knowledge of the structure, we can say little about general comparative statics effects. We should emphasize that we are concerned with testing for the existence of a comparative statics effect; the counterfactual prediction of the effects of policies remains virtually impossible without substantial information about equilibrium selection.

In this paper, we restrict our attention to economic models that exhibit *complementarities* between explanatory and dependent variables. In such models, despite the possible presence of multiple equilibria, a monotone comparative statics (MCS) prediction holds: there is a smallest and a largest equilibrium, and these change monotonically with explanatory variables (Milgrom and Roberts, 1994; Villas-Boas,

[1]Consider Kreps (1990), for example: "*There are … lots of Nash equilibria to this game. Which one is the 'solution'? I have no idea and, more to the point, game theory isn't any help. Some (important) sorts of games have many equilibria, and the theory is of no help in sorting out whether any one is the 'solution' and, if one is, which one is.*"

1997). The paper's main contribution is to show how MCS arguments translate into observable restrictions on the conditional quantiles of the dependent variable.

Our framework is as follows: similar to Jovanovic (1989), we start with an underlying economic model relating dependent and explanatory variables. We disturb the model by adding an unobservable disturbance term that captures individual heterogeneity, or other unaccounted random features. The assumptions we impose on the resulting structure are fairly weak: we allow for unknown structural function, unknown equilibrium selection, and reasonably general correlation between the disturbance and the explanatory variable. Our main result is that MCS produces testable implications on the (small and large) quantiles of the dependent variable.[2] The result does not assume, nor require estimating, an equilibrium selection procedure.[3]

The intuition behind is fairly simple. Consider a model in which there are complementarities between explanatory and dependent variables. When the generated equilibrium is unique, then the model can be globally implicitly solved and the resulting reduced form is such that the dependent variable increases in the explanatory variable. This property translates into first order stochastic dominance among distributions: *all* conditional quantiles of the dependent variable are increasing functions of the explanatory variable. When the model generates multiple equilibria, the above implicit function arguments fail to hold globally. It remains, however, the MCS property of the extremal equilibria. By focusing on regions in which the monotonicity of equilibria holds, we still obtain that *tail* (small and large) conditional quantiles of the endogenous variable increase in the explanatory variable. Testing for complementarities is thus possible by examining the behavior of extreme conditional tails of the dependent variable.

Our method applies to a large class of economic models with continuous dependent variables. These are: models of individual decision making in which the equilibrium

[2]An early test for MCS can be found in Athey and Stern (1998) in the context of firms' choice of organizational form. This prior work, however, does not address equilibrium problems.

[3]Understandably, estimating the structural parameters requires additional parametric assumptions on the equilibrium selection.

values are the solutions of an extremum problem, and one-dimensional equilibrium models where equilibria are fixed points. Since the dependent variable is continuous, our findings complement those developed by the growing literature on discrete games with multiple equilibria (Bresnahan and Reiss, 1990, 1991; Berry, 1992; Tamer, 2003; Ciliberto and Tamer, 2004; Aguirregabiria and Mira, 2007).[4]

The next section discusses equilibrium multiplicity in Berry, Levinsohn, and Pakes's (1999) influential empirical model of price-setting with differentiated products. In Section 3 we introduce the setup, and present our results. We conclude in Section 4 with a discussion and possible extensions of our approach.

2. Example

We now present a simplified version of Berry, Levinsohn, and Pakes's (1995) model of price competition with differentiated products. We use this model for two purposes: first, to illustrate the challenges posed by equilibrium multiplicity, even in popular and well-behaved economic models. Second, to argue that our methods provide useful tools for policy analysis in these models. Concretely, we discuss the analysis of the Japanese "Voluntary Export Restraint" (VER) policy for automobile exports published in Berry, Levinsohn, and Pakes (1999) (BLP hereafter).

In our version of the BLP model there are two firms, each producing one good. Firm 1 is foreign and Firm 2 is a home firm. Following BLP, we model the VERs as (tax) increases in firms' marginal costs. Firm i sets the price p_i of its product and obtains profits $\Pi_i(p_i, p_{-i}, VER) = (p_i - c_i - \lambda \mathrm{VER}_i) D_i(p_i, p_{-i})$, where $D_i(p_i, p_{-i})$ is the demand for Firm i's good when its competitor sets a price p_{-i}, c_i is i's marginal cost, VER_i is a dummy variable for the VER, and λ is the corresponding tax per unit of i's production. The firms are assumed to choose prices $p_i^* \in [0, \bar{p}\,]$ which—given the price p_{-i} set by their competitor—maximize their profits: $p_i^* = \mathrm{argmax}_{p_i \in [\,0, \bar{p}\,]} \Pi_i(p_i, p_{-i}, \mathrm{VER}_i)$. Only Firm 1 (the foreign firm) is subject to the VER and we denote by $\mathrm{VER} = \mathrm{VER}_1$ the VER dummy.

[4]Unlike in these papers, however, our methods can only be applied to discuss comparative statics effects, and are silent about other structural features of the model.

Let $p_1^* = \beta_1^c(p_2, \mathrm{VER})$ and $p_2^* = \beta_2^c(p_1, 0)$ denote the best responses (reactions) of Firms 1 and 2 as determined by profit maximization. In our application, the existence of maximizers p_i^* directly follows from the continuity of Π_i and compactness of $[0, \bar{p}\,]$; in general, however, the maximizers need not be unique so we allow the best responses β_i^c to be correspondences (or set-valued functions). Hereafter, we denote β_i the best response function that Firm i selects from β_i^c; by the maximum theorem we can take β_i to be continuous.

2.1. **BLP Model with Multiple Equilibria.** We focus our analysis on the composed best response function for foreign firm (Firm 1): $\beta_1(\beta_2(p_1, 0), \mathrm{VER})$. The equilibrium price $p = p_1^*$ set by Firm 1 is determined by the fixed-point condition:

$$\beta_1\left(\beta_2(p, 0), \mathrm{VER}\right) - p = 0. \tag{1}$$

Without additional restrictions on the demand functions, a solution p to the above equilibrium condition need not be unique. Not only are the known conditions for uniqueness very strong (Gabay and Moulin, 1980; Caplin and Nalebuff, 1991) there is a sense in which games generally tend to have large numbers of equilibria. In a model of randomly generated games, McLennan (2005) shows that the mean number of equilibria grows exponentially with the number of strategies. Games of strategic complements, which are especially relevant for our paper, tend to have particularly large numbers of equilibria (Takahashi, 2005).

As pointed out by Berry, Levinsohn, and Pakes (1995, 1999), the BLP model in particular is not guaranteed to have a unique equilibrium. For example, Milgrom and Roberts (1990) establish uniqueness for the linear demand, CES, logit, or translog models (under additional parameter constraints). Simple departures from these models result in multiple equilibria for p. Echenique and Komunjer (2007b) make this point in the context of a logit model with conditional heteroskedasticity.

We now argue that, despite equilibrium multiplicity, it is still possible to evaluate the impact of the VER on the prices in Equation (1). If VER and prices are complements, then any selection from β_1^c is monotone increasing in VER. To

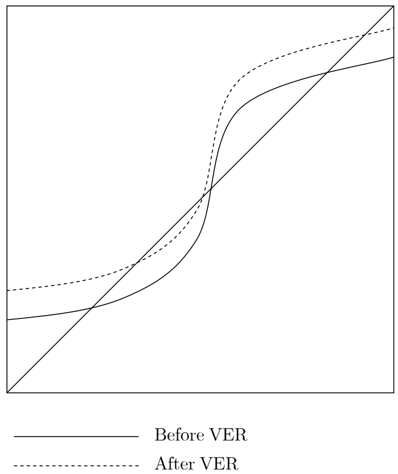

Figure 1: Monotone Comparative Statics of Equilibria.

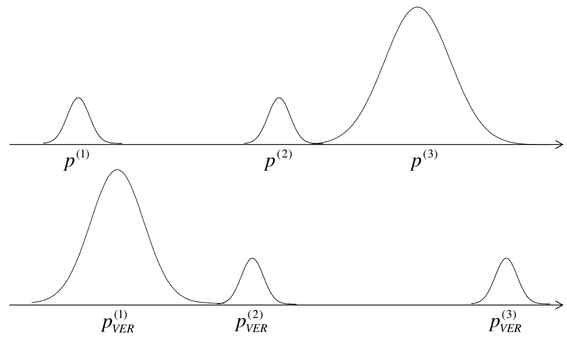

Figure 2: Firm 1 equilibria are $\{p^{(1)}, p^{(2)}, p^{(3)}\}$ before and $\{p^{(1)}_{\mathrm{VER}}, p^{(2)}_{\mathrm{VER}}, p^{(3)}_{\mathrm{VER}}\}$ after VER.

see this, note that $\partial^2\Pi_1(p_1, p_2, \mathrm{VER})/\partial p_1\,\partial\mathrm{VER} = -\lambda\partial D_1(p_1, p_2)/\partial p_1$ has the same sign as λ provided the demand functions D_i are everywhere decreasing in p_i, i.e. $\partial D_i(p_i, p_{-i})/\partial p_i < 0$. When $\lambda > 0$, Π_1 has strictly increasing differences (is strictly supermodular) in (p_1, VER). Every selection from the best response $p_1^* = \beta_1^c(p_2, \mathrm{VER})$ is then monotone increasing in VER (by Milgrom and Shannon's (1994) monotone selection theorem). Hence, the comparative statics effects take the form of a monotone comparative statics (MCS) prediction: if the implicit tax on exports λ is positive, then the VER will cause the extremal price equilibria to increase. Figure 1 illustrates this effect. Evaluating the price impact of VER in the BLP framework is then equivalent to testing for the presence of MCS.

There are two difficulties in testing for MCS. First, we need to work with a stochastic version of the model (1). In presence of unobserved heterogeneity, the observed prices will no longer be discretely distributed over equilibrium sets. Instead, they will have mixture distributions (in Section 3 we show how such mixtures arise). Since we allow for unknown equilibrium selection, the exact probabilities in the mixture remain unknown. This is the source of the second difficulty: certain equilibrium selections may induce smaller observations with the VER than without it thus working against the positive effect of the VER. For example, assume that the price of Firm 1's product is like the one in Figure 2. Here, the average foreign car price *decreases* in presence of the VER. This simple example shows that MCS has no general implications on the conditional mean of prices given VER. All we have to work with is the MCS property: that in the presence of the VER the smallest equilibrium and the largest equilibrium have increased.

2.2. **Observable Implications of MCS.** Write $y \equiv \ln p$, $x \equiv$ VER and let $r(y,x) \equiv \ln\big(\beta_1\left(\beta_2(\exp y, 0), x\right)\big) - y$. Then, the equilibrium values for y solve the equation:

$$r(y,x) = 0. \tag{2}$$

We first obtain a stochastic version of the model based on the equilibrium condition (2). Let Y be a scalar random variable whose realizations correspond to the log-prices $y = \ln p$, let X be the VER dummy with realizations x. Consider the structural model $r(Y,X) = U$, in which U is a scalar disturbance term. Different realizations of U induce values of Y that deviate from the exact equilibrium condition. The continuity and limit behavior of r (which we discuss in Section 3.1) guarantee that the disturbed equilibrium condition $r(y,x) = u$ always has at least one solution.

We now study the comparative statics effects on the log-prices Y following the introduction of the VER based on the model $r(Y,X) = U$. For this, first note that, since β_1 is increasing in x, so is r. This property (which we later label Assumption S2) states that X and Y are complements. The existence of complementarities between the VER and the log-prices is the starting point of our comparative statics analysis.

Our main result provides simple conditions under which the effect of extremal equilibria will prevail *for large (and small)* values of the dependent variable. The conditions are simple and do not restrict the equilibrium selection rule. When they are satisfied, the effect of VER on extremal equilibria translates into testable implications on some large (and small) enough quantile of the distribution of log-prices.

The conditions (found in Assumptions S3/S3') restrict the dependence of U on X, and the tail behavior of U. The first of those properties is an identification condition:

we use it to prevent the variations in U from exactly canceling out with an increase in X. This condition is easily satisfied if U and X are independent, for example. Independence is, however, stronger than needed; Section 4.1.2 contains examples in which U and X are correlated in a reasonably general way. The role of the second condition is to ensure that any increase in X eventually translates into an increase of large enough conditional quantiles of Y given X. Its key feature is to only restrict the distribution of U given X without placing assumptions on the equilibrium selection procedure. Broadly speaking, this condition rules out heavy-tailed distributions; Section 3.2 contains a detailed discussion on the distributions that we allow for.

3. Structural Model and Results

3.1. **Structure.** We consider a structural equation given by:

$$r(Y, X) = U, \tag{3}$$

where $r : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is specified by economic theory.[5] The variables that enter the structural model in (3) are: a dependent variable $Y \in \mathbb{R}$, an explanatory variable $X \in \mathcal{X} \subseteq \mathbb{R}$, and a disturbance to the system $U \in \mathbb{R}$. When the structural function r is parameterized by a finite dimensional parameter θ in Θ, one can write $r(Y, X, \theta) = U$ in Equation (3). We assume that X and Y are observable, but U is not; U can be thought of as unaccounted heterogeneity in the model.

We have in mind the structural equations derived from two classes of economic models. One class predicts equilibrium values y based on a first-order condition $r(y, x) = 0$; these are single-person decision models, such as models solved by a social planner. A second class predicts equilibrium values y based on a fixed-point condition $r(y, x) = \rho(y, x) - y = 0$, as in the BLP model in Section 2.

Given the function r, the structural econometric model is built by introducing the disturbance term U in the underlying economic model. Different realizations of U

[5] As in Matzkin (1994, 2005), we consider structural equations in which U is additively separable. The methods developed here are not suited for the non-separable problem $\tilde{r}(Y, X, U) = 0$. In such cases the framework in Echenique and Komunjer (2007a) may still be applied.

induce values of Y that deviate from the equilibria predicted by the economic model. The disturbance U has a clear interpretation as the extent to which a realized Y violates the exact (undisturbed) equilibrium condition.

When Equation (3) determines Y as a function $Y = m(X, U)$, the distribution of the disturbance U conditional on the explanatory variable X, denoted $F_{U|X}$, determines unambiguously the conditional distribution of Y, denoted $F_{Y|X}$. We say that $F_{Y|X}$ is generated by the *structure* $S = (r, F_{U|X})$. On the other hand, when Equation (3) has multiple solutions, a complete specification of the structure must include a rule that selects a particular realization y from the set of solutions. Such an *equilibrium-selection rule* can depend on the realized values of X and U.

We assume that for any x in the support $\mathcal{X}$ of X, $F_{U|X=x}$ has a strictly positive density $f_{U|X=x}$ on $\mathbb{R}$. The variable X can be discrete or continuous. Let the equilibrium set as the set of solutions to (3) when $X = x$ and $U = u$: let $(x, u) \in \mathcal{X} \times \mathbb{R}$ and $\mathcal{E}_{xu} = \{y \in \mathbb{R} : r(y, x) = u\}$. We work with the following assumption.

Assumption S1. (i) The function $r : \mathbb{R} \times \mathcal{X} \to \mathbb{R}$ is continuous; (ii) for any $x \in \mathcal{X}$, $\lim_{y\to-\infty} r(y, x) = +\infty$ and $\lim_{y\to+\infty} r(y, x) = -\infty$; (iii) for any $(x, u) \in \mathcal{X} \times \mathbb{R}$, $\mathcal{E}_{xu}$ is a finite set. We write $\mathcal{E}_{xu} = \{\xi_{1xu}, ..., \xi_{n_x xu}\}$ $(\xi_{1xu} \leqslant \ldots \leqslant \xi_{n_x xu})$ with $n_x = \mathrm{Card}(\mathcal{E}_{xu})$.

Assumptions S1.i and S1.ii are standard. S1.ii is akin to an Inada condition; in particular, S1.i and S1.ii imply, by the Intermediate Value Theorem, that a solution to Equation (3) always exists. Assumption S1.iii requires r not to be constant over any subintervals. By using suitable arguments from differential topology, S1.iii can be shown to hold generically (see Mas-Colell, Whinston, and Green (1995) for examples of these arguments). That the number of equilibria only depends on the explanatory variable X is not a serious restriction; it can simply be satisfied by duplicating elements of the equilibrium set until its cardinality no longer depends on U.

We now show that the BLP example in Section 2 satisfies all our assumptions. Given that the best-response functions β_i are continuous, so is r (Assumption S1.i). That the demand functions D_i are everywhere positive implies the limit conditions on r in S1.ii.[6] Finally, when the distribution of U is absolutely continuous, the set of solutions to $r(y,x)=u$ will be finite with probability 1 (Assumption S1.iii).

We specify the selection rule as follows: let $\mathcal{P}_{xu}$ be a probability distribution over $\mathcal{E}_{xu}$, which assigns probabilities $\{\pi_{1x},\ldots,\pi_{n_x x}\}$ to outcomes $\{\xi_{1xu},\ldots,\xi_{n_x xu}\}$, such that $\pi_{1x}>0$ and $\pi_{n_x x}>0$. For a given x, different realizations u can affect the support of $\mathcal{P}_{xu}$, but not the probabilities assigned to different outcomes in the support. For example, $\mathcal{P}_{xu}$ might assign equal probabilities across all elements of $\mathcal{E}_{xu}$. The conditional distribution of Y is then obtained as follows.

Proposition 1. *Assume S1 holds, and fix a selection rule $\mathcal{P}_{XU}$. Then, for any $x\in\mathcal{X}$ there are distribution functions $F_{iY|X=x}(y)=\int_{-\infty}^{+\infty}1\!\!1(\xi_{ixu}\leqslant y)f_{U|X=x}(u)du$, for $1\leqslant i\leqslant n_x$, such that, $j\geqslant i$ implies that $F_{jY|X=x}$ first-order stochastically dominates $F_{iY|X=x}$. And, for any $y\in\mathbb{R}$, $F_{Y|X=x}(y)=\sum_{i=1}^{n_x}\pi_{ix}F_{iY|X=x}(y)$.*

When multiple equilibria exist, $F_{Y|X}$ is generated by the structure $S=(r,F_{U|X},\mathcal{P}_{XU})$, which now includes the additional element $\mathcal{P}_{XU}$. Proposition 1 shows that under S the conditional distribution of the dependent variable has a mixture form. When equilibrium is unique, i.e. when the structural function r is monotone decreasing, the results of Proposition 1 reduce to the usual expression of the image distribution $F_{Y|X}$ of Y given X: $\bar{F}_{Y|X=x}(y)=F_{U|X=x}(r(y,x))$, where we use $\bar{F}_{Y|X}\equiv 1-F_{Y|X}$ to denote the conditional distribution tail of Y.

In general, the structure S may not be known. We work with a class of structures that share a qualitative feature: they exhibit *complementarities* between the explanatory variable X and the dependent variable Y.

Assumption S2. $r(y,x)$ is monotone increasing in x on $\mathbb{R}$.

Assumption S2 says that X and Y are complements. Such complementarity usually follows from a supermodularity property of the primitive model. An example is

[6] Note that, if $D_i(p_i,p)>0$ for all (p_i,p_{-i}) then, $\hat{p}_i<c_i+\lambda\mathrm{VER}_i$ implies that $\Pi_i(\hat{p}_i,p_{-i},VER)<0=\Pi_i(c_i+\lambda\mathrm{VER}_i,p_{-i},VER)$. So $\beta_i(p_{-i},\mathrm{VER}_i)\geqslant c_i+\lambda\mathrm{VER}_i$, for all values of p_{-i}. As $y\to-\infty$, $\beta_1(\beta_2(\exp y,0),\mathrm{VER})\to\beta_1(\beta_2(0,0),\mathrm{VER})\geqslant c_1$ and so $r(y,x)\to+\infty$. Moreover, for any value of p_{-i}, $\beta_i(p_{-i},\mathrm{VER}_i)\leqslant\bar{p}$. So $\beta_1(\beta_2(\exp y,0),\mathrm{VER})$ remains bounded as $y\to+\infty$ and $r(y,x)\to-\infty$.

the BLP model in Section 2, in which Assumption S2 follows from the strict supermodularity of firms' profits in their own prices and the VER. The key feature of S2 is that it implies the MCS property: the extremal equilibria of $r(y, x) = 0$ increase with x (Figure 1). We now review briefly some of the many economic models that fall into our framework.

3.1.1. *Individual decision maker.* Consider models of individual decision making, in which the dependent variable is one-dimensional, and determined through the first-order condition of an optimization problem. An important class of such models are the ones solved by a social-planning problem, such as growth and macroeconomic models in Barro and Sala-I-Martin (2003) and Ljungqvist and Sargent (2004). Other examples include models of firms' investment choices used for testing if investment is sensitive to Tobin's q (Hayashi, 1982; Hayashi and Inoue, 1991).

3.1.2. *One-dimensional equilibrium.* Consider one-dimensional equilibrium models where equilibria are fixed points. For example, in a two-player game one can compose the two players' best-response functions, similarly to how we dealt with BLP's model in Section 2. As a consequence, duopoly models generally have the structure we need. Cournot n-firm oligopoly models also reduce to a one-dimensional equilibrium model by an aggregation procedure as described by Amir (1996). One can thus examine if entry of additional firms to a market causes a decrease in prices as in Amir and Lambson (2000). Additional examples can be found in overlapping-generations models, and two-good general equilibrium models.

3.2. **Main Result.** The presence of complementarities between X and Y is the basis of our main result: we show that an increase in x implies an increase in all the sufficiently large (and small) quantiles of $F_{Y|X=x}$. The result will follow from combining Assumption S2 with restrictions on U.

Consider x_1 and x_2 in $\mathcal{X}$ with $x_1 < x_2$. Let $n_1 = n_{x_1}$ and $n_2 = n_{x_2}$. How does the MCS property translate into observable implications on $\bar{F}_{Y|X=x_1}$ and $\bar{F}_{Y|X=x_2}$? Recall that $\bar{F}_{Y|X} = 1 - F_{Y|X}$ is the conditional distribution tail of Y. Let $\pi_{1i} = \pi_{ix_1}$

and $\pi_{2j} = \pi_{jx_2}$. Using the mixture result in Proposition 1 and focusing on the largest equilibria, we then have:

$$\begin{aligned} \frac{\bar{F}_{Y|X=x_1}(y)}{\bar{F}_{Y|X=x_2}(y)} &= \frac{\bar{F}_{n_1Y|X=x_1}(y)}{\bar{F}_{n_2Y|X=x_2}(y)} \frac{\sum_{i=1}^{n_1} \pi_{1i}[\bar{F}_{iY|X=x_1}(y)/\bar{F}_{n_1Y|X=x_1}(y)]}{\sum_{j=1}^{n_2} \pi_{2j}[\bar{F}_{jY|X=x_2}(y)/\bar{F}_{n_2Y|X=x_2}(y)]} \\ &\leqslant \frac{\bar{F}_{n_1Y|X=x_1}(y)}{\bar{F}_{n_2Y|X=x_2}(y)} \frac{1}{\pi_{2n_2}}, \end{aligned} \tag{4}$$

where the second inequality follows because $\pi_{2n_2} > 0$, $\bar{F}_{jY|X=x_2}(y)/\bar{F}_{n_2Y|X=x_2}(y) > 0$, and because stochastic dominance implies $\bar{F}_{iY|X=x_1}(y) \leqslant \bar{F}_{n_1Y|X=x_1}(y)$.

The upper bound in Equation (4) involves the probability of the largest equilibrium π_{2n_2}—on which we place no restrictions other than being positive—as well as the ratio of the distributions $\bar{F}_{n_1Y|X=x_1}$ and $\bar{F}_{n_2Y|X=x_2}$. These distributions are unknown and depend on the locations of the largest equilibria; hence they are difficult to control. A careful change of variables, however, transforms the problem so that (in the limit) the behavior of their ratio depends solely on the properties of r and $F_{U|X}$.

Lemma 2. *Under S1 and S2, and given $(y_0, x) \in \mathbb{R} \times \mathcal{X}$, we have $\mathbb{1}(\xi_{n_x x u} \leqslant y) = \mathbb{1}(u \leqslant r^{\mathrm{e}}(y,x))$ for any $y \geqslant y_0$, where $r^{\mathrm{e}}(y,x)$ is the non-increasing envelope of $r(y,x)$ on $[y_0, +\infty)$, i.e. $r^{\mathrm{e}}(y,x) = \inf\{q(y) : q$ is non-increasing on $[y_0, +\infty)$ and $q(y) \geqslant r(y,x)$ for all $y \in [y_0, +\infty)\}$.*

The idea in Lemma 2 is to consider a non-increasing transformation r^{e} which coincides with r around the largest equilibrium (see Figure 3). For $y \geqslant y_0$ then:

$$\frac{\bar{F}_{n_1Y|X=x_1}(y)}{\bar{F}_{n_2Y|X=x_2}(y)} = \frac{\int_{-\infty}^{r^{\mathrm{e}}(y,x_1)} f_{U|X=x_1}(u)du}{\int_{-\infty}^{r^{\mathrm{e}}(y,x_2)} f_{U|X=x_2}(u)du} = \frac{F_{U|X=x_1}(r^{\mathrm{e}}(y,x_1))}{F_{U|X=x_2}(r^{\mathrm{e}}(y,x_2))}. \tag{5}$$

Now, how the increase in the largest equilibria translates into $F_{Y|X=x_1}$ and $F_{Y|X=x_2}$, depends on two factors: (i) the limit behavior or $r^{\mathrm{e}}(y,x_1)$ and $r^{\mathrm{e}}(y,x_2)$ as y grows, and (ii) the limit behavior of the distribution $F_{U|X}$. On (i), recall that, by S2, $r(y,x)$ is monotone increasing in x. Hence $r(y,x_1) \leqslant r(y,x_2)$, which given the continuity and limit conditions in S1 implies $r^{\mathrm{e}}(y,x_1) \leqslant r^{\mathrm{e}}(y,x_2)$ for all $y \in [y_0, +\infty)$. We allow for two cases. Each case requires an assumption on $F_{U|X}$.

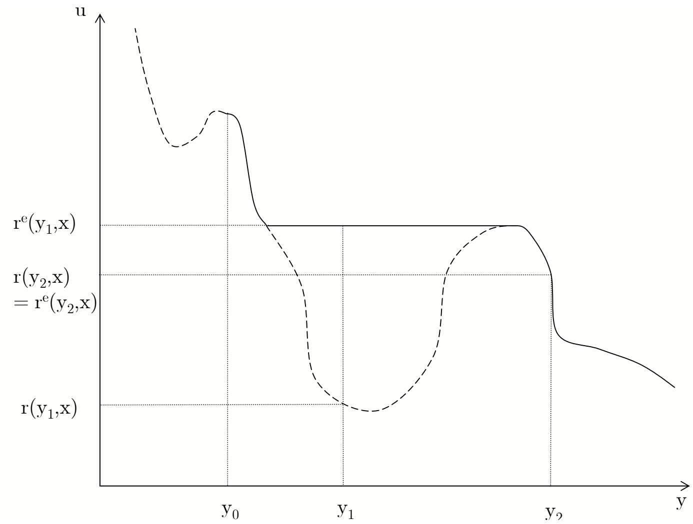

Figure 3: Plots of $y \mapsto r(y,x)$ (dashed line) and $y \mapsto r^{\mathrm{e}}(y,x)$ (solid line).

Assumption S3. (i) $\lim_{y\to+\infty}[r(y,x_1)/r(y,x_2)] = \lambda$ with $\lambda > 1$; (ii) for $\lambda > 1$, $\lim_{u\to-\infty} F_{U|X=x_1}(\lambda u)/F_{U|X=x_1}(u) = 0$; (iii) $F_{U|X=x_1}(u)/F_{U|X=x_2}(u)$ is bounded as $u \to -\infty$.

Assumption S3'. (i) $\lim_{y\to+\infty}[r(y,x_1) - r(y,x_2)] = \delta$ with $\delta < 0$; (ii) for $\delta < 0$, $\lim_{u\to-\infty} F_{U|X=x_1}(u+\delta)/F_{U|X=x_1}(u) = 0$; (iii) $F_{U|X=x_1}(u)/F_{U|X=x_2}(u)$ is bounded as $u \to -\infty$.

In S3.i, we control the limit ratio of $r(y,x_1)$ to $r(y,x_2)$; in S3'.i, we control the difference between $r(y,x_1)$ and $r(y,x_2)$, as in the BLP example in Section 2.[7] Assumption S3.ii prevents the left tail of the distribution $F_{U|X=x_1}$ from being too heavy. Letting $V \equiv -U$, S3.ii can be restated as: $\lim_{v\to+\infty} \bar{F}_{V|X=x_1}(\lambda v)/\bar{F}_{V|X=x_1}(v) = 0$ for any $\lambda > 1$, where $\bar{F}_{V|X} \equiv 1 - F_{V|X}$ denotes the tail of the conditional distribution of V given X. Put in words, Assumption S3.ii requires that the distribution of $-U$ be *rapidly varying* at $+\infty$. Rapid variation is a well-known condition in the statistics of extreme values, and is satisfied by a large variety of distributions whose tail behavior ranges from moderately heavy (log-normal, heavy-tailed Weibull) to light (exponential, Gamma, normal). Assumption S3'.ii is more restrictive; it prevents the left tail of $F_{U|X=x_1}$ from decaying at a rate slower than (or equal to) that of an exponential. S3.ii' is satisfied in distributions such as the normal or light-tailed

[7]We have $\lim_{y\to+\infty}[r(y,0) - r(y,\mathrm{VER})] = \lim_{p_1\to+\infty} \ln\left(\beta_1(\beta_2(p_1,0),0)/\beta_1(\beta_2(p_1,0),\mathrm{VER})\right) = \ln\left(\beta_1(\beta_2(\bar{p},0),0)/\beta_1(\beta_2(\bar{p},0),\mathrm{VER})\right) \equiv \delta < 0$, since β_1 is increasing in VER for any value of p_2.

Weibull. Both S3.ii and S3'.ii exclude the distributions with power-like decaying tails (Student-t, Pareto). Finally, Assumption S3.iii (S3'.iii) ensures that $F_{U|X=x_2}$ does not decrease towards 0 faster than $F_{U|X=x_1}$. This property is trivially satisfied when U is independent of X, and accommodates some interesting cases where U and X are dependent (see Section 4.1.2).

Assumption S3 implies that the last term in Equation (5) converges to 0 as y grows. Indeed, let $\lambda_1 \in (1, \lambda)$: then by S3.i there is $y_1 \in \mathbb{R}$ such that $r(y, x_1) \leqslant \lambda_1 r(y, x_2)$ whenever $y \geqslant y_1$ and hence $r^{\mathrm{e}}(y, x_1) \leqslant \lambda_1 r^{\mathrm{e}}(y, x_2)$. As $F_{U|X}$ is increasing, we have:

$$\begin{aligned} \lim_{y\to+\infty} \frac{F_{U|X=x_1}(r^{\mathrm{e}}(y,x_1))}{F_{U|X=x_2}(r^{\mathrm{e}}(y,x_2))} &\leqslant \lim_{y\to+\infty} \frac{F_{U|X=x_1}(\lambda_1 r^{\mathrm{e}}(y,x_2))}{F_{U|X=x_2}(r^{\mathrm{e}}(y,x_2))} \\ &= \lim_{y\to+\infty} \underbrace{\left[\frac{F_{U|X=x_1}(\lambda_1 r^{\mathrm{e}}(y,x_2))}{F_{U|X=x_1}(r^{\mathrm{e}}(y,x_2))}\right]}_{A} \underbrace{\left[\frac{F_{U|X=x_1}(r^{\mathrm{e}}(y,x_2)))}{F_{U|X=x_2}(r^{\mathrm{e}}(y,x_2))}\right]}_{B}. \end{aligned} \tag{6}$$

By S1.ii, r goes to $-\infty$ as y gets large, and so does its envelope r^{e}; hence the term B in (6) remains bounded. Using the rapid variation in S3.ii, the term A goes to 0 as y increases, and so does the product $A \times B$. As a result,

$$\lim_{y\to+\infty} \frac{F_{U|X=x_1}(r^{\mathrm{e}}(y,x_1))}{F_{U|X=x_2}(r^{\mathrm{e}}(y,x_2))} = 0. \tag{7}$$

Similarly, under Assumption S3'.i, $d(y) = r(y, x_1) - r(y, x_2)$ converges to $\delta < 0$. Let $\delta_1 \in (\delta, 0)$ and $y_1' \in \mathbb{R}$ be such that, for $y \geqslant y_1'$, we have $d(y) < \delta_1$. Hence, $d^{\mathrm{e}}(y) \leqslant \delta_1$ where $d^{\mathrm{e}}(y) = r^{\mathrm{e}}(y, x_1) - r^{\mathrm{e}}(y, x_2)$. Noting that:

$$\lim_{y\to+\infty} \frac{F_{U|X=x_1}(r^{\mathrm{e}}(y,x_1))}{F_{U|X=x_2}(r^{\mathrm{e}}(y,x_2))} \leqslant \lim_{y\to+\infty} \underbrace{\left[\frac{F_{U|X=x_1}(r^{\mathrm{e}}(y,x_2)+\delta_1)}{F_{U|X=x_1}(r^{\mathrm{e}}(y,x_2))}\right]}_{A} \underbrace{\left[\frac{F_{U|X=x_1}(r^{\mathrm{e}}(y,x_2)))}{F_{U|X=x_2}(r^{\mathrm{e}}(y,x_2))}\right]}_{B},$$

and using the same reasoning as previously, we again get the limit result in Equation (7). Combining the latter with Equations (4) and (5) then shows that:

$$\lim_{y\to+\infty} \frac{\bar{F}_{Y|X=x_1}(y)}{\bar{F}_{Y|X=x_2}(y)} = 0. \tag{8}$$

The statement in (8) is crucial. It says that for large enough values of y, $x_1 \leqslant x_2$ implies that the corresponding conditional distributions are ordered. This ordering of large enough conditional quantiles of Y given X holds under very weak restrictions on the equilibrium selection; recall that we only needed the probability of the largest equilibrium to be positive. We have thus shown:

Theorem 3. *Assume S1, S2, and either S3 or S3' hold. Fix a selection rule* $\mathcal{P}_{XU}$*. Let* $(x_1, \ldots, x_N) \in \mathcal{X}^N$ *be such that:* $x_1 \leqslant \ldots \leqslant x_N$*. Then, there exists* $\bar{y}_N \in \mathbb{R}$ *such that for all* $y \geqslant \bar{y}_N$, $\bar{F}_{Y|X=x_1}(y) \leqslant \ldots \leqslant \bar{F}_{Y|X=x_N}(y)$*. Equivalently, there exists* $\bar{\alpha}_N \in (0,1)$ *such that for all* $\alpha \in [\bar{\alpha}_N, 1)$, $F^{-1}_{Y|X=x_1}(\alpha) \leqslant \ldots \leqslant F^{-1}_{Y|X=x_N}(\alpha)$*.*

The previous analysis has focused on quantiles with probabilities close to one, but an analogous result continues to hold for probabilities close to zero.

4. Discussion

Theorem 3 derives observable implications of models with complementarities between the dependent variable Y and the explanatory variable X, which are valid despite the possible presence of multiple equilibria. These implications come in the form of order restrictions on the extreme (high and low) quantiles of Y conditional on X. We now discuss important features and possible limitations of the results of Theorem 3 when used for testing the presence of MCS.

4.1. Main Features.

We first discuss the applicability of our results.

4.1.1. *Robustness to Identification Failures.* We have shown that the MCS property has implications for the conditional quantiles of Y given X. Given a sample of observations on the dependent and explanatory variables, these quantiles are by definition identified and consistently estimable using standard nonparametric methods. In particular, no additional restrictions on the structural function r are needed for estimation. Consequently, the results of Theorem 3 can be used to test for complementarities whether or not the structural function r is identified.[8]

4.1.2. *Unobserved Heterogeneity.* Given that Theorem 3 does not require the structural function r to be identified or estimable, its results are fairly robust to departures

[8]Primitive conditions for identification of the structural function r are discussed in Chesher (2003), Newey and Powell (2003), and Matzkin (2005), for example.

from independence or mean independence conditions between the latent disturbance U and the explanatory variable X. As a consequence, our testable implications apply even in models in which U is *endogenous*. In particular, under the assumptions of Theorem 3, the individual heterogeneity U can be correlated with the explanatory variable X in a reasonably general way.

Say that conditional on X, U is normally distributed with mean $\mu(X)$ and variance σ^2. When $\mu(x)$ is non-increasing in x, a simple application of L'Hôpital's rule shows that Assumptions S3.ii and S3.iii hold. A simple example would be the one in which X and U are jointly normally distributed with a non-positive correlation coefficient. A positive correlation between X and U, under which Assumption S3.iii fails, prevents the econometrician from learning anything about MCS property. The intuition behind is simple: following an increase in X, U can in those cases increase so as to decrease the extremal equilibria.

In addition to being correlated with the explanatory variable, we allow U to be *heteroskedastic* conditional on X. Say that given X, U is normally distributed with mean 0 and variance $\sigma^2(X)$. If $\sigma^2(x)$ is non-decreasing in x, then Assumption S3.iii holds. Therefore a normal disturbance whose conditional variance increases with the equilibrium level satisfies our Assumption S3.

4.2. **Limitations.** We now caution for possible limitations of our approach.

4.2.1. *Tail Observations and Robustness to Outliers.* Theorem 3 suggests that one can use observations from the extreme (high and low) quantiles of Y conditional on X in order to test for the presence of MCS. Such a test shall obviously be affected by the presence of outliers. When the latter are caused by mismeasurements, methods proposed in Chen, Hong, and Tamer (2005), for example, can be used to filter the errors prior to applying the test. Unless outliers are easy to detect, one should be careful when considering very large (or small) quantiles of the dependent variable. In particular, the results of Theorem 3 lend themselves to the study of cases where X can take some relatively small number of values for which large numbers of observations of Y are available. Evaluations of policy effects, such as those following the VER, are one such example: typically X then takes on two values.

4.2.2. *Continuous Explanatory Variable.* The cutoff level $\bar{y}_N$ in Theorem 3 is conditional on a realization of the sample of explanatory variables, $(x_1, \ldots, x_N) \in \mathcal{X}^N$. This is not a problem in applications in which the explanatory variables are treated as given. In some situations, however, an unconditional version of Theorem 3 is needed. The latter follows easily when the explanatory variables are discrete: it suffices to apply the reasoning in Section 3.2 to all the points in $\mathcal{X}$. When the explanatory variables are *continuous*, we need to include an extra step which will ensure that x's do not get too close: given a random sample $(X_1, \ldots, X_N)$ drawn from F_X, consider the joint distribution of the $N-1$ spacings between the consecutive order statistics $(X_1^N, \ldots, X_N^N)$. Fix any $\varepsilon > 0$, and let $\delta_N > 0$ be such that the probability of all spacings being greater than δ_N, is greater or equal than $1-\varepsilon$. Applying the reasoning in Section 3.2 to x and $x + \delta_N$ we get the following corollary to Theorem 3:

Corollary 4. *Assume S1, S2, and either S3 or S3' hold. Fix a selection rule $\mathcal{P}_{XU}$. Given $\varepsilon > 0$, there exists $\bar{y}_N \in \mathbb{R}$ such that for all $y \geqslant \bar{y}_N$, $\Pr\{\bar{F}_{Y|X_1^N}(y) \leqslant \ldots \leqslant \bar{F}_{Y|X_N^N}(y)\} \geqslant 1-\varepsilon$. Equivalently, there exists $\bar{\alpha}_N \in (0,1)$ such that for all $\alpha \in [\bar{\alpha}_N, 1)$, $\Pr\{F^{-1}_{Y|X_1^N}(\alpha) \leqslant \ldots \leqslant F^{-1}_{Y|X_N^N}(\alpha)\} \geqslant 1-\varepsilon$.*

In a sense, Corollary 4 gives a stochastic version of the orderings in Theorem 3.

4.2.3. *Test Implementation.* Finally, the conditional distributions (and quantiles) of the dependent variable are typically *unknown* and need to be estimated from the data. A statistical test of the orderings in Theorem 3 and its Corollary 4 can then be constructed by deriving the asymptotic distribution of the conditional quantile estimators—the key is to derive the latter by imposing assumptions on the distributions $F_{U|X}$ while maintaining our agnosticism about the equilibrium selection $\mathcal{P}_{XU}$. When using the asymptotics, however, one needs to control the speed at which the probability $\bar{\alpha}_N$ increases (or decreases) relative to the sample size N. See Echenique and Komunjer (2007a) for results, albeit in a somewhat different framework.

References

Aguirregabiria, V., and P. Mira (2007): "Sequential Estimation of Dynamic Discrete Games," *Econometrica*, 75, 1–53.

Amir, R. (1996): "Cournot Oligopoly and the Theory of Supermodular Games," *Games and Economic Behavior*, 15, 132–148.

Amir, R., and V. E. Lambson (2000): "On the Effects of Entry in Cournot Markets," *Review of Economic Studies*, 67(2), 235–254.

Arrow, K. J., and F. H. Hahn (1971): *General Competitive Analysis*. San Francisco, Holden-Day.

Athey, S., and S. Stern (1998): "An Empirical Framework for Testing Theories About Complementarity in Organizational Design," NBER Working Paper 6600.

Barro, R. J., and X. Sala-I-Martin (2003): *Economic Growth*. MIT Press.

Berry, S. T. (1992): "Estimation of a Model of Entry in the Airline Industry," *Econometrica*, 60(4), 889–917.

Berry, S. T., J. Levinsohn, and A. Pakes (1995): "Automobile Prices in Market Equilibrium," *Econometrica*, 63(4), 841–890.

——— (1999): "Voluntary Export Restraints on Automobiles: Evaluating a Trade Policy," *American Economic Review*, 89(3), 400–430.

Bresnahan, T. F., and P. C. Reiss (1990): "Entry in Monopoly Markets," *Review of Economic Studies*, 57(4), 531–553.

——— (1991): "Empirical Models of Discrete Games," *Journal of Econometrics*, 48, 57–81.

Caplin, A., and B. Nalebuff (1991): "Aggregation and Imperfect Competition: On the Existence of Equilibrium," *Econometrica*, 59, 25–59.

Chen, X., H. Hong, and E. Tamer (2005): "Measurement Error Models with Auxiliary Data," *Review of Economic Studies*, 72, 343–366.

Chesher, A. (2003): "Identification in Nonseparable Models," *Econometrica*, 71, 1405–1441.

Ciliberto, F., and E. Tamer (2004): "Market Structure and Multiple Equilibria in the Airline Industry," Northestern University and University of Virginia.

Echenique, F., and I. Komunjer (2007a): "A Test for Monotone Comparative Statics," Caltech HSS Working Paper 1278.

——— (2007b): "Testing Models with Multiple Equilibria by Quantile Methods," Caltech HSS Working Paper 1244R.

Gabay, D., and H. Moulin (1980): "On the Uniqueness and Stability of Nash-equilibria in Noncooperative Games," in *Applied Stochastic Control in Econometrics and Management Science*, ed. by P. K. A. Bensoussan, and C. Tapiero, pp. 271–293. North Holland.

Hayashi, F. (1982): "Tobin's Marginal q and Average q: A Neoclassical Interpretation," *Econometrica*, 50(1), 213–224.

Hayashi, F., and T. Inoue (1991): "The Relation Between Firm Growth and Q with Multiple Capital Goods: Theory and Evidence from Panel Data on Japanese Firms," *Econometrica*, 59(3), 731–753.

Jovanovic, B. (1989): "Observable Implications of Models with Multiple Equilibria," *Econometrica*, 57(6), 1431–1437.

Kreps, D. M. (1990): *Game Theory and Economic Modeling*. Oxford University Press.

Ljungqvist, L., and T. J. Sargent (2004): *Recursive Macroeconomic Theory*. MIT Press (Second Edition).

Mas-Colell, A., M. D. Whinston, and J. R. Green (1995): *Microeconomic Theory*. Oxford University Press.

Matzkin, R. L. (1994): "Restrictions of Economic Theory in Nonparametric Methods," in *Handbook of Econometrics, Volume IV*, ed. by R. F. Engle, and D. L. McFadden, chap. 42, pp. 2523–2558. Elsevier Science.

——— (2005): "Identification in Nonparametric Simultaneous Equations," Northwestern University.

McLennan, A. (2005): "The Expected Number of Nash Equilibria of a Normal Form Game," *Econometrica*, 73(1), 141–174.

Milgrom, P., and J. Roberts (1990): "Rationalizability, Learning and Equilibrium in Games with Strategic Complementarities," *Econometrica*, 58(6), 1255–1277.

——— (1994): "Comparing Equilibria," *American Economic Review*, 84(3), 441–459.

Milgrom, P., and C. Shannon (1994): "Monotone Comparative Statics," *Econometrica*, 62(1), 157–180.

Newey, W. K., and J. L. Powell (2003): "Instrumental Variable Estimation of Nonparametric Models," *Econometrica*, 71, 1565–1578.

Takahashi, S. (2005): "The Number of Pure Nash Equilibria in a Random Game with Nondecreasing Best Responses," Mimeo, Harvard University.

Tamer, E. (2003): "Incomplete Simultaneous Discrete Response Model with Multiple Equilibria," *Review of Economic Studies*, 70(1), 147–165.

Villas-Boas, J. M. (1997): "Comparative Statics of Fixed Points," *Journal of Economic Theory*, 73(1), 183–198.

SUPPLEMENTARY APPENDIX WITH PROOFS

Appendix A. Proofs

Proof of Proposition 1. For any $(y,x)\in\mathbb{R}\times\mathcal{X}$, $F_{Y|X=x}(y)=\int_{-\infty}^{+\infty}\mathcal{P}_{xu}(y)f_{U|X=x}(u)du$ with $\mathcal{P}_{xu}(y)=\sum_{i=1}^{n_x}\pi_{ix}\mathbb{I}(\xi_{ixu}\leqslant y)$, where $\mathbb{I}$ denotes the standard indicator function: For any event A in $\mathcal{B}$ where $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}$, $\mathbb{I}(A)=1$ if A is true, and 0 otherwise. Combining all of the above we get:

$$F_{Y|X=x}(y)=\sum_{i=1}^{n_x}\pi_{ix}\int_{-\infty}^{+\infty}\mathbb{I}(\xi_{ixu}\leqslant y)f_{U|X=x}(u)du.$$

For any $x\in\mathcal{X}$ and any $1\leqslant i\leqslant n_x$, let $F_{iY|X=x}(y)=\int_{-\infty}^{+\infty}\mathbb{I}(\xi_{ixu}\leqslant y)f_{U|X=x}(u)du$ for all $y\in\mathbb{R}$. Then $F_{iY|X=x}(y):\mathbb{R}\to\mathbb{R}$ is right-continuous, $\lim_{y\to-\infty}F_{iY|X=x}(y)=0$, $\lim_{y\to+\infty}F_{iY|X=x}(y)=1$, and $F_{iY|X=x}$ is nondecreasing in y. Hence, $F_{iY|X=x}$'s are distribution functions and the conditional distribution of the dependent variable can be written as in Proposition 1. Moreover, for any $(y,x)\in\mathbb{R}\times\mathcal{X}$ we have $F_{iY|X=x}(y)-F_{jY|X=x}(y)=\int_{-\infty}^{+\infty}\mathbb{I}(\xi_{ixu}\leqslant y<\xi_{jxu})f_{U|X=x}(u)du\geqslant 0$ whenever $\xi_{jxu}\geqslant\xi_{ixu}$, i.e. $F_{jY|X=x}(y)\leqslant F_{iY|X=x}(y)$ whenever $j\geqslant i$. So, $F_{jY|X=x}$ first-order stochastically dominates $F_{iY|X=x}$ for any $j\geqslant i$. $\square$

Proof of Lemma 2. Fix $(y_0,x)\in\mathbb{R}\times\mathcal{X}:$ continuity and limit conditions on $r(y,x)$ in S1 then ensure that the envelope $r^{\mathrm{e}}(y,x)$ is well defined on $[y_0,+\infty)$. Now consider $y\geqslant y_0$. That $\mathbb{I}(\xi_{n_xxu}\leqslant y)=\mathbb{I}(u\leqslant r^{\mathrm{e}}(y,x))$ follows from showing that $r^{\mathrm{e}}(\xi_{n_xxu},x)=r(\xi_{n_xxu},x)$, as r^e is non-increasing and ξ_{n_xxu} is the largest equilibrium. We proceed in two steps. First, we show that for all $y>\xi_{n_xxu}$ we have $r(\xi_{n_xxu},x)>r(y,x)$. If that were not the case then there would exist a $y'>\xi_{n_xxu}$ such that $r(\xi_{n_xxu},x)\leqslant r(y',x)$. But this is incompatible with ξ_{n_xxu} being the largest equilibrium: we would have $u\leqslant r(y',x)$, so given the limit condition S1.ii on r at $+\infty$ there would be an equilibrium larger that ξ_{n_xxu}. Second, we show that $r^{\mathrm{e}}(\xi_{n_xxu},x)=r(\xi_{n_xxu},x)$. By definition of r^e, we have $r^{\mathrm{e}}(\xi_{n_xxu},x)\geqslant r(\xi_{n_xxu},x)$, so we need to rule out that the strict inequality holds. We again reason by contradiction: assume that $r^{\mathrm{e}}(\xi_{n_xxu},x)>r(\xi_{n_xxu},x)$. From the first step we know that $r(\xi_{n_xxu},x)>r(y,x)$ for all $y>\xi_{n_xxu}$. Then, consider the function which coincides with $r^e(y,x)$ for $y<\xi_{n_xxu}$ and with

$\min\{r^e(y,x), r(y,x)\}$ for $y \geqslant \xi_{n_x x u}$. This function is non-increasing, larger than r, and smaller than r^e at $\xi_{n_x x u}$, which is impossible by the definition of r^e. $\square$