

**ONLINE APPENDIX TO HOW TO CONTROL
CONTROLLED SCHOOL CHOICE
(NOT FOR PUBLICATION)**

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APPENDIX B. PROOFS OF THEOREMS 1-6

The following rationality axiom simply says that a rejected student may be made unavailable without affecting the set of chosen students. It has been used before in the matching context by Blair (1988), Alkan (2002), and Alkan and Gale (2003); and by Aygün and Sönmez (2013) for markets with contracts. It is satisfied by all of our choice models. For example, gross substitutes and monotonicity are sufficient as are gross substitutes and acceptance.

Axiom B.1. *Choice rule C satisfies **irrelevance of rejected students (IRS)** if $C(S') \subseteq S \subseteq S'$ implies that $C(S) = C(S')$.*

Proof of Theorem 4. Suppose that C satisfies the axioms. We shall prove that it is generated by an ideal point. To this end, we show that there exist an ideal point z^* and a strict priority \succ such that the choice function created by these coincides with C . The result follows essentially from Lemma 1 above. We start with the following lemma, which establishes that C also satisfies IRS.

Lemma B.1. *If C satisfies GS and Mon, then it also satisfies IRS.*

Proof. Let $C(S') \subseteq S \subseteq S'$. By GS, $C(S) \supseteq C(S')$. Since $S \subseteq S'$, we have $\xi(S) \leq \xi(S')$ and by Mon, $\xi(C(S)) \leq \xi(C(S'))$. This together with $C(S) \supseteq C(S')$ imply that $C(S') = C(S)$, so C satisfies IRS. \square

Define f as follows. For any $x \leq \xi(\mathcal{S})$, let S be such that $x = \xi(S)$ and let $f(x) = \xi(C(S))$. By distribution-monotonicity we know that $\xi(S) = \xi(S') \Rightarrow \xi(C(S)) = \xi(C(S'))$, so the particular choice of S does not matter; thus f is well defined. Moreover, when $y \leq x$ we have $f(y) \leq f(x)$, again by

Date: March 26, 2014.

distribution-monotonicity. So f is a monotone increasing function. In addition, $f(x) \leq x$ and $\|f(x)\| \leq q$, so f is within budget. Let z^* be as defined in the proof of Lemma 1. Since $f(z^*) = z^*$, we have that $\|z^*\| \leq q$.

Define a binary relation R by saying that $s R s'$ if $\tau(s) = \tau(s')$ and there is some $S \ni s, s'$ such that $s \in C(S)$ and $s' \notin C(S)$. We shall prove that R is transitive.

Lemma B.2. *If C satisfies GS, t-WARP and IRS, then R is transitive.*

Proof. Let $s R s'$ and $s' R s''$; we shall prove that $s R s''$. Let S' be such that $s', s'' \in S'$, $s' \in C(S')$, and $s'' \notin C(S')$. Consider the set $S' \cup \{s\}$. First, note that $s \in C(S' \cup \{s\})$. The reason is that if $s \notin C(S' \cup \{s\})$ then $C(S' \cup \{s\}) = C(S') \ni s'$, by IRS. Thus $s' R s$, in violation of t-WARP. Second, note that $s'' \notin C(S' \cup \{s\})$, as $s'' \notin C(S')$ and C satisfies gross substitutes. \square

The relation R is transitive. Thus it has an extension to a linear order \succ over S . For any S , and any $s, s' \in S$ with $\tau(s) = \tau(s')$ we have that $s \succ s'$ when $s \in C(S)$ while $s' \notin C(S)$.

By Lemma 3, that C satisfies gross substitutes implies that f satisfies gross substitutes. In addition, f is also monotone increasing and within budget, as was shown above. Therefore, f is generated by an ideal point rule with z^* by Lemma 1. Then C is generated by the ideal point z^* and priority order \succ .

Conversely, let C be generated by an ideal point z^* and \succ . It is immediate that C satisfies t-WARP. Define f as above. Here, f is well defined because for any S and S' such that $\xi(S) = \xi(S') = x$, $\xi(C(S))$ is the closest vector to z^* among those in $B(x)$ and $\xi(C(S'))$ is the closest vector to z^* among those in $B(x)$. Therefore, $\xi(C(S)) = \xi(C(S'))$ and so f is well defined.

To show that C satisfies distribution-monotonicity, let $y = \xi(S)$ and $x = \xi(S')$ such that $y \leq x$. By Lemma 2, $f(x) = x \wedge z^*$ and $f(y) = y \wedge z^*$. Then, $f(x) = x \wedge z^* \leq y \wedge z^* = f(y)$, and, therefore, $\xi(C(S)) \leq \xi(C(S'))$. Hence, C satisfies distribution-monotonicity.

To see that C satisfies gross substitutes, let $s \in S \subseteq S'$, $\tau(s) = t$, $\xi(S) = y$ and $\xi(S') = x$. As we have shown above, $f(x) = x \wedge z^*$ and $f(y) = y \wedge z^*$. If $f(y)_t \geq f(x)_t$, then more type t students are chosen in S compared to S' . Since $s \in C(S')$, and C is generated by an ideal point, we derive that $s \in C(S)$. On

the other hand, if $f(y)_t < f(x)_t$, then $f(y)_t < z_t^*$ since $f(x)_t = (x \wedge z^*)_t \leq z_t^*$. Since $f(y)_t = (y \wedge z^*)_t$, we derive that $f(y)_t = y_t$. That means all type t students are chosen from S , so $s \in C(S)$. Hence, C satisfies gross substitutes.

Proof of Theorem 5. For any $x \leq \xi(\mathcal{S})$, let $F(x) \equiv \{\xi(C(S)) : \xi(S) = x\}$ and

$$\hat{f}(x) = \bigwedge_{f(x) \in F(x)} f(x).$$

The proof requires the following lemmas.

Lemma B.3. *Let C satisfy GS. If $y \in \mathbf{Z}_+^d$ is such that $\hat{f}(y)_t < y_t$ then $\hat{f}(y + e_{t'})_t < y_t + 1_{t=t'}$*

Proof. Let y and t be as in the statement of the lemma. Let S be such that $\xi(S) = y$ and $\xi(C(S))_t < \xi(S)_t = y_t$. Such a set S exists because $\hat{f}(y)_t < y_t$. Let $s' \notin S$ be an arbitrary student with $\tau(s') = t'$. Note that

$$\emptyset \neq S^t \setminus C(S)^t \subseteq (S \cup \{s'\})^t \setminus C(S \cup \{s'\})^t,$$

as C satisfies GS. Then we cannot have $\xi(C(S \cup \{s'\}))_t = y_t + 1_{t=t'}$ because that would imply $(S \cup \{s'\})^t \setminus C(S \cup \{s'\})^t = \emptyset$. Then

$$y_t + 1_{t=t'} > \xi(C(S \cup \{s'\}))_t \geq \hat{f}(y + e_{t'})_t.$$

□

Lemma B.4. *If C satisfies GS and acceptance, then it also satisfies IRS.*

Proof. Let $C(S') \subseteq S \subseteq S'$. By GS, $C(S) \supseteq C(S')$. Since $S \subseteq S'$, acceptance implies $|C(S)| \leq |C(S')|$. This together with $C(S) \supseteq C(S')$ imply that $C(S') = C(S)$, so C satisfies IRS. □

Suppose that C satisfies the axioms. Using Lemma B.3, we can construct the vector r of minimum quotas as follows. Let $\bar{x} = \xi(\mathcal{S})$. The lemma implies that if $\hat{f}(y_t, \bar{x}_{-t})_t < y_t$ then $\hat{f}(y'_t, \bar{x}_{-t})_t < y'_t$ for all $y'_t > y_t$. Then there is $r_t \in \mathbf{N}$ such that $y_t > r_t$ if and only if $\hat{f}(y_t, \bar{x}_{-t})_t < y_t$. This uses the assumption we made on the cardinality of \mathcal{S}^t , which ensures that $\hat{f}(y)_t < y_t$ if y_t is large enough. Note that we may have $r_t = 0$.

First we prove that $S \subseteq \mathcal{S}$ with $|S^t| \leq r_t$ then $S^t = C(S)^t$. Observe that, for any x and t , $\hat{f}(r_t, x_{-t}) = r_t$. To see this note that if there is x and t

such that $\hat{f}(r_t, x_{-t}) < r_t$ then Lemma B.3 would imply that $\hat{f}(r_t, \bar{x}_{-t}) < r_t$, in contradiction with the definition of r . In fact, we can say more: For any x , t , and y_t , if $y_t \leq r_t$ then $\hat{f}(r_t, x_{-t}) = r_t$ and Lemma B.3 imply that $\hat{f}(y_t, x_{-t}) = y_t$. Therefore, letting $S \subseteq \mathcal{S}$ with $|S^t| \leq r_t$ we have that

$$|C(S)^t| \geq \hat{f}(y)_t = y_t,$$

where $y = \xi(S)$. Since $y_t = |S^t| \geq |C(S)^t|$ we have that $S^t = C(S)^t$.

Second we prove that, if $|S^t| > r_t$, then $|C(S)^t| \geq r_t$. Let $\tilde{S} = C(S)$. Assume, towards a contradiction, that $|\tilde{S}^t| < r_t$. Let $S' = \tilde{S} \cup S''$, where $S'' \subseteq S^t \setminus \tilde{S}^t$ is such that $|S''^t| = r_t$. By Lemma B.4, C satisfies IRS, so $C(S') = C(S)$. Thus,

$$\hat{f}(\xi(S'))_t \leq |C(S')^t| = |C(S)^t| < r_t.$$

Since $\xi(S')_t = |S'^t| = r_t$, we obtain a contradiction with the definition of r_t above.

Consider the following binary relation. Let $s \succ^* s'$ if there is S , at which $\{s\} = \{s, s'\} \cap C(S)$ and $\{s, s'\} \subseteq S$, and either $\tau(s) = \tau(s')$ or $\tau(s)$ is saturated at S . By the adapted strong axiom of revealed preference, \succ^* has a linear extension \succeq to \mathcal{S} .

Third we prove that C is consistent with \succeq , as stated in the definition. Let $s \in C(S)$ and $s' \in S \setminus C(S)$. If $\tau(s) = \tau(s')$ then $s \succ^* s'$ by definition of \succ^* ; hence $s \succeq s'$. If $\tau(s) \neq \tau(s')$ then we need to consider the case when $|S^t| > r_t$ where $t = \tau(s)$. The construction of r_t implies that $r_t = \hat{f}(|S^t|, \bar{x}_{-t}) < |S^t|$. Therefore, there exists $S' \subseteq \mathcal{S}$ such that if

$$S' = S^t \cup \left(\bigcup_{i \neq t} S^i \right)$$

then $S'^t \setminus C(S')^t \neq \emptyset$. Thus t is saturated at S . Since $s \in C(S)$ and $s' \in S \setminus C(S)$, we get $s \succeq s'$, as \succeq extends \succ^* .

It remains to show that if C is generated by reserves, then it satisfies the axioms. It is immediate that it satisfies Acceptance and S-SARP.

To see that it satisfies gross substitutes, let $S \subseteq S'$ and $s \in S \setminus C(S)$. Then $|S^{\tau(s)}| > r_{\tau(s)}$, so $|S'^{\tau(s)}| > r_{\tau(s)}$. Moreover, $s \in S \setminus C(S)$ implies that there are $r_{\tau(s)}$ students in $S^{\tau(s)}$ ranked above s . So s could only be admitted at the

second step in the construction of C . Let $C^{(1)}(S)$ be the set of students that are accepted in the first step, S^* be the set of students that are considered in the second step and q^* be the number of remaining seats to be allocated in the second step. Again, $s \in S \setminus C(S)$ implies that there are q^* students ranked above s in S^* . Consider the following procedure for S' . In the first step for each t we accept $\xi(C^{(1)}(S))_t$ highest ranked students of type t . And in the second step we consider all remaining students. It is clear that s cannot be admitted in the first step since $S'^{\tau(s)} \supseteq S^{\tau(s)}$ and that there are at least $r_{\tau(s)}$ students ranked above s in $S^{\tau(s)}$. Moreover, in the second step of the new procedure, there are more higher ranked students of each type compared to S^* , so s can also not be admitted in the second step since there are only q^* seats left. If s cannot be admitted with this procedure, then it cannot be in $C(S')$ because for each $t \neq \tau(s)$, $\xi(C^{(1)}(S))_t \leq r_t$. Therefore, $s \in S' \setminus C(S')$.

Proof of Theorem 6. Suppose that C satisfies the axioms. We start by showing that C is generated by quotas.

Let $r_t \equiv \max_{S \in \mathcal{S}} |C(S)^t|$. We need the following lemma.

Lemma B.5. *Suppose $S' \subseteq \mathcal{S}^t$. If $|C(S')| < \min\{q, |S'|\}$ then $|C(S')| = r_t$.*

Proof. Since $r_t = \max_{S \in \mathcal{S}} |C(S)^t|$, there exists a set \bar{S} such that $|C(\bar{S})^t| = r_t$. By GS, we can choose \bar{S} such that $\bar{S} \subseteq \mathcal{S}^t$ and $\bar{S} = C(\bar{S})$ (simply choose $C(\bar{S})^t$ to be the set in question). Now let S' be a set of students as in the statement of the lemma. Suppose towards a contradiction that $|C(S')| < r_t$.

Note that $|C(S')| < \min\{q, |S'|, r_t\}$ and $|C(\bar{S})| = r_t$. So RM implies that $|\bar{S}| > |S'|$.

Let $P \subseteq \bar{S}$ be a set of cardinality $|S'|$. By GS, $\bar{S} = C(\bar{S})$ implies that $P = C(P)$, so $|C(P)| = |S'| > |C(S')|$. A contradiction to RM. \square

In addition, let \succ^* be defined as follows: $s \succ^* s'$ if there exists $S \supseteq \{s, s'\}$ such that $s \in C(S)$, $s' \notin C(S)$ and either $\tau(s) = \tau(s')$ or $\tau(s')$ is demanded in S . By D-SARP, \succ^* has a linear extension \succ to \mathcal{S} .

To show that C is generated by quotas we need to show three things. First, we need $|C(S)^t| \leq r_t$ for every $S \subseteq \mathcal{S}$. This is immediate by construction of r_t .

Second we show that if $s \in C(S)$, $s' \in S \setminus C(S)$ and $s' \succ s$, then it must be the case that $\tau(s) \neq \tau(s')$ and $|C(S)^{\tau(s')}| = r_{\tau(s')}$. If $\tau(s) = \tau(s')$, then $s \succ^* s'$ and $s \succ s'$, which is a contradiction with the fact that \succ is an extension of \succ^* . So $\tau(s) \neq \tau(s')$. To prove that $|C(S)^{\tau(s')}| = r_{\tau(s')}$ suppose, towards a contradiction, that $|C(S)^{\tau(s')}| \neq r_{\tau(s')}$, so $|C(S)^{\tau(s')}| < r_{\tau(s')}$.

We shall prove that $\tau(s')$ is demanded in S , which will yield the desired contradiction by D-SARP, as \succ is an extension of \succ^* . Let $S' \equiv S^{\tau(s')}$. We consider three cases.

- First, $|C(S')| = q$ then $|C(S)^{\tau(s')}| < |C(S')^{\tau(s')}|$ (as $s \in C(S)$ and $\tau(s) \neq \tau(s')$), so $\tau(s')$ is demanded in S .
- Second, consider the case when $|C(S')| < q$ and $|C(S')| < |S'|$. Then, by Lemma B.5, $|C(S')| = r_{\tau(s')}$, so $|C(S')| > |C(S)^{\tau(s')}|$. Hence $\tau(s')$ is demanded in S .
- Third, consider the case when $|C(S')| < q$, and $|C(S')| = |S'|$. Then $|C(S')| > |C(S)^{\tau(s')}|$, as $s' \in S^{\tau(s')} \setminus C(S)^{\tau(s')}$. Thus $\tau(s')$ is demanded in S .

In all three cases we conclude that $s \succ^* s'$. Since \succ is a linear extension of \succ^* , we get $s \succ s'$, a contradiction.

Finally, we need to show that if $s \in S \setminus C(S)$, then either $|C(S)| = q$ or $|C(S)^{\tau(s)}| = r_{\tau(s)}$. Suppose that $|C(S)| < q$. We need $|C(S)^{\tau(s)}| = r_{\tau(s)}$. Let $S' \equiv S^{\tau(s)}$. By RM, $|C(S)^{\tau(s)}| \geq |C(S')|$, so $|C(S')| < q$ since $|C(S)| < q$. Similarly $|C(S')| < |S'|$, because otherwise $|C(S)^{\tau(s)}| \geq |C(S')|$ would imply $C(S)^{\tau(s)} = S'$; a contradiction since $s \in S \setminus C(S)$. We have established $|C(S')| < \min\{q, |S'|\}$, so by Lemma B.5 we get $|C(S')| = r_{\tau(s)}$.

To finish the proof, suppose that C is generated by quotas. Then it is easy to see that C satisfies D-SARP, RM and IRS. We show that it also satisfies GS. Suppose that $s \in S \subseteq S'$ and $s \in C(S')$. For each type t , let $S(t; r_t) \subseteq S^t$ be the r_t highest ranked type t students in S (if $|S^t| \leq r_t$ then $S(t; r_t) = S^t$). Define $S'(t; r_t)$ analogously. Since $s \in C(S')$, we have $s \in S'(\tau(s), r_{\tau(s)})$ and the ranking of s in $\cup_t S'(t; r_t)$ is no more than q . Since $S \subseteq S'$, the preceding statements also hold for S instead of S' , which implies that $s \in C(S)$.

Proof of Theorem 1. Let the choice rule be C .

(\Leftarrow **direction**) First observe that within-type \succ -compatibility implies t-WARP: Suppose for contradiction that there is a violation of t-WARP. Thus, there exist students s, s' of the same type t and sets S, S' with $s, s' \in S \cap S'$ such that $s \in C(S) \setminus C(S')$ and $s' \in C(S') \setminus C(S)$. By within-type \succ -compatibility, we get $s \succeq s'$ because $s \in C(S)$, $s' \in S \setminus C(S)$ and $\tau(s) = \tau(s')$. Similarly, we also get $s' \succeq s$ because $s' \in C(S')$, $s \in S' \setminus C(S')$ and $\tau(s) = \tau(s')$. But this is a contradiction since \succeq is a strict priority and s and s' must be different students. Therefore, choice rule C satisfies t-WARP and, consequently, Theorem 4 implies that the C is generated by an ideal point. Let z^* be the associated vector and \succ' be the associated priority.

We claim that the ideal point choice rule, say C' , generated with parameters z^* and \succ is the same as C . Suppose, for contradiction, that there exists set S such that $C(S) \neq C'(S)$. Since both are ideal point choice rules with the same vector z^* , we get that for every type t , $|C(S)^t| = |C'(S)^t|$. Therefore, there exist s, s' with $\tau(s) = \tau(s')$ such that $s \in C(S) \setminus C'(S)$ and $s' \in C'(S) \setminus C(S)$. Since $s' \in C'(S)$, $s \in S \setminus C'(S)$ and $\tau(s) = \tau(s')$ we get $s' \succeq s$ by construction of C' . On the other hand, C satisfies within-type \succ -compatibility, so $s \in C(S)$, $s' \in S \setminus C(S)$ and $\tau(s) = \tau(s')$ imply $s \succeq s'$. This is a contradiction since \succeq is a strict priority and s cannot be equal to s' . Hence, $C = C'$, so C is generated by an ideal point with \succ .

(\Rightarrow **direction**) Suppose that C is generated by ideal point rule with z^* and \succ . That C satisfies GS and Mon are shown in the proof of Theorem 4. By construction, C trivially satisfies within-type \succ -compatibility.

Proof of Theorem 2. Let C be the choice rule.

(\Leftarrow **direction**) First, we show that saturated \succ -compatibility implies S-SARP. Suppose, for contradiction, that there exist sequences $\{s_k\}_{k=1}^K$ and $\{S_k\}_{k=1}^K$, of students and sets of students, respectively, such that, for all k

- (1) $s_{k+1} \in C(S_{k+1})$ and $s_k \in S_{k+1} \setminus C(S_{k+1})$;
- (2) $\tau(s_{k+1}) = \tau(s_k)$ or $\tau(s_{k+1})$ is saturated at S_{k+1}

(using addition mod K). If $\tau(s_{k+1}) = \tau(s_k)$, then $\tau(s_k)$ is saturated in S_{k+1} , so saturated \succ -compatibility implies that $s_{k+1} \succ s_k$. Otherwise, if $\tau(s_{k+1}) \neq \tau(s_k)$, then $\tau(s_{k+1})$ is saturated at S_{k+1} . By saturated \succ -compatibility, we

get $s_{k+1} \succ s_k$. In both cases, $s_{k+1} \succ s_k$ for all k . Because \succ is strict, we get a contradiction. Therefore, C satisfies S-SARP. Then by Theorem 5, C is generated by reserves with $(r_t)_{t \in T}$ and \succ' . Let C' be the choice rule generated by reserves with $(r_t)_{t \in T}$ and \succ . We claim that $C = C'$.

Suppose, for contradiction, that $C \neq C'$. By construction, $|C(S)| = |C'(S)|$. Just like in the proof of Theorem 3, there exist types t and t' such that $|(C(S) \setminus C'(S))^t| > |(C'(S) \setminus C(S))^t| = 0$ and $|(C'(S) \setminus C(S))^{t'}| > |(C(S) \setminus C'(S))^{t'}| = 0$. Let $s \in (C(S) \setminus C'(S))^t$ and $s' \in (C'(S) \setminus C(S))^{t'}$. Since $s' \in S \setminus C(S)$ and $|(C(S) \setminus C'(S))^{t'}| = 0$ we get $|(C(S) \cap C'(S))^{t'}| = r_{t'}$. Thus, $|C'(S)^{t'}| > r_{t'}$, which implies that $s' \succeq s$ by construction. On the other hand, for choice rule C , $\tau(s')$ is saturated at S since $s' \in (S \setminus C(S))^{t'}$ (in the definition let $S' \equiv S$). Then saturated \succ -compatibility implies that $s \succeq s'$. This is a contradiction since \succ is a strict priority and s is different from s' . Therefore, C is generated by reserves with $(r_t)_{t \in T}$ and \succ .

(\Rightarrow **direction**) Suppose that C is generated by reserves with \succ . In the proof of Theorem 6, we show that C satisfies GS and acceptance. In addition, it is trivial that C satisfies saturated \succ -compatibility.

Proof of Theorem 3. Let the choice rule be C .

(\Leftarrow **direction**) First, we show that within-type \succ -compatibility and demanded \succ -compatibility imply D-SARP. Suppose, for contradiction, that there is a violation of D-SARP. Then there exist sequences $\{s_k\}_{k=1}^K$ and $\{S_k\}_{k=1}^K$, of students and sets of students, respectively, such that, for all k

- (1) $s_{k+1} \in C(S_{k+1})$ and $s_k \in S_{k+1} \setminus C(S_{k+1})$;
- (2) $\tau(s_{k+1}) = \tau(s_k)$ or $\tau(s_k)$ is demanded in S_{k+1} .

(using addition mod K). If $\tau(s_{k+1}) = \tau(s_k)$ then within-type \succ -compatibility implies that $s_{k+1} \succ s_k$. Otherwise, if $\tau(s_{k+1}) \neq \tau(s_k)$ then $\tau(s_k)$ is demanded in S_{k+1} . By demanded \succ -compatibility, $s_{k+1} \succ s_k$. In both cases, $s_{k+1} \succ s_k$ for all k , which is a contradiction. Thus, C satisfies D-SARP. Theorem 6 implies that C is generated by quotas with $(r_t)_{t \in T}$ and \succ' . Let C' be the choice rule generated by quotas with $(r_t)_{t \in T}$ and \succ . We claim that $C = C'$.

Suppose for contradiction that there exists set S such that $C(S) \neq C'(S)$. By construction, $|C(S)| = |C'(S)|$. Therefore, $|C(S) \setminus C'(S)| = |C'(S) \setminus C(S)| > 0$. If there exist $s \in C(S) \setminus C'(S)$ and $s' \in C'(S) \setminus C(S)$ such that $\tau(s) =$

$\tau(s')$, then we get a contradiction as in the proof of Theorem 3. Assume that such students do not exist. Then there exist types t and t' such that $|(C(S) \setminus C'(S))^t| > |(C'(S) \setminus C(S))^t| = 0$ and $|(C'(S) \setminus C(S))^{t'}| > |(C(S) \setminus C'(S))^{t'}| = 0$. Let $s \in (C(S) \setminus C'(S))^t$ and $s' \in (C'(S) \setminus C(S))^{t'}$. These imply that $|C'(S)^t| < r_t$ and as a result $s' \succeq s$ by construction of C' . On the other hand, it is easy to see that for choice rule C , t' is demanded in S because for $S' \equiv S^{t'}$ we have $\min\{r_{t'}, |S'|\} = |C(S')| > |C(S)^{t'}|$. Since C satisfies demanded \succ -compatibility, we get $s \succeq s'$. This is a contradiction since \succeq is a strict priority and s is different from s' . Therefore, C is generated by quotas with $(r_t)_{t \in T}$ and \succ .

(\Rightarrow **direction**) Suppose that C is generated by quotas with \succ . In the proof of Theorem 6 we show that C satisfies GS and RM. In addition, it is trivial that C satisfies within-type \succ -compatibility and demanded \succ -compatibility.

APPENDIX C. A GENERAL COMPARATIVE STATIC

Definition C.1. *Choice rule C is **path independent** if for every S and S' , $C(S \cup S') = C(S \cup C(S'))$.*

Definition C.2. *A choice rule is an **expansion** of another choice rule if, for any set of students, any student chosen by the latter is also chosen by the former. (C' is an expansion of C is for every set S , $C'(S) \supseteq C(S)$.)*

For matching markets, stability has proved to be a useful solution concept because mechanisms that find stable matchings are successful in practice (Roth, 2008). Moreover, finding stable matchings is relatively easy. In particular, the deferred acceptance algorithm (DA) of Gale and Shapley (1962) finds a stable matching, and DA has other attractive properties.¹ Therefore, it also serves as a recipe for market design. For example, it has been adapted by the New York and Boston school districts (see Abdulkadiroğlu, Pathak, Roth, and Sönmez (2005) and Abdulkadiroğlu, Pathak, and Roth (2009)). For completeness, we provide a description of the student-proposing deferred acceptance algorithm.

Deferred Acceptance Algorithm (DA)

¹For a history of the deferred acceptance algorithm, see Roth (2008).

Step 1: Each student applies to her most preferred school. Suppose that S_c^1 is the set of students who applied to school c . School c tentatively admits students in $C_c(S_c^1)$ and permanently rejects the rest. If there are no rejections, stop.

Step k : Each student who was rejected at Step $k - 1$ applies to her next preferred school. Suppose that S_c^k is the set of new applicants and students tentatively admitted at the end of Step $k - 1$ for school c . School c tentatively admits students in $C_c(S_c^k)$ and permanently rejects the rest. If there are no rejections, stop.

The algorithm ends in finite time since at least one student is rejected at each step. When choice rules are path independent, DA produces the student-optimal stable matching (Roth, 1984; Aygün and Sönmez, 2013; Chambers and Yenmez, 2013). Therefore, the student-optimal stable mechanism (SOSM) coincides with DA.

Theorem C.1. *Suppose that for each school c , C_c is path independent and C'_c is a path-independent expansion of C_c . Then all students weakly prefer the outcome of SOSM with $(C'_c)_{c \in C}$ to the outcome with $(C_c)_{c \in C}$.*

Proof. We start with the following lemma.

Lemma C.1. *If C satisfies GS and (c, S) blocks a matching μ , then for every $s \in S \setminus \mu(c)$, $(c, \{s\})$ blocks μ .*

Proof. Since (c, S) blocks μ , we have $S \subseteq C_c(\mu(c) \cup S)$. Let $s \in S \setminus \mu(c)$, by substitutability $s \in C(\mu(c) \cup S)$ implies $s \in C(\mu(c) \cup \{s\})$. Therefore, $(c, \{s\})$ blocks μ . \square

Since we use two different choice rule profiles and stability depends on the choice rules, we prefix the choice rule profile to stability, individual rationality and no blocking to avoid confusion. For example, we use C -stability, C -individual rationality and C -no blocking.

DA produces the student-optimal stable matching (Roth and Sotomayor, 1990). Denote the student-optimal stable matching with C and C' by μ and μ' , respectively. Since $C_c(\mu(c)) = \mu(c)$ by C -individual rationality of μ by every school c , $C'_c(\mu(c)) \supseteq C_c(\mu(c))$ by the assumption, and $C'_c(\mu(c)) \subseteq \mu(c)$

by definition of the choice rule we get $C'(\mu(c)) = \mu(c)$. Therefore, μ is also C' -individually rational for schools. Since student preference profile is fixed, μ is also C' -individually rational for students. If μ is a C' -stable matching, then μ' Pareto dominates μ since μ' is the student-optimal C' -stable matching. Otherwise, if μ is not a C' -stable matching, then there exists a C' -blocking pair. Whenever there exists such a blocking pair, there also exists a blocking pair consisting a school and a student by Lemma C.1. In such a situation, we apply the following improvement algorithm. Let $\mu^0 \equiv \mu$.

Step k : Consider blocking pairs involving school c_k and students who would like to switch to c_k , say $S_{c_k}^k \equiv \{s : c_k \succ_s \mu^{k-1}(s)\}$. School c_k accepts $C'_{c_k}(\mu^{k-1}(c_k) \cup S_{c_k}^k)$ and rejects the rest of the students. Let $\mu^k(c_k) \equiv C'_{c_k}(\mu^{k-1}(c_k) \cup S_{c_k}^k)$ and $\mu^k(c) \equiv \mu^{k-1}(c) \setminus C'_{c_k}(\mu^{k-1}(c_k) \cup S_{c_k}^k)$ for $c \neq c_k$. If there are no more blocking pairs, then stop and return μ^k , otherwise go to Step $k + 1$.

We first prove by induction that no previously admitted student is ever rejected in the improvement algorithm. For the base case when $k = 1$ note that $C'_{c_1}(\mu(c_1) \cup S_{c_1}^1) \supseteq C_{c_1}(\mu(c_1) \cup S_{c_1}^1)$ by assumption and $C_{c_1}(\mu(c_1) \cup S_{c_1}^1) = \mu(c_1)$ since μ is C -stable. Therefore, $C'(\mu(c_1) \cup S_{c_1}^1) \supseteq \mu(c_1)$, which implies that no students are rejected at the first stage of the algorithm. Assume, by mathematical induction hypothesis, that no students are rejected during Steps 1 through $k - 1$ of the improvement algorithm. We prove that no student is rejected at Step k . There are two cases to consider.

First, consider the case when $c_n \neq c_k$ for all $n \leq k - 1$. Since μ is C -stable, we have $C_{c_k}(\mu(c_k) \cup S_{c_k}^k) = \mu(c_k)$ (as students in $S_{c_k}^k$ prefer c_k to their schools in μ). By assumption, $C'_{c_k}(\mu(c_k) \cup S_{c_k}^k) \supseteq C_{c_k}(\mu(c_k) \cup S_{c_k}^k)$ which implies $C'_{c_k}(\mu(c_k) \cup S_{c_k}^k) \supseteq \mu(c_k)$. Since $\mu(c_k) \supseteq \mu^{k-1}(c_k)$ we have $C'_{c_k}(\mu^{k-1}(c_k) \cup S_{c_k}^k) \supseteq \mu^{k-1}(c_k)$ by substitutability. In this case no student is rejected at Step k .

Second, consider the case when $c_k = c_n$ for some $n \leq k - 1$. Let n^* be the last step smaller than k in which school c_k was considered. Since each student's match is either the same or improved at Steps 1 through $k - 1$, we have $\mu^{n^*-1}(c_k) \cup S_{c_k}^{n^*} \supseteq \mu^{k-1}(c_k) \cup S_{c_k}^k$. By construction, $\mu^{n^*}(c_k) = C'_{c_k}(\mu^{n^*-1}(c_k) \cup S_{c_k}^{n^*})$ which implies $\mu^{k-1}(c_k) \subseteq C'_{c_k}(\mu^{k-1}(c_k) \cup S_{c_k}^k)$ by substitutability and the

fact that $\mu^{n^*}(c_k) \supseteq \mu^{k-1}(c_k)$ (since n^* is the last step before k in which school c_k is considered). Therefore, no student is rejected at Step k .

Since no student is ever rejected by the improvement algorithm, it ends in a finite number of steps. Moreover, the resulting matching does not have any C' -blocking pair. By construction, it is also C' -individually rational. This shows that there exists a C' -stable matching that Pareto dominates μ . Since μ' is the student-optimal C' -stable matching, we have that μ' Pareto dominates μ for students. \square

APPENDIX D. INDEPENDENCE OF AXIOMS IN THEOREMS 1-6

Here, we check the independence of axioms that are used in Theorems 1-6. The following axiom is useful in our examples below.

Axiom D.1. *Choice rule C satisfies the **strong axiom of revealed preference (SARP)** if there are no sequences $\{s_k\}_{k=1}^K$ and $\{S_k\}_{k=1}^K$, of students and sets of students, respectively, such that, for all k*

$$(1) \ s_{k+1} \in C(S_{k+1}) \text{ and } s_k \in S_{k+1} \setminus C(S_{k+1}).$$

(using addition mod K).

SARP is stronger than both D-SARP and S-SARP.

Axioms in Theorem 4.

Example 1 (GS, t-WARP but not Mon). Let $\mathcal{S} = \{s_1, s_2, s_3\}$, $q = 2$, and $\tau(s_1) = \tau(s_2) = \tau(s_3) = t$. Consider the following choice function: $C(s_1, s_2, s_3) = C(s_1, s_2) = C(s_1, s_3) = C(s_1) = \{s_1\}$, $C(s_2, s_3) = \{s_2, s_3\}$, $C(s_2) = \{s_2\}$, and $C(s_3) = \{s_3\}$.² Clearly, C satisfies both GS and t-WARP. But it fails Mon since $|\{s_1, s_2, s_3\}^t| \geq |\{s_2, s_3\}^t|$ but $|C(s_1, s_2, s_3)^t| < |C(s_2, s_3)^t|$.

Example 2 (t-WARP, Mon but not GS). Let $\mathcal{S} = \{s_1, s_2, s_3\}$, $\tau(s_1) = \tau(s_2) = t_1$, $\tau(s_3) = t_2$. Consider the following choice function: $C(s_1, s_2) = \{s_1\}$ and $C(S) = S$ for the remaining S . C satisfies t-WARP and Mon. But it fails GS because $s_2 \in C(s_1, s_2, s_3)$ and $s_2 \notin C(s_1, s_2)$.

²For ease of notation we write $C(s_i, \dots, s_j)$ for $C(\{s_i, \dots, s_j\})$.

Example 3 (Mon, GS but not t-WARP). Let $\mathcal{S} = \{s_1, s_2, s_3, s_4\}$, $q = 2$, and $\tau(s_1) = \tau(s_2) = \tau(s_3) = \tau(s_4) = t$. Consider the following choice function: $C(s_1, s_2, s_3, s_4) = C(s_1, s_2, s_3) = C(s_1, s_2, s_4) = \{s_1, s_2\}$, $C(s_1, s_3, s_4) = \{s_1, s_3\}$, $C(s_2, s_3, s_4) = \{s_2, s_4\}$, and $C(S) = S$ for the remaining S . C satisfies Mon and GS. But it fails t-WARP because $s_3 \in C(s_1, s_3, s_4) \setminus C(s_2, s_3, s_4)$ and $s_4 \in C(s_2, s_3, s_4) \setminus C(s_1, s_3, s_4)$.

Axioms in Theorem 5.

Example 4 (GS, S-SARP but not Acceptance). Consider the choice function in Example 1. C satisfies both GS and SARP (and hence S-SARP). But it fails acceptance since $|C(s_1, s_2, s_3)| = 1 < 2 = q$.

Example 5 (S-SARP, Acceptance but not GS). Let $S = \{s_1, s_2, s_3, s_4\}$ and $T = \{t_1, t_2\}$. Suppose that s_1 and s_2 are of type t_1 and the rest of type t_2 . Let the capacity of the school be 2 and the choice be:

$$C(S) = \begin{cases} S & \text{if } |S| \leq 2 \\ \{s_1, s_2\} & \text{if } \{s_1, s_2\} \subseteq S \\ \{s_3, s_4\} & \text{otherwise.} \end{cases}$$

Note that C violates GS because $s_1 \notin C(s_1, s_3, s_4)$ while $s_1 \in C(s_1, s_2, s_3, s_4)$. However, C satisfies acceptance and S-SARP. Acceptance is obvious. To see that it satisfies S-SARP, let R be the revealed preference relation, where $x R y$ if there is S such that $x \in C(S)$ and $y \in S \setminus C(S)$, and either x and y are of the same type or the type of x is saturated in S .

We can only infer $x R y$ when there is S with $S \setminus C(S) \neq \emptyset$. So we can focus in S with $|S| \geq 3$. There are four such sets. When $|S^{t_1}| = 2$ we have $S^{t_1} \setminus C(S)^{t_1} = \emptyset$, so t_1 is never saturated at any S with $|S^{t_1}| = 2$. Therefore we cannot infer any $x R y$ from any S with $\{s_1, s_2\} \subseteq S$. Thus we are only left with the facts that

$$\{s_3, s_4\} = C(s_1, s_3, s_4) = C(s_2, s_3, s_4).$$

That is, $s_3 R s_1$, $s_3 R s_2$, $s_4 R s_1$, and $s_4 R s_2$. Such R is acyclic. So S-SARP is satisfied.

Example 6 (Acceptance, GS but not S-SARP). Consider choice function C introduced in Example 3. We showed that C satisfies GS but fails t-WARP. Since S-SARP is stronger than t-WARP, S-SARP is also violated. It is easy to check that C also satisfies acceptance.

Axioms in Theorem 6.

Example 7 (GS, D-SARP but not RM). Consider the choice function in Example 1. C satisfies both GS and SARP (and hence D-SARP). But it fails RM since $s_2 \in \{s_1, s_2\} \setminus C(s_1, s_2)$ and $|C(s_1, s_2)| < q = 2$ but $|C(s_1, s_2)^t| < |C(s_2, s_3)^t|$.

Example 8 (D-SARP, RM but not GS). Let $\mathcal{S} = \{s_1, s_2, s_3, s_4\}$, $q = 2$, and $\tau(s_1) = \tau(s_2) = \tau(s_3) = t_1$ and $\tau(s_4) = t_2$. Consider the following choice function: $C(s_1, s_2, s_3, s_4) = C(s_1, s_2, s_3) = \{s_1, s_2\}$, $C(s_1, s_2, s_4) = C(s_1, s_3, s_4) = \{s_1, s_4\}$, $C(s_2, s_3, s_4) = \{s_2, s_4\}$, $C(s_1, s_2) = C(s_1, s_3) = \{s_1\}$, $C(s_2, s_3) = \{s_2\}$, and $C(S) = S$ for the remaining S .

Let \succ be defined as follows: $s \succ s'$ if there exists $S \supseteq \{s, s'\}$ such that $s \in C(S)$, $s' \notin C(S)$ and either $\tau(s) = \tau(s')$ or $\tau(s')$ is demanded in S . We consider every set of students from which a student is rejected and deduce that $s_1 \succ s_2 \succ s_3, s_4$. Since there are no cycles, D-SARP is satisfied. It is easy to see that RM is also satisfied. To see that GS fails, note $s_2 \in C(s_1, s_2, s_3, s_4)$ and $s_2 \notin C(s_1, s_2, s_4)$.

Example 9 (RM, GS but not D-SARP). Consider choice function C introduced in Example 3. C satisfies GS but it fails t-WARP. Since D-SARP is stronger than t-WARP, D-SARP is also not satisfied. In addition, C also satisfies acceptance, which implies RM.

Axioms in Theorem 1.

Example 10 (GS, within-type \succ -compatibility but not Mon). Consider choice function C introduced in Example 1. Let \succ be as follows: $s_1 \succ s_2 \succ s_3$. Clearly, C satisfies both GS and within-type \succ -compatibility. But it fails Mon.

Example 11 (Within-type \succ -compatibility, Mon but not GS). Consider choice function C introduced in Example 2. Let \succ be as follows: $s_1 \succ s_2 \succ s_3$. Clearly, C satisfies both within-type \succ -compatibility and Mon. But it fails GS.

Example 12 (Mon, GS but not within-type \succ -compatibility). Consider choice function C introduced in Example 3. It satisfies Mon and GS but fails t-WARP. Therefore, it fails within-type \succ -compatibility for any \succ .

Axioms in Theorem 2.

Example 13 (GS, saturated \succ -compatibility but not acceptance). Consider the choice function in Example 1. Let \succ be as follows: $s_1 \succ s_2 \succ s_3$. C satisfies GS and saturated \succ -compatibility. But it fails acceptance since $|C(s_1, s_2, s_3)| = 1 < 2 = q$.

Example 14 (Saturated \succ -compatibility, acceptance but not GS). Consider the choice function in Example 5. Let \succ be as follows: $s_3 \succ s_4 \succ s_1 \succ s_2$. It is clear by the argument in Example 5 that C satisfies saturated \succ -compatibility because \succ agrees with the revealed preference constructed therein. In addition, C also satisfies acceptance. But it fails GS as shown in Example 5.

Example 15 (Acceptance, GS but not saturated \succ -compatibility). Consider the choice function in Example 3 but suppose that all students have different types. C satisfies acceptance and GS. But it fails saturated \succ -compatibility: $\tau(s_3)$ is saturated in $\{s_1, s_3, s_4\}$, $s_3 \in C(s_1, s_3, s_4)$ and $s_4 \notin C(s_1, s_3, s_4)$ imply $s_3 \succ s_4$. On the other hand, $\tau(s_4)$ is saturated in $\{s_2, s_3, s_4\}$, $s_4 \in C(s_2, s_3, s_4)$ and $s_3 \notin C(s_2, s_3, s_4)$ imply $s_4 \succ s_3$. Therefore, C cannot satisfy saturated \succ -compatibility for any \succ .

Axioms in Theorem 3.

Example 16 (GS, within-type \succ -compatibility, demanded \succ -compatibility but not RM). Consider the choice function in Example 1. Let \succ be as follows: $s_1 \succ s_2 \succ s_3$. As argued in Example 6, C satisfies GS but fails RM. Moreover, it satisfies within-type \succ -compatibility and demanded \succ -compatibility.

Example 17 (Within-type \succ -compatibility, demanded \succ -compatibility, RM but not GS). Consider the choice function in Example 8. Let \succ be as follows: $s_1 \succ s_2 \succ s_3 \succ s_4$. As argued in Example 8, C satisfies RM but fails GS. Clearly it satisfies within-type \succ -compatibility. Let us now check saturated \succ -compatibility: the only sets in which a lower priority student is chosen over a higher priority student are $\{s_1, s_2, s_4\}$, $\{s_1, s_3, s_4\}$, and $\{s_2, s_3, s_4\}$. But s_2 is not demanded for $\{s_1, s_2, s_4\}$, s_3 is not demanded for $\{s_1, s_3, s_4\}$ and s_3 is not demanded for $\{s_2, s_3, s_4\}$. Therefore, demanded \succ -compatibility is also satisfied.

Example 18 (Demanded \succ -compatibility, RM, GS but not within-type \succ -compatibility). Consider the choice function in Example 3. C satisfies GS as argued in Example 3 and it satisfies RM as argued in Example 9. Since there is only one type and this type is never demanded in a set, C also satisfies demanded \succ -compatibility for any \succ . However, it fails within-type \succ -compatibility because it fails t-WARP as shown in Example 3.

Example 19 (RM, GS, within-type \succ -compatibility but not demanded \succ -compatibility). Consider the choice function in Example 3 but suppose that all students have different types. C satisfies GS as argued in Example 3 and it satisfies RM as argued in Example 9. It trivially satisfies within-type \succ -compatibility because all students have different types. But it fails demanded \succ -compatibility for any \succ because $\tau(s_4)$ is demanded in $\{s_1, s_3, s_4\}$, so we need $s_3 \succ s_4$. On the other hand, $\tau(s_3)$ is demanded in $\{s_2, s_3, s_4\}$, so we need $s_4 \succ s_3$.

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