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Ordinal notions of submodularity

Christopher P. Chambers and Federico Echenique*

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Abstract

We consider several ordinal formulations of submodularity, defined for arbitrary binary relations on lattices. Two of these formulations are essentially due to David Kreps (A Representation Theorem for "Preference for Flexibility", *Econometrica*, 1979) and one is a weakening of a notion due to Paul Milgrom and Chris Shannon (Monotone Comparative Statics, *Econometrica*, 1994). We show that any reflexive binary relation satisfying either of Kreps's definitions also satisfies Milgrom and Shannon's definition, and that any transitive and monotonic binary relation satisfying the Milgrom and Shannon condition satisfies both of Kreps's conditions. Keywords: quasisupermodularity, quasisubmodularity, comparative statics, submodularity. JEL classification: C65.

1 Introduction

The purpose of this note is to clarify the relationship between several ordinal notions of submodularity that have appeared in the literature. The notions of submodularity are properties of a binary relation on a lattice; we consider general lattices. Two well-known notions, originally due to Kreps (1979), appear in the decision-theory literature, while another well-known notion, due to Milgrom and Shannon (1994), appears in the literature on monotone comparative statics.

*Division of the Humanities and Social Sciences, California Institute of Technology, Pasadena CA 91125, USA. Email: `chambers@hss.caltech.edu` (Chambers) and `fede@hss.caltech.edu` (Echenique)

While Epstein and Marinacci (2007) have shown that Kreps's two notions are equivalent, under the additional assumptions of monotonicity and transitivity, it appears that there has not been any work pointing out other relations between the concepts. The decision-theory literature and monotone comparative statics literature remain relatively disjoint. We show that Milgrom and Shannon's notion is generally the weakest condition of the three, but that, for monotonic and transitive binary relations, the three notions coincide. Our contribution here is to explore the relation between the different concepts, in the hope of unifying the theory.

The notions of Kreps are also special in that they can be defined for any join-semilattice, and do not require a meet operation. We also explore the implications of the notions on join semilattices.

2 The model and results

A **partial order** on a set X is a binary relation which is reflexive, antisymmetric, and transitive.[1] A **lattice** is a partially ordered set for which every pair of elements x, y, possesses a greatest lower bound according to $\leq$ (the meet) and a least upper bound according to $\leq$ (the join). These elements are denoted $x \wedge y$ and $x \vee y$ respectively. A **join semilattice** is a partially ordered set for which every pair of elements $x, y \in X$ possesses a least upper bound according to $\leq$.

Let $(X, \leq)$ be a lattice, and suppose that R is a binary relation on X. Say that R is **quasisubmodular** if for all $x, y \in X$, $(x \wedge y) R x$ implies $y R (x \vee y)$, with a corresponding statement for P.[2] Say that R is **monotonic** if for all $x, y \in X$, $x \geq y$ implies $x \, R \, y$.

Under monotonicity, quasisubmodularity is equivalent to the statement that for all $x, y \in X$, $(x \wedge y) \; I \; x$ implies $y \; I \; (x \vee y)$. Thus, we will say that R is **weakly quasisubmodular** if for all $x, y \in X$, $(x \wedge y) \; I \; x$ implies $y \; I \; (x \vee y)$. Say that R is **modular** if for all $x, y \in X$, $x \; I \; (x \vee y)$ implies $x \vee z \, I \, (x \vee y) \vee z$ (Epstein and Marinacci refer to the property as *Generalized Kreps*). Lastly, define $x \geq^* y$ to mean that $x \; I \; (x \vee y)$.[3] Say that R is

[1] Reflexive: for all $x \in X$, $x \, R \, x$
Antisymmetric: for all $x, y \in X$, if $x \, R \, y$ and $y \, R \, x$, then $x = y$
Transitive: for all $x, y, z \in X$, if $x \, R \, y$ and $y \, R \, z$, then $x \, R \, z$

[2] As usual, P denotes the asymmetric part of R whereas I denotes the symmetric part.

[3] This order initially appears in Kreps.

transitive submodular if $\geq^*$ is transitive.

Modularity and transitive submodularity make their first appearance in Kreps. Quasisubmodularity appears somewhat later, in the work of Milgrom and Shannon. Indeed; their paper actually focuses on the dual of quasisubmodularity (quasisupermodularity).

A proof of the following result appears in Epstein and Marinacci (2007) (Theorem 6). A weaker result, applying only to the join semilattice of nonempty subsets, appears in Kreps (1979). In Epstein and Marinacci's paper, completeness, transitivity, and monotonicity of R are assumed throughout, but a reading of their proof establishes which properties of R are essential.

Proposition (Kreps, Epstein and Marinacci). *Suppose that $(X, \leq)$ is a lattice. If R is reflexive, then transitive submodularity implies modularity. If R is monotonic and transitive, then modularity implies transitive submodularity.*

Proposition 1. *Suppose that $(X, \leq)$ is a lattice. If R satisfies modularity, then it satisfies weak quasisubmodularity. If R is reflexive, then transitive submodularity implies weak quasisubmodularity.*

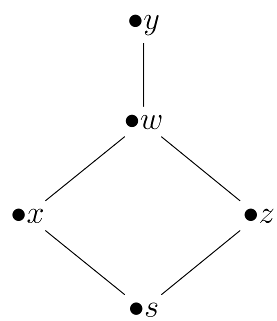
Proof. Suppose that R satisfies modularity, and let $(x \wedge y)\ I\ x$. We will show that $y\ I\ (x \vee y)$. As $x = (x \wedge y) \vee x$, we conclude by hypothesis that $(x \wedge y)\, I\, (x \wedge y) \vee x$. By modularity, $(x \wedge y) \vee y\, I\, ((x \wedge y) \vee x) \vee y$. Now, $y = (x \wedge y) \vee y$ and $((x \wedge y) \vee x) \vee y = x \vee y$ so that $yI\ (x \vee y)$. Now, suppose that R is reflexive and transitive submodular. Suppose that $x\ I\ (x \wedge y)$. We will show that $y\, I\, (x \vee y)$. Clearly, as $y \geq (x \wedge y)$, we have $y = y \vee (x \wedge y)$, so by reflexivity, $y \geq^* (x \wedge y)$. Similarly, $x = x \vee (x \wedge y)$. Hence, $x\, I\, (x \wedge y)$ implies $(x \wedge y) \geq^* x$. As $y \geq^* (x \wedge y) \geq^* x$, transitive submodularity establishes that $y \geq^* x$, so that $y\ I\ (x \vee y)$. □

We present an example to show that, when R is not reflexive, transitive submodularity does not imply weak quasisubmodularity.

Example 2. Let $X = \{0, 1\}^2$, and let $\leq$ be the usual order. Define R by $(0, 0)\, R\, (0, 1)$, $(0, 1)\, R\, (0, 0)$, $(0, 0)\, R\, (0, 0)$ and $(0, 1)\, R\, (0, 1)$. Then it is clear that R is not weakly quasisubmodular as $(1, 1)$ is unrelated to $(1, 0)$, yet $(0, 0)\ I\ (0, 1)$. However R is transitive submodular. One can check that $\geq^*$ coincides with R in this case, and is hence transitive.

Even for a complete and transitive R, the converse statements to Proposition 1 do not hold. We present an example of a weakly quasisubmodular R which fails both modularity and transitive submodularity. This R is not monotonic, so the example (and Kreps, Epstein, and Marinacci's result) motivate our Proposition 4 below.

Example 3. Consider the lattice X shown in the Hasse diagram below. The points x, y, z, w and s are ordered by their position in the diagram: larger elements are higher in the diagram.



Consider the relation R induced by the function u which takes the value 0 on $\{w, s\}$ and 1 on $\{x, y, z\}$. Note that R is not monotonic. It is clear that R is weakly quasisubmodular, as the only unordered elements of X are x and z, and $x\,P\,x \wedge z$. We show that R is neither modular nor transitive submodular. First note that $x \vee y = y$, so $x\;I\;(x \vee y)$. But $x \vee z = w$ and $x \vee z \vee y = y$, so $y\;P\;w$ implies that $\neg\,(x \vee z\;I\;x \vee y \vee z)$. Thus R is not modular. Second, note that $x \geq^* y$ and $y \geq^* z$, as u is 1 on $\{x, y, z\}$, $x \vee y = y$ and $z \vee y = y$. But $x \vee z = w$, and $u(w) = 0$. So $\neg\,(x\;I\;x \vee z)$ and thus R is not transitive submodular.

Proposition 4. *Suppose that* $(X, \leq)$ *is a lattice. If* R *is transitive and monotonic, then weak quasisubmodularity implies i) modularity and ii) transitive submodularity.*

Proof. Suppose that R is weakly quasisubmodular. We shall establish that it is modular. To this end, suppose that $x\;I\;(x \vee y)$. We will show that $x \vee z\;I\;(x \vee y) \vee z$. As $(x \vee y) \wedge (x \vee z) \geq x$, monotonicity of R implies $(x \vee y) \wedge (x \vee z)\,R\,x$. As $x\;I\;(x \vee y)$, transitivity of R implies that $(x \vee y) \wedge (x \vee z)\,R\,(x \vee y)$. By monotonicity of R, $(x \vee y) \wedge (x \vee z)\,I\,(x \vee y)$. By weak quasisubmodularity, $(x \vee z) I (x \vee y) \vee (x \vee z) = (x \vee y) \vee z$. So $x \vee z I (x \vee y) \vee z$ and modularity is satisfied. Now, suppose that R is transitive, monotonic, and weakly quasisubmodular. We will establish that it is transitive submodular. Let $x, y, z \in X$ such that $x \geq^* y$ and $y \geq^* z$; i.e. $x \, I \, (x \vee y)$ and $y \, I \, (y \vee z)$. We will establish that $x \, I \, (x \vee z)$. First, note that $y \leq (x \vee y) \wedge (y \vee z) \leq (y \vee z)$. By monotonicity, and as $y \, I \, (y \vee z)$, we may conclude that $(y \vee z) \, I \, (y \vee z) \wedge (x \vee y)$. By weak quasisubmodularity, we conclude that $(y \vee z) \vee (x \vee y) \, I \, (x \vee y)$. Hence, $(x \vee y) \, I \, (x \vee y \vee z)$. Moreover, $x \, I \, (x \vee y)$, so that $x \, I \, (x \vee y \vee z)$ by transitivity of R. Now, $x \leq (x \vee z) \leq (x \vee y \vee z)$. Conclude by monotonicity that $x \, I \, (x \vee z)$, so that $x \geq^* z$. □

Alternatively, part *ii*) of Proposition 4 follows as for any transitive monotonic R, modularity implies transitive submodularity. Quasisubmodularity is a generalization of the notion of submodularity which is meaningful for functions. For a lattice $(X, \leq)$, say a function $u : X \to \mathbb{R}$ is **submodular** if for all $x, y \in X$, $u(x \wedge y) + u(x \vee y) \leq u(x) + u(y)$. A **representation** of R is a function $u : X \to \mathbb{R}$ for which for all $x, y \in X$

$$x \, R \, y \Longrightarrow u(x) \geq u(y)$$
$$x \, P \, y \Longrightarrow u(x) > u(y) \, .$$

In a recent paper, Chambers and Echenique (2007) have shown the following:

Theorem 5. *Suppose that $(X, \leq)$ is a finite lattice, and that R is monotonic and quasisubmodular. If R has a representation, it has a submodular representation.*

The notions of modularity and transitive submodularity are of interest specifically because they are meaningful for join semilattices, whereas quasisubmodularity is not. A typical example of a join semilattice which is not a lattice is the set of nonempty subsets of some global set. Indeed, this is the environment inspiring the seminal work of Kreps (1979). Other natural examples include the positive probability events of a probability space, and nonzero measurable functions.

One method in which the preceding results can be used is to go back and forth between different environments, establishing connections between environments where quasisubmodularity has dominated, and where modularity has dominated. For example, characterizations of quasisubmodular functions based on monotone comparative statics appear in Milgrom and Shannon (1994). An alternative characterization is due to Chambers and Echenique (2007). Epstein and Marinacci (2007) use modularity to characterize mutual absolute continuity of probability measures in the multiple priors model.[4] Their result can therefore be reformulated in terms of the characterization results due to Milgrom and Shannon, or Chambers and Echenique.

For a join semilattice $(X, \leq)$, we can meaningfully define $u : X \to \mathbb{R}$ to be **submodular** if for all $x, y \in X$ for which $x \wedge y$ exists,

$$u(x \wedge y) + u(x \vee y) \leq u(x) + u(y).$$

Theorem 6. *Suppose that $(X, \leq)$ is a finite join semilattice, and that R is monotonic and modular. If R has a representation, it has a submodular representation.*

Proof. We extend $(X, \leq)$ to a lattice. Consider some $x^* \notin X$, and define $\leq^*$ on $X \cup \{x^*\}$ by $x \leq^* y \Leftrightarrow x \leq y$ for all $x, y \in X$, and $x^* \leq^* y$ for all $y \in X \cup \{x^*\}$. Note that x^* is now a lower bound for every element of $X \cup \{x^*\}$. Moreover, every pair $x, y \in X \cup \{x^*\}$ retains a least upper bound according to $\leq^*$. By a well-known result (see page 23 in Birkhoff (1967)) $(X \cup \{x^*\}, \leq^*)$ is therefore a lattice. Extend R similarly; so that for all $x, y \in X$, $x \; R^* \; y \iff x \; R \; y$ and $y \; R^* \; x^*$ for all $y \in X \cup \{x^*\}$. Note that R^* is monotonic. Moreover, R^* is also modular (this follows as $x^* \; I \; (x^* \vee y)$ implies $y = x^*$). We may conclude that R^* is quasisubmodular with respect to $(X \cup \{x^*\}, \leq^*)$. Lastly, note that as R has a representation, so does R^*. Therefore, R^* has a submodular representation. Hence for all $x, y \in X$ for which $x \wedge y$ exists, $u(x \wedge y) + u(x \vee y) \leq u(x) + u(y)$. □

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[4]Epstein and Marinacci, strictly speaking, do not consider a lattice; but it is easy to construct a lattice of equivalence classes in their framework. Epstein and Marinacci implicitly also show that typical multiple priors utility representations are not submodular (even though they are the infimum of submodular functions).

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