LECTURE NOTES ON ORDERS AND MONOTONE COMPARATIVE STATICS

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Contents

0. Disclaimer

I've used these lecture notes to teach from for a few iterations of SS211, the second-year theory sequence at Caltech. I wrote them for me to teach from. The notes are terse, and surely have many typos and mistakes. Some proofs are incomplete, and I am probably missing many references.

The interested reader may want to consult the more readable Topkis (1998), Vives (1999), Amir (2005), or Vives (2008).

In a few places, I follow Topkis closely, and in general, I've used the arguments in the original papers.

1. Introduction

Labor demand slopes down.

Consider the neoclassical theory of the firm. A firm is defined by a production function $f : \mathbf{R}_+ \to \mathbf{R}$ which specifies production as a function of labor input: $f(l)$ is how much (of a single good) the firm produces when it employs l units of labor. The price of labor is a wage $w \geq 0$, and firm sells its product at a price p. Thus the profit of the firm when it employs l units of labor is

$$\pi(w, l) = pf(l) - wl.$$

Labor demand is $l(w) = \text{argmax}_{l\geq 0} pf(l) - wl$. To do comparative statics, suppose we can apply the implicit function theorem. To this end, we need f to be smooth, and assume an interior solution to the profit maximization problem. From the first order condition $pf'(l(w)) - w = 0$ we obtain: $pf''(l(w))l'(w) - 1 =$. Meaning that $l'(w) = 1/pf''(l)$. Then it would seem that downward sloping labor demand hinges on f being concave.

As we shall see this idea is wrong. I'm going to re-consider labor demand in Example 47. For now let's ponder the following observation.

Proposition 1. *Let $f : \mathbf{R}^2 \to \mathbf{R}$ be twice differentiable, with*

$$\frac{\partial^2 f(x,t)}{\partial x \partial t} \geq 0,$$

and suppose that $a, b : \mathbf{R} \to \mathbf{R}$ are monotone increasing, with $a(t) < b(t)$ for all t. Then there is

$$x^*(t) \in \mathit{argmax}\{f(x,t) : x \in [a(t), b(t)]\}$$

that is monotone increasing.

Proof. Since f is continuous, $\text{argmax}\{f(x,t) : x \in [a(t), b(t)]\}$ is nonempty and compact; let $x^*(t)$ be its largest element. For any $t, t' \in \mathbf{R}$ with $t < t'$, note that

$$x = \min\{x^*(t), x^*(t')\} \in [a(t), b(t)]$$

because $a(t) \leq x \leq x^*(t) \leq b(t)$. Then $f(x^*(t)), t) - f(x,t) \leq 0$

and any $x \leq x^*(t)$ we have that Let $t < t'$ □

2. Preliminary definitions

When n is a positive integer and X is a set, let $[n]$ denote $\{1,\dots,n\}$ and X^n the cartesian product $\underbrace{X\times\cdots\times X}_{ntimes}$. The n-dimensional Euclidean space is $\mathbf{R}^n$.

For $x,y\in\mathbf{R}^n$ we say that $x\leq y$ if $x_i\leq y_i$ for $i\in[n]$. That $x<y$ if $x\leq y$ and $x\neq y$. Finally that $x\ll y$ if $x_i<y_i$ for all $i\in[n]$. The symbols $\geq$, $>$ and $\gg$ have the obvious meaning.

For a measure space $(\Theta,\mathcal{T})$, $\Delta(\Theta,\mathcal{T})$ is the set of all probability measures on $(\Theta,\mathcal{T})$. Usually, $\mathcal{T}$ is understood and I just write $\Delta(\Theta)$.

For $\Theta\subseteq\mathbf{R}^n$, a Borel set, I use the Borel σ-algebra without explicitly mentioning it each time. In this case, I identify probability measures μ with their associated cumulative distribution functions $F_\mu(x)=\mu(\{\theta\in\Theta:\theta\leq x\})$. ***Probability distribution*** refers to a cumulative distribution function.

$\mathbf{E}$ denotes mathematical expectation.

2.1. PO sets.

A binary relation $\leq$ on a set X is

- ***reflexive*** if $(\forall x\in X)(x\leq x))$,
- ***antisymmetric*** if $(\forall x,y\in X)(x\leq y \text{ and } y\leq x\implies x=y)$
- ***transitive*** if
$$(\forall x,y,z\in X)(y\leq x \text{ and } z\leq y\implies z\leq x)),$$
- a ***partial order*** if it is reflexive, antisymmetric and transitive,
- a ***linear order*** if $(\forall x,y\in X)(x\leq y \text{ or } y\leq x)$

A pair $(X,\leq)$, where X is a set and $\leq$ is a partial order on X, is called a ***partially ordered set***, or a PO set.

When $x\leq y$ and $x\nleq y$ we write $x<y$.

A set $[x,y]=\{z\in X:x\leq z\leq y\}$ is an ***interval***. Similarly, we use the notation $[x,y)=\{z\in X:x\leq z<y\}$, $(x,y)=\{z\in X:x<z<y\}$, and $(x,y]=\{z\in X:x<z\leq y\}$.

A PO set is ***locally finite*** if all intervals are finite sets.

Examples of PO sets:

(1) When $\leq$ is the usual partial order on $\mathbf{R}^n$, $(\mathbf{R}^n,\leq)$ is a PO set.

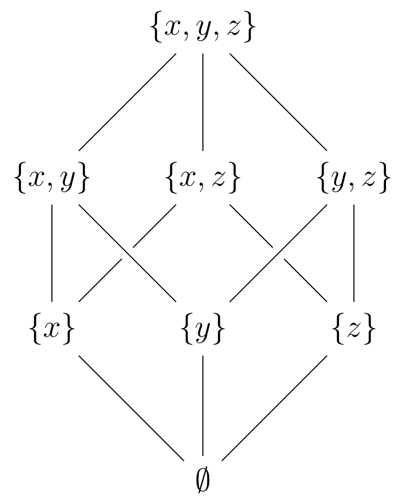

FIGURE 1. Hasse diagram for $(2^{\{x,y,z\}}, \subseteq)$

(2) More generally, if $(X_a, \leq_a)$, for a in some index set A, then $\leq$ on $\times_{a \in A} X_a$ defined by $(x_a) \leq (y_a)$ iff $x_a \leq_a y_a$ for all a is a PO set. This is called the ***product PO set***, and $\leq$ the ***component-wise order***.
(3) A special case of (2) is the set of all function X^A when $(X, \leq)$ is a PO set and we say that $f \leq g$ if $f(a) \leq g(a)$ for all $a \in A$.
(4) If Ω is a nonempty set, $(2^\Omega, \subseteq)$ is a PO set.
(5) Let $I \subseteq \mathbf{R}$ be a Borel set, and $\mathcal{F}$ be the set of all cdfs of Borel probability measures on I. Then $\mathcal{F}$ ordered by $F \leq G$ iff $G(x) \leq F(x)$ for all $x \in I$ is a PO set. The order is ***first-order stochastic dominance*** (FOSD).
(6) Let V be a real vector space and P be a pointed convex cone in V (meaning that if $x, y \in P$ $\alpha \in (0,1)$ and $\lambda > 0$ then $\lambda x \in P$ and $\alpha x + (1-\alpha)y \in P$; and that if $x, -x \in P$ then $x = 0$). Then P defines a partial order by $x \leq y$ iff $y - x \in P$.
(7) Let X be the set of natural numbers. Say that $x \leq y$ iff x divides y.

Finite PO sets are represented by a ***Hasse diagram***. For example consider $(2^X, \subseteq)$, the power set of $X = \{x, y, z\}$ ordered by set inclusion:

2.2. **Lattices.** Let $(x, \leq)$ be a PO set and $A \subseteq X$ be a subset of X. Then $A^u = \{x \in X : \forall z \in A, z \leq x\}$ is the set of ***upper bounds*** of A; and $A^l = \{x \in X : \forall z \in A, x \leq z\}$ is the set of ***lower bounds*** of A.

The ***least upper bound***, or ***supremum*** of A, if it exists, is a smallest element of A^u. So the supremum of A, denoted $\sup A$, is $x \in A^u$ such that

$x \leq z$ for all $z \in A^u$. Similarly, the ***greatest lower bound***, $\inf A$, is $x \in A^l$ such that $z \leq x$ for all $z \in A^l$.

For sets with two elements, we use a special notation to denote infimum and supremum. For $x, y \in X$, we let $x \vee y = \sup\{x, y\}$ $x \wedge y = \inf\{x, y\}$. We term $x \vee y$ the ***join*** of x and y; and $x \wedge y$ the ***meet*** x and y. Figure 2 illustrates the join and meet of two vectors in $\mathbf{R}^2$.

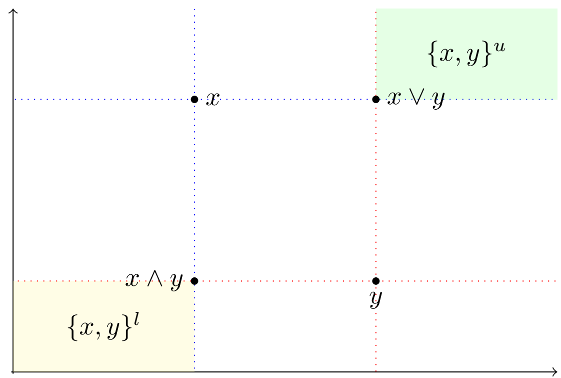

FIGURE 2. Join and meet in $\mathbf{R}^n$

Definition 2. *A PO set $(X, \leq)$ is a **lattice** if, for all $x, y \in X$, $x \wedge y$ and $x \vee y$ exist in X.*

Some properties of join and meet:

- $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$;
- $x \leq y$ iff $x = x \wedge y$;
- $x \leq y$ iff $y = x \vee y$;
- $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ and $x \vee (y \vee z) = (x \vee y) \vee z$

Remark 3. Let $(X, \leq)$ be a lattice. For any $x, y \in X$. $x \vee (x \wedge y) = x$ $x \wedge (x \vee y) = x$

Example 4. A PO set $(X, \leq)$ is ***linearly ordered*** if, for all $x, y \in X$, $x \leq y$ or $y \leq x$. Any linearly ordered PO set is a lattice.

Example 5. A ***rectangle*** in $\mathbf{R}^n$ is a subset $\prod_{i=1}^n [a_i, b_i]$ of $\mathbf{R}^n$, with $a_i < b_i$ for all $i = 1, \ldots, n$. A rectangle is a lattice; see Figure **??**

A lattice $(X, \leq)$ is ***complete*** if for all $B \subseteq A$, $\inf B$ and $\sup B$ exists in A.[1]

A lattice $(X, \leq)$ is ***distributive*** if $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.[2]

[1]This terminology is potentially confusing. In economics, linear orders are often termed complete.

[2]This is equivalent to $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

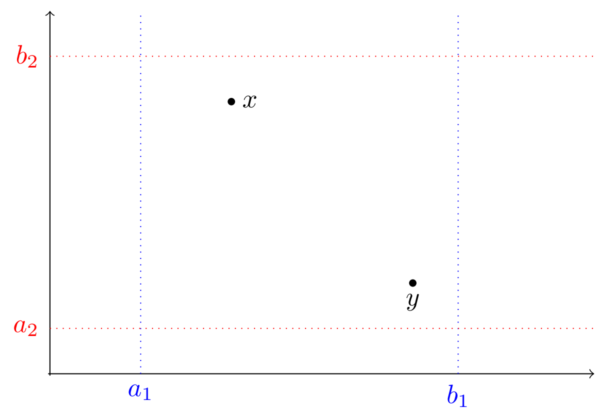

FIGURE 3. A rectangle in $\mathbf{R}^n$

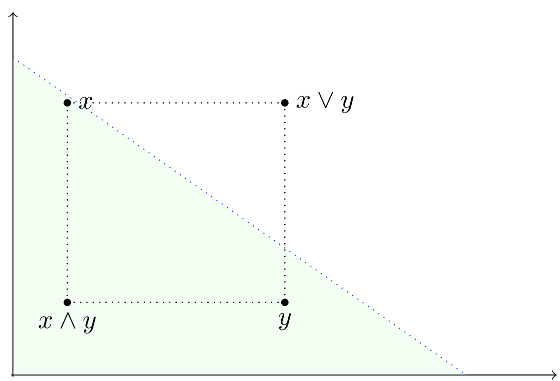

FIGURE 4. A budget set in $\mathbf{R}^n$

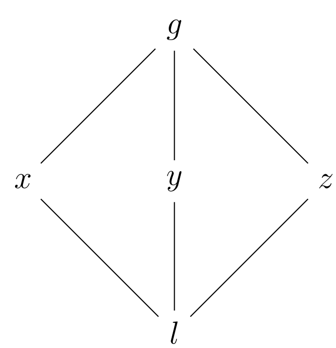

FIGURE 5. Non-distributive lattice

An example of a non-distributive lattice is $X = \{l, x, y, z, g\}$ with $l \leq \{x, y, z\} \leq g$. The Hasse diagram is in Figure 5. We have that

$$(x \vee y) \wedge z = g \wedge z = z \neq l = (x \wedge z) \vee (y \wedge z),$$

and thus the lattice $(X, \leq)$ is not distributive.

Any power-set lattice is distributive (and up to a homomorphism all distributive lattices are lattices of sets, see Birkhoff's representation theorem).

Let $(X, \leq)$ be a lattice and $A \subseteq X$. We say that A is a ***sublattice*** of $(X, \leq)$ if, for all $x, y \in A$, $x \wedge y, x \vee y \in A$.

Let A be a sublattice of $(X, \leq)$. We say that A is ***subcomplete*** if, for all $B \subseteq A$, $\inf B, \sup B \in A$.

3. Order-interval topology

Let $(X, \leq)$ be a lattice. For any $x \in X$, denote by $\downarrow x$ the set of $z \in X$ with $z \leq x$; $\downarrow x$ is called the ***principal ideal*** generated by x. Similarly, the ***principal dual ideal*** generated by x is the set $\uparrow x = \{z \in X : x \leq z\}$.

A closed interval of X is a subset of X that is

- $\emptyset$ or X;
- $\downarrow x$ or $\uparrow x$, for some $x \in X$;
- $[x, y] = \uparrow x \cap \downarrow y$, for $x, y \in X$.

The ***order interval topology*** is the topology having as sub-basis the closed intervals of X. In other words, it is the topology for which the closed sets are intersections of finite unions of closed intervals.

The following result is due to Frink (1942).

Theorem 6. *If $(X, \leq)$ is a complete lattice then it is compact in the order-interval topology.*

Proof. We only go through the main step in the proof. We need to establish that any collection of closed sets with the finite intersection property has nonempty intersection, but it turns out that it is sufficient to do so for families of basic closed sets. So let $\{F_a : a \in A\}$ be a collection of basic closed sets with the property that any finite sub-collection has nonempty intersection. We shall prove that $\cap_a F_a$ is nonempty.

First we prove the claim for the case when each F_a is a closed interval. Let $[\underline{x}_a, \bar{x}_a]$ for $a \in A$ be a collection of closed intervals in X with $\cap_{i \in \{1,\ldots,n\}} [\underline{x}_{a_i}, \bar{x}_{a_i}] = \emptyset$ for each finite subcollection $\{a_1, \ldots, a_n\}$. Let $\underline{y} = \sup\{\underline{x}_a : a \in A\}$ and $\bar{y} = \inf\{\bar{x}_a : a \in A\}$.

For any $a, a' \in A$, there is some $z \in [\underline{x}_a, \bar{x}_a] \cap [\underline{x}_{a'}, \bar{x}_{a'}]$, so $\underline{x}_a \leq z \leq \bar{x}_{a'}$. So any $\underline{x}_a$ is a lower bound on the set of $\bar{x}_{a'}$; thus $\underline{x}_a \leq \bar{y}$. Since $\bar{y}$ is an upper bound on the set of x_a, $\underline{y} \leq \bar{y}$. Thus

$$\{\underline{y}, \bar{y}\} \subseteq \cap_{a \in A} [\underline{x}_a, \bar{x}_a].$$

Each basic set is the finite union of closed intervals. So if F_a is a collection of basic closed sets so that each finite subset F_{a_i}, $i = 1, \ldots, n$ has nonempty intersection, then we the previous argument implies that $\cap_a F_a \neq \emptyset$. □

Lemma 7. *If $\{x^k\}$ is a monotone sequence in a complete lattice X then $\{x^k\}$ is convergent and $\lim_k x_k = \bigvee_k x_k$ if the sequence is monotone increasing, while $\lim_k x_k = \bigwedge_k x_k$ if it is monotone decreasing.*

Proof. We will only do the proof for the case of a monotone increasing sequence. Let A be the range of $\{x^k\}$ and $x^* = \sup A = \bigvee_k x_k$. Note that x^* is also the supremum of the range of any subsequence of $\{x^k\}$ since by monotonicity the range of any subsequence has the same set of upper bounds. Let V be any neighborhood of x^*. The claim is that eventually $x_k \in V$ for all k. Since V^c is closed and the closed order intervals are a sub-basis for the closed sets in the order interval topology, there is a collection $\{\cup_{m=1}^{n_i} [a_m^i, b_m^i] : i \in I\}$ with $V^c = \cap_{i \in I} \cup_{m=1}^{n_i} [a_m^i, b_m^i]$ (this is without loss of generality since any a_m^i or b_m^i may be $\inf X$ or $\sup X$ because X is complete). But $x^* \notin V^c$ so there is $j \in I$ with $x^* \notin \cup_{m=1}^{n_j} [a_m^j, b_m^j]$. Now, for any $m = 1 \ldots n_j$ there can only be a finite number of elements of $\{x^k\}$ in $[a_m^j, b_m^j]$. To see this note that if there is a subsequence $\{x_{k_l}\}$ with $a_m^j \preceq x_{k_l} \preceq b_m^j$ for all $l \in \mathbf{N}$ then $a_m^j \preceq x_{k_l} \preceq x^*$ and b_m^j is an upper bound on the subsequence so $x^* \preceq b_m^j$. Hence $x^* \in [a_m^j, b_m^j]$, a contradiction. Since n_j is finite, there can only be a finite number of elements of $\{x^k\}$ in $\cup_{m=1}^{n_j} [a_m^j, b_m^j]$. Hence, eventually,

$$x_k \notin \cap_{i \in I} \cup_{m=1}^{n_i} [a_m^i, b_m^i] = V^c.$$

Since V was an arbitrary neighborhood, we conclude that $x_k \to x^*$. □

Theorem 8. *Let $X = \mathbf{R}^n$, endowed with the usual order $\leq$. A sublattice $L \subseteq X$ is subcomplete iff it is compact.*

Proof. Let $\pi_i : X \to \mathbf{R}$ be the projection onto the ith coordinate. Let $L \subseteq X$ be a compact sublattice of X and let $A \subseteq L$. Denote by $\bar{A}$ the closure of A in L. Let

$$x^i \in \operatorname{argmax}\{\pi_i(x) : x \in \bar{A}\}$$

(note that x^i exists because L is compact), and let

$$x^* = \bigvee_{i=1}^{n} x^i = (x_i^i)_{i=1}^n.$$

Observe that each $x^i \in L$, so $x^* \in L$ as L is a sublattice. We claim that $x^* = \sup A$. First, x^* is an upper bound on A because if $z \in A$ then

$z_i \leq x_i^i = x_i^*$. So $z \leq x^*$. Second, x^* is the least upper bound because if z is an upper bound on A then z is also an upper bound on $\bar{A}$ (because if y^k is a sequence in A with $y^k \to y$ then $y^k \leq z$ for all k, which implies that $y \leq z$). So $x^i \in \bar{A}$ means that $x_i^i \leq z_i$. Thus $x^* \leq z$.

Conversely, let L be a subcomplete sublattice (nonempty). Since L is a subset of $\mathbf{R}^n$ it is enough to show that L is closed and bounded. Since L is subcomplete, it is bounded. So lets show that it is closed. Let $\{x^n\}$ be a sequence in L and let $x^* = \lim_{n\to\infty} x^n$. First note that, for each k,

$$z^k = \sup\{x^k, x^{k+1}, \ldots\} \in L,$$

as L is subcomplete. Note also that

$$x^{**} = \inf\{z^1, z^2, \ldots\} \in L,$$

also by the subcompleteness of L. We have that $z^k \geq z^{k+1}$ for each k, so $x^{**} = \lim_{k\to\infty} z^k$. At the same time, $\lim z^k = \lim x^k = x^*$. So the uniqueness of limits gives that $x^* = x^{**} \in L$. □

4. Möbius inverse

Suppose that $(X, \leq)$ is locally finite.

Define the function $\mu : X \times X \to \mathbf{R}$ by induction as follows. Let $\mu(x, x) = 1$. Let $\mu(x, y) = 0$ if $x \not\leq y$. Suppose that $\mu(x, z)$ has been defined for all $z \in [x, y)$, and let $\mu(x, y) = -\sum_{z\in[x,y)} \mu(x, z)$. The function μ is called the ***Möbius function*** of the PO set $(X, \leq)$.

Theorem 9. *Let $(x, \leq)$ be a locally finite PO set. Let $f : X \to \mathbf{R}$ be a function and $p \in X$ such that $f(x) = 0$ unless $p \leq x$. Then*

$$g(x) = \sum_{y\leq x} f(y)$$

implies that

$$f(x) = \sum_{y\leq x} g(y)\mu(x, y)$$

Example 10. Let X be the set of all integers. Then $\mu(x, y) = 1$ if $x = y$, $\mu(x, y) = -1$ if $y = x + 1$, and $\mu(x, y) = 0$ otherwise.

This means that $\sum_{y\leq x} g(y)\mu(x, y) = g(x) - g(x - 1)$. In this case, the theorem is a discrete fundamental theorem of calculus,

Proof. Consider the class of functions on $X \times X$ for which $h(x, y) \neq 0$ implies that $x \leq y$. We can define the product for such functions by $(hl)(x, y) =$

$\sum_{z\in[x,y]} h(x,z)l(z,y)$. The unit for this product is the Kronecker delta, $\delta(x,y) = 1$ if $x = y$ and 0 otherwise.[3]

The ***zeta function*** for $(x, \leq)$ is defined to be $\zeta(x,y) = 1$ if $x \leq y$ and 0 otherwise. Observe that

$$\sum_{x\leq z\leq y} \mu(x,z)\zeta(z,y) = \delta(x,y).^4$$

Suppose that $g(y) = \sum_{z\leq y} f(z)$. Then:

$$\begin{aligned}
\sum_{y\leq x} g(y)\mu(x,y) &= \sum_{y\leq x}\left(\sum_{z\leq y} f(z)\right)\mu(x,y) \\
&= \sum_{y\leq x}\sum_{z} \zeta(z,y)f(z)\mu(x,y) \\
&= \sum_{z} f(z)\sum_{y\leq x}\zeta(z,y)\mu(x,y) \\
&= \sum_{z} f(z)\delta(z,x) \\
&= f(x),
\end{aligned}$$

where we have used that $\sum_{y\leq x}\zeta(z,y)\mu(x,y) = \sum_{z\leq y\leq x}\zeta(z,y)\mu(x,y) = \delta(z,y)$. □

Example 11. (Inclusion-Exclusion principle) Let $(X, \leq)$ be the power set of some finite set, with $\leq$ being defined by set inclusion. Then

$$\mu(x,y) = (-1)^{|y|-|x|},$$

for $x \leq y$.

Theorem 9 is useful for solving various counting problems. For example suppose you want to know how many functions there are from $\{1,\ldots,n\}$ onto $\{1,\ldots,k\}$. This is difficult to count directly. However, the set of all functions from $\{1,\ldots,n\}$ to $A \subseteq \{1,\ldots,k\}$ is just $|A|^n$. We can use this simple fact to solve the problem. Let $f(A)$ be the number of functions from $\{1,\ldots,n\}$ onto $A \subseteq \{1,\ldots,k\}$, and $g(A)$ be the number of functions from $\{1,\ldots,n\}$ into $A \subseteq \{1,\ldots,k\}$. So $g(A) = |A|^n$, but also $g(A) = \sum_{B\subseteq A} f(B)$. Therefore,

$$f(A) = \sum_{B\subseteq A}(-1)^{|A-B|}|B|^n.$$

[3]Note that $(h\delta)(x,y) = \sum_{z\in[x,y]} h(x,z)\delta(z,y) = h(x,y)\delta(y,y) = h(x,y)$. That's what it means for δ to be the unit.

[4]So μ and ζ are inverses of each other.

Hence the number of functions from $\{1,\dots,n\}$ onto $\{1,\dots,k\}$ is

$$\sum_{B\subseteq\{1,\dots,k\}}(-1)^{k-|B|}\,|B|^n=\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}i^n.$$

This kind of application of Theorem 9 is often called the inclusion-exclusion principle.

Example 12. (Dempster-Shafer) Let Ω be a finite set. A function $m:2^\Omega\to[0,1]$ with the properties that $m(\emptyset)=0$ and $\sum_{A\subseteq\Omega}m(A)=1$ is called a ***probability assignment***. The function $b(A)=\sum_{B\subseteq A}m(B)$ is the ***belief function*** associated to m. Then we have

$$m(A)=\sum_{B\subseteq A}(-1)^{|A\setminus B|}b(B).$$

That is, given a belief function we can recover probabilistic assignments via Möbius inversion.

4.1. **Random utility.** Let X be a finite set and L the set of all total orders on X. A ***stochastic choice function*** is a mapping

$$p:X\times 2^X\to[0,1]$$

with the property that if $p(x,A)>0$ then $x\in A$ and $\sum_{x\in A}p(x,A)=1$.

A stochastic choice function p is a ***random utility function*** is there exists $\lambda\in\Delta(L)$ such that

$$p(x,A)=\lambda(\{\succeq\in: x\succeq y \text{ for all } y\in A\}).$$

The following theorem is due to Falmagne (1978). The proof using Möbius inversion is due to Fiorini (2004).

Theorem 13. *p is a random utility function iff*

$$0\leq\sum_{A\subseteq S\subseteq X}(-1)^{|S\setminus A|}p(x,S)$$

for all x and A.

$$\pi(x,A)=\lambda(\{\succeq: X\setminus A\succ x\succeq A\})$$

$$p(x,A)=\sum_{A\subseteq S\subseteq X}\pi(x,S)$$

By Möbius inversion (Theorem 9):

$$\pi(x, A) = \sum_{A\subseteq S\subseteq X} (-1)^{|S\setminus A|} p(x, S). \tag{1}$$

$p_{\succeq}(x, A) = 1$ if $x \succeq A$ and 0 otherwise.

Consider the lattice $(2^X, \subseteq)$. For each $A \subseteq X$ and $x \in X \setminus A$, let $Q(x, A)$ denote the pair $(A, A \cup \{x\}) \in 2^X \times 2^X$. Let Q denote the set of all such pairs.

To each stochastic choice function p we associate $f(p) \in [0, 1]^Q$ by letting the value of $f(p)$ at $Q(A, x)$ by $\pi(x, A)$. For each linear order $\succeq$ we consider $f(p_{\succeq})$ and note that if

$$x_1 \succ x_2 \succ \cdots x_{|X|}$$

then $f(p_{\succeq})$ assigns value 1 to $(\emptyset, x_{|X|})$, $(\{x_{|X|}\}, \{x_{|X|-1})$, ...and 0 to all other elements of Q.

Now observe that p is a random utility function iff $f(p)$ lies in the convex hull of $\{f(p_{\succeq}) : \succeq \in L\}$.

Consider a matrix A that has a row for every element of Q and a column for every $\succeq$. Let $f(p_{\succeq})$ define each column of A. [INCOMPLETE]

5. Strong Set Order

Let $(X, \leq)$ be a lattice. For $A, B \subseteq X$, say that A is ***smaller than*** B ***in the strong set order*** (or ***induced set order***; and denoted by $A \sqsubseteq B$) if

$$x \in A \text{ and } y \in B \Longrightarrow x \wedge y \in A \text{ and } x \vee y \in B.$$

Lemma 14. *$\sqsubseteq$ is antisymmetric and transitive on $2^X \setminus \{\emptyset\}$.*

Proof. Let $A \sqsubseteq B$ and $B \sqsubseteq A$. Let $x \in A$ and $y \in B$. Then $x \wedge y \in A$ and $x \vee y \in B$, as $A \sqsubseteq B$. Then $x = (x \vee y) \wedge x \in B$ and $y = y \vee (x \wedge y) \in A$, as $B \sqsubseteq A$. Thus $A = B$.

Let $A \sqsubseteq B$ and $B \sqsubseteq C$. Let $x \in A$ and $y \in C$. Choose any $z \in B$. Note that

$$x \vee y = x \vee ((y \wedge z \vee y)) = (x \vee (y \wedge z)) \vee y.$$

Now, $y \wedge z \in B$, as $B \sqsubseteq C$ Then $x \vee (y \wedge z) \in B$, as $A \sqsubseteq B$. So $x \vee y \in C$, as $B \sqsubseteq C$.

Similarly,

$$x \wedge y = (x \wedge (x \vee z)) \wedge y = x \wedge ((x \vee z) \wedge y).$$

Now, $(x \vee z) \in B$, so $(x \vee z) \wedge y \in B$. Then $x \wedge y \in A$ $\square$

Observe that $A \sqsubseteq A$ iff A is a sublattice of $(X, \leq)$. If we denote by $L(X)$ the set of all nonempty sublattices of $(X, \leq)$, we obtain:

Theorem 15. *$(L(X), \sqsubseteq)$ is a PO set.*

Let T and X be sets. A function $\phi : T \to 2^X \setminus \{\emptyset\}$ is a ***correspondence***. A function $f : T \to X$ with $f(t) \in \phi(t)$ is a ***selection*** from ϕ.

If $(X, \leq)$ and $(T, \leq')$ are PO sets, a function $f : T \to X$ is ***monotone increasing***.

Theorem 16. *Let $(X, \leq)$ be a complete lattice and $(T, \leq')$ be a PO set. Suppose that $\{S_t : t \in T\}$ is a collection of subsets of X such that*

- *S_t is a subcomplete sublattice*
- *$t \leq t' \implies S_t \sqsubseteq S_{t'}$*

Then the functions $t \mapsto \inf S_t$ and $t \mapsto \sup S_t$ are monotone increasing selections from the correspondence $t \mapsto S_t$.

Proof. First, $\inf S_t, \sup S_t \in S_t$ as S_t is a subcomplete sublattice. So these functions are selections.

Let $t \leq t'$, Since $\inf S_t \in S_t$ and $\inf S_{t'} \in S_{t'}$ we know that $\inf S_t \wedge \inf S_{t'} \in S_t$. By definition of inf, then $\inf S_t \leq \inf S_t \wedge \inf S_{t'}$. But we also have $\inf S_t \wedge \inf S_{t'} \leq \inf S_t$ by definition of $\wedge$. So $\inf S_t \wedge \inf S_{t'} = \inf S_t$, which means that $\inf S_t \leq \inf S_{t'}$. The argument for $t \mapsto S_t$ is similar. $\square$

6. Supermodular functions and increasing differences

Let $(X, \leq)$ be a lattice and $(T, \leq')$ be a PO set.

A function $f : X \times T \to \mathbf{R}$ has ***increasing differences*** if, for all $x, x' \in X$ and $t, t' \in T$ with $x < x'$ and $t < t'$

$$f(x', t) - f(x, t) \leq f(x', t') - f(x, t').$$

(Put differently, if the function $t \mapsto f(x', t) - f(x, t)$ is monotone increasing, for each $x, x' \in X$ with $x < x'$.)

A function $f : X \times T \to \mathbf{R}$ has ***strictly increasing differences*** if, for all $x, x' \in X$ and $t, t' \in T$ with $x < x'$ and $t < t'$

$$f(x', t) - f(x, t) < f(x', t') - f(x, t').$$

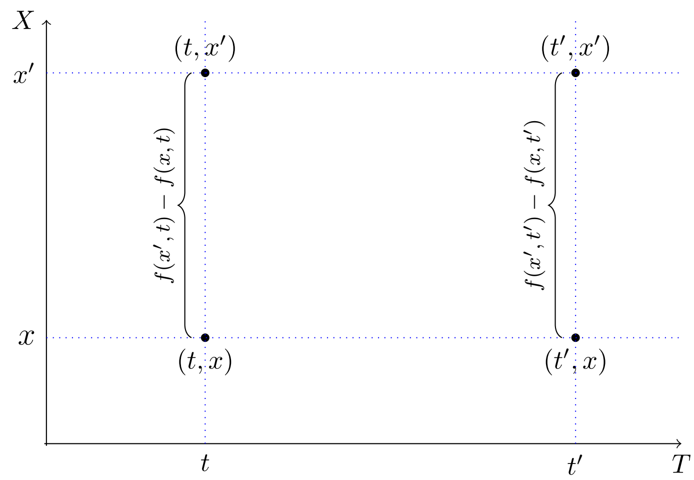

FIGURE 6. Increasing differences in $\mathbf{R}^n$

We often say that the function has increasing differences "in (x, t)." Observe that a function has increasing differences in (x, t) iff it has increasing differences in (t, x).

Figure 6 has an illustration of the increasing difference property in the case when $X = \mathbf{R}$ and $T = \mathbf{R}$. In the figure, increasing differences asks that $f(x', t) - f(x, t) \leq f(x', t') - f(x, t')$, as $x < x'$ and $t < t'$.

Definition 17. *A function $f : X \to \mathbf{R}$ is **supermodular** if, for all $x, x' \in X$,*

$$f(x) + f(x') \leq f(x \vee x') + f(x \wedge x').$$

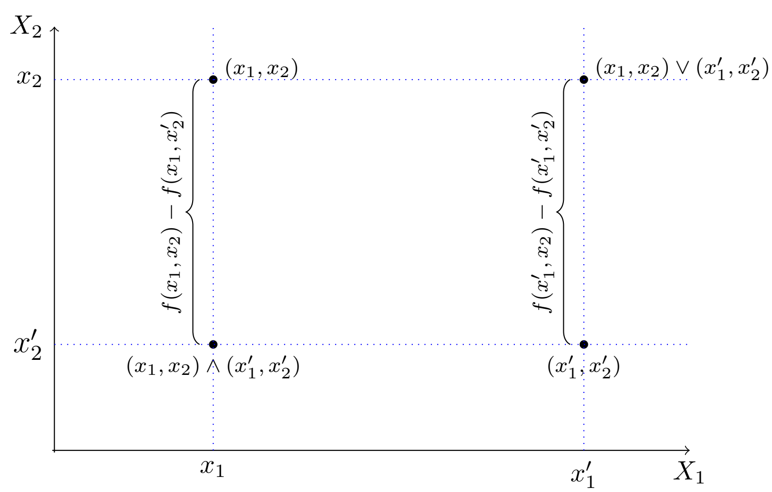

FIGURE 7. Supermodularity in $\mathbf{R}^n$

Figure **??** is an illustration of supermodularity with $X = X_1 \times X_2$. In the figure, we have that $(x_1, x_2) \wedge (x_1', x_2') = (x_1, x_2')$ and $(x_1, x_2) \vee (x_1', x_2') = (x_1', x_2)$. Supermodularity requires that

$$f(x) - f(x \vee x') \leq f(x \wedge x') - f(x'),$$

which in the figure means that

$$f(x_1, x_2) - f(x_1, x_2') \leq f(x_1', x_2) - f(x_1', x_2').$$

So in this case, supermodularity is the same as increasing differences in each dimension. This is generally true, in the sense provided by Theorem 20 below.

Supermodularity is often viewed in economics as a notion of complementarity because of the connection to increasing differences. In Figure **??**, think of f as production from two inputs, or the utility of consumption from two goods. Supermodularity says that any "marginal" increase along the X_2 dimension is aided by increases in the X_1 dimension. The two inputs, or the two consumption goods, are therefore complements.

Remark 18. If f and g are supermodular functions, and $\lambda > 0$, then $f + g$ and λf are supermodular functions. There is, however, a supermodular f and strictly increasing $h : \mathbf{R} \to \mathbf{R}$ such that $h \circ f$ is not supermodular. Supermodularity is a "cardinal" property.

Two elements $x, x' \in X$ are ***unordered*** if $x \not\leq x'$ and $x' \not\leq x$. A function $f : X \to \mathbf{R}$ is ***strictly supermodular*** if, for all unordered $x, x' \in X$,

$$f(x) + f(x') < f(x \vee x') + f(x \wedge x').$$

Theorem 19. *Let X_α be a lattice for all $\alpha \in A$. Let X be a sublattice of $\prod_{\alpha \in A} X_\alpha$. If $f : X \to \mathbf{R}$ is supermodular, then f has increasing differences.*

Theorem 20. *Let X_1 and X_2 be lattices and $f : X_1 \times X_2 \to \mathbf{R}$. If*

- *$x_1 \mapsto f(x_1, x_2)$ is supermodular for all $x_2 \in X_2$*
- *$x_2 \mapsto f(x_1, x_2)$ is supermodular for all $x_1 \in X_1$*
- *f has increasing differences,*

then f is supermodular.

Proof.

$$\begin{aligned} f(x \vee x') = & \underbrace{f(x_1 \vee x_1', x_2 \vee x_2') - f(x_1 \vee x_1', x_2)}_{A} \\ & \underbrace{+ f(x_1 \vee x_1', x_2) - f(x_1, x_2)}_{B} \\ & + f(x_1, x_2) \end{aligned}$$

By supermodularity and increasing differences:

$$A \geq f(x_1 \vee x_1', x_2') - f(x_1 \vee x_1', x_2' \wedge x_2) \geq f(x_1', x_2') - f(x_1', x_2' \wedge x_2)$$
$$B \geq f(x_1', x_2) - f(x_1 \wedge x_1', x_2) \geq f(x_1', x_2' \wedge x_2) - f(x_1 \wedge x_1', x' \wedge x_2).$$

Hence,

$$\begin{aligned} f(x \vee x') &\geq f(x_1', x_2') - f(x_1', x_2' \wedge x_2) \\ &+ f(x_1', x_2' \wedge x_2) - f(x_1 \wedge x_1', x' \wedge x_2) \\ &+ f(x_1, x_2) \\ &= f(x_1', x_2') + f(x_1, x_2) - f(x_1 \wedge x_1', x' \wedge x_2); \end{aligned}$$

establishing supermodularity. □

More generally:

Theorem 21. *Let $(X_i, \leq_i)$ be a lattice for $i = 1, \ldots, n$. If $f : \times_{i=1}^n X_i \to \mathbf{R}$ has increasing differences in (x_i, x_j), for $i \neq j$, and $x_i \mapsto f(x_i, x_{-i})$, for all i is supermodular, then f is supermodular.*

Corollary 22. *Let $X \subseteq \mathbf{R}^n$ be a lattice under the usual order on $\mathbf{R}^n$. If $f : X \to \mathbf{R}$ has increasing differences in any two variables then it is supermodular.*

There is a dual notion to supermodularity: A function $f : X \to \mathbf{R}$ is ***submodular*** if $-f$ is supermodular.

6.1. **Supermodular capacities.** Let Ω be a finite set. A ***capacity*** is a monotone increasing function $v : 2^\Omega \to \mathbf{R}$ such that $v(\emptyset) = 0$ and $v(\Omega) = 1$. The motivation for studying capacities is that they represent non-additive "probability" assessments.

A capacity is called ***convex*** if it is supermodular. For $A \subseteq \Omega$, let δ_A denote the difference operator (mapping capacities to capacities) relative to A, defined as $(\delta_A v)(B) = v(A \cup B) - v(B \setminus A)$.

Proposition 23. *A capacity v is* ***convex*** *iff $\delta_A(\delta_{A'} v) \geq 0$ for all $A, A' \subseteq \Omega$.*

Proof. Consider $\delta_A(\delta_{A'} v)$

$$\begin{aligned} (\delta_A \delta_{A'} v)(B) &= (\delta_{A'} v)(A \cup B) - (\delta_{A'} v)(B \setminus A) \\ &= v((A \cup B) \cup A') - v((A \cup B) \setminus A') \\ &- [v((B \setminus A) \cup A') - v((B \setminus A) \setminus A')] \end{aligned}$$

Let $C, D \subseteq \Omega$. Let $B = C \cap D$, $A = C \setminus B$ and $A' = D \setminus B$. Then $C \cup D = (A \cup B) \cup A'$, $(B \setminus A) \setminus A' = C \cap D$, $(A \cup B) \setminus A' = C$, and $(B \setminus A) \cup A'$. So $(\delta_A \delta_{A'} v)(B) \geq 0$ iff $v(C \cup D) - v(C) - v(D) + v(C \cap D) \geq 0$. □

Denote by $\Delta(\Omega)$ the set of all probability measures on Ω.

The ***core*** of a capacity v is the set

$$c(v) = \{p \in \Delta(\Omega) : \forall A \subseteq \Omega, p(A) \geq v(A)\}.$$

Proposition 24. *If v is a convex capacity, then $c(v) \neq \emptyset$.*

Proof. Let $p \in \Delta(\Omega)$ be defined by enumerating the elements of Ω as $\omega_1, \ldots, \omega_K$, setting $p(\omega_1) = v(\{\omega_1\})$, and $p(\omega_k) = v(\{\omega_1, \ldots, \omega_k\}) - v(\{\omega_1, \ldots, \omega_{k-1}\})$ for $k = 2, \ldots, K$. We shall prove that p is in the core of v.

Let $A \subseteq \Omega$ be nonempty. Let ω_k be the first element (in the enumeration of Ω) that is not in A. Then $A \cup \{\omega_1, \ldots, \omega_k\} = A \cup \omega_k$ and $A \cap \{\omega_1, \ldots, \omega_k\} = \{\omega_1, \ldots, \omega_{k-1}\}$ So

$$p(\{\omega_k\}) = v(\{\omega_1, \ldots, \omega_k\}) - v(\{\omega_1, \ldots, \omega_{k-1}\}) \leq v(A \cup \omega_k) - v(A).$$

Thus $p(A) + p(\{\omega_k\}) \leq v(A \cup \omega_k) - v(A) + p(A)$, or

$$p(A \cup \{\omega_k\}) - v(A \cup \omega_k) \leq p(A) - v(A).$$

Repeat this argument by adding the first element of Ω that is not in $A \cup \{\omega_k\}$, say ω_l, and so on until we obtain that

$$\begin{aligned} 0 = 1 - 1 = p(\Omega) - v(\Omega) &\leq p(A \cup \{\omega_k, w_l\}) - v(A \cup \{\omega_k, w_l\}) \\ &\leq p(A \cup \{\omega_k\}) - v(A \cup \{\omega_k\}) \\ &\leq p(A) - v(A). \end{aligned}$$

So $p(A) \geq v(A)$, and hence $p \in c(v)$. □

Proposition 25. *If v is a convex capacity, then $v(A) = \inf\{p(A) : p \in c(v)\}$.*

It is possible to define an integral of functions $f : \Omega \to \mathbf{R}$ with respect to any capacity. The definition is

$$\int f d\nu = \int_{-\infty}^{0} [\nu(\{\omega : f(\omega) \geq t\}) - 1] dt + \int_{0}^{\infty} \nu(\{\omega : f(\omega) \geq t\}) dt.$$

When v is a convex capacity then the integral becomes:

$$\int f dv = \min\{\int f dp : p \in c(v).\}$$

In fact a capacity is convex iff it has nonempty core, and the integral has the above form.

6.2. **Supermodularity of probability measures on R.** Let $\Delta(A)$ be the set of Borel probability measures on a Borel set $A \subseteq \mathbf{R}$. Each $p \in \Delta(A)$ is identified with a cdf F_p. Suppose that $h : A \to \mathbf{R}$ is bounded and measurable. Then

$$p \mapsto \int h(x) dp(x)$$

is supermodular. (In fact it is both super and submodular.)

To see this, let $A_p = \{x \in A : F_p \leq F_q\}$ and $A_q = A \setminus A_p$. Then

$$\begin{aligned}\int_A h(x) dF_{p \vee q}(x) + \int_A h(x) dF_{p \wedge q}(x) &= \int_{A_p} h(x) dF_p(x) + \int_{A_q} h(x) dF_q(x) \\ &+ \int_{A_p} h(x) dF_q(x) + \int_{A_q} h(x) dF_p(x) \\ &= \int_A h(x) dF_p(x) + \int_A h(x) dF_q(x)\end{aligned}$$

7. Lattices and supermodular functions in $\mathbf{R}^n$

Proposition 26. *Let $X_1 \subseteq \mathbf{R}^n$ and $X_2 \subseteq \mathbf{R}^m$ be open sets and $f : X_1 \times X_2 \to \mathbf{R}$ be twice differentiable. If $f(x_1, x_2)$ has increasing differences, then*

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_{1,i} \partial x_{2,j}} \geq 0$$

for $i = 1, \ldots, n$ and $j = 1, \ldots, m$.

Proposition 27. *Let $X_i \subseteq \mathbf{R}$ be an open set, and $X = \times_{i=1}^n X_i \subseteq \mathbf{R}^n$. Let $f : X \to \mathbf{R}$ be twice continuously differentiable. If*

$$\frac{\partial^2 f(x)}{\partial x_i \partial x_j} \geq 0$$

for $i \neq j$ then f has increasing differences in (x_i, x_j); and if this inequality holds for all $i \neq j$ then f is supermodular.

Corollary 28. *Let $X \subseteq \mathbf{R}^n$ be open and a lattice, and let $f : X \to \mathbf{R}$ be twice continuously differentiable. Then f is supermodular iff*

$$\frac{\partial^2 f(x)}{\partial x_i \partial x_j} \geq 0$$

for all $i \neq j$.

Example 29. The following functions are seen to be supermodular:

- $f(x_1, x_2) = x_1 x_2$
- $f(x_1, \ldots, x_n) = K \prod x_i^{a_i}$, $K > 0$ and $a_i > 0$

- $\log[D_1(p_1, p_2)(p_1 - c)]$, when
$$p_2 \mapsto \frac{\partial \log D_1(p_1, p_2)}{\partial p_1}$$
is monotone increasing. When D_1 is a demand function (for "differentiated products") this means that demand elasticity
$$\epsilon = -\frac{\partial \log D_1(p_1, p_2)}{\partial log p_1}$$
is decreasing in p_2.
- $f(x_1, x_2) = g(x_1 - x_2)$, when g is a concave function. In particular, in the previous example, $\log[D_1(p_1, p_2)(p_1 - c)]$ is supermodular in (p_1, c) as the log is concave.

Theorem 30. *If $(X, \leq)$ is a lattice and X is compact, then $(X, \leq)$ is a complete lattice.*

Proof. Proceed by induction on n, the dimension of $\mathbf{R}^n$. First, if $n = 1$ then the result is obviously true. Suppose that the result is true for all dimension $k \leq n - 1$.

Let $\pi_{n-1} : \mathbf{R}^n \to \mathbf{R}^{n-1}$ be the projection onto the first $n - 1$ coordinates, and $\pi_1 : \mathbf{R}^n \to \mathbf{R}$ be the projection onto the last coordinate, of $\mathbf{R}^n$.

Let $A \subseteq X$ be nonempty. Then
$$\pi_{n-1}A = \{y \in \mathbf{R}^{n-1} : (y, z) \in A, \text{ some } z \in \mathbf{R}\}$$
$$\pi_1 A = \{z \in \mathbf{R} : (y, z) \in A, \text{ some } y \in \mathbf{R}^{n-1}\}$$

By the inductive hypothesis, there is $\bar{y} = \sup \pi_{n-1}A \in \pi_{n-1}X$ and $\bar{z} = \sup \pi_1 A \in \pi_1 X$. □

8. Quasi-supermodularity and single crossing

Supermodularity is a cardinal property (see Remark 18), which can be a problem. Supermodularity is used to capture economic phenomena that affect the behavior of optimizing agents; but optimization is guided by ordinal, not cardinal, properties. For example, comparative statics are determined by the ordinal properties of the objective function being optimized.

An ordinal theory of supermodularity and comparative statics was proposed by Milgrom and Shannon (1994). The main concepts involved are the single-crossing property, an ordinal generalization of increasing differences, and quasi-supermodularity, an ordinal notion of supermodularity.

Let $(X, \leq)$ and $(T, \leq)$ be PO sets. A function $f : X \times T \to \mathbf{R}$ satisfies the ***single-crossing property*** in (x, t) ($x \in X$ and $t \in T$) if, for all $x, x' \in X$ and $t, t' \in T$.

$$f(x,t) \leq f(x',t) \Longrightarrow f(x,t') \leq f(x',t')$$
$$f(x,t) < f(x',t) \Longrightarrow f(x,t') < f(x',t')$$

when $x \leq x'$ and $t \leq t'$. Moreover, if

$$f(x,t) \leq f(x',t) \Longrightarrow f(x,t') < f(x',t')$$

for $x < x'$ and $t < t'$ then we say that f satisfies the ***strict single crossing property***.

Remark 31. If $f : X \times T \to \mathbf{R}$ has increasing differences in (x, t) then it satisfies the single crossing property.

Let $(X, \leq)$ be a lattice. A function $f : X \to \mathbf{R}$ is ***quasi-supermodular*** if, for all $x, x' \in X$

$$f(x \wedge x') \leq f(x') \Longrightarrow f(x) \leq f(x \vee x')$$
$$f(x \wedge x') < f(x') \Longrightarrow f(x) < f(x \vee x')$$

Remark 32. If $f : X \to \mathbf{R}$ is supermodular then it is quasi-supermodular.

The following result is due to Chambers and Echenique (2009). It says that quasi-supermodular functions that are monotone are ordinally equivalent to a supermodular function.

Theorem 33. *Let $(X, \leq)$ be a finite lattice. A function $f : X \to \mathbf{R}$ is monotone increasing and quasi-supermodular iff there is a strictly monotone increasing function $g : \mathbf{R} \to \mathbf{R}$ such that $g \circ f : X \to \mathbf{R}$ is monotone increasing and supermodular.*

Proof. One direction of the theorem is obvious. So suppose that $f : X \to \mathbf{R}$ is monotone increasing and quasi-supermodular. Let $f(X) = \{z_1, \ldots, z_N\}$ with $z_j < z_{j+1}$ (possible since X is finite). Define $g : f(X) \to \mathbf{R}$ by $g(z_j) = 2^{j-1}$. Note that g is strictly monotone increasing, so that $h = g \circ f$ is monotone increasing. Note that h is quasi-supermodular. Let $x, y \in X$. If x and y are ordered under $\leq$ then $h(x) + h(y) = h(x \wedge y) + h(x \vee y)$. So suppose that they are not ordered. By monotonicity, $h(x \wedge y) \leq h(x)$ and $h(y) \leq h(x \vee y)$.

If $h(x \wedge y) = h(x)$ then $h(y) \leq h(x \vee y)$ implies that $h(x) + h(y) \leq h(x \wedge y) + h(x \vee y)$. And if $h(y) = h(x \vee y)$ then quasi-supermodularity of h implies that $h(x \wedge y) = h(x)$, and we are again done.

So suppose that $h(x \wedge y) < h(x)$ and $h(y) < h(x \vee y)$. Let $j \in \{1, \ldots, N\}$ be such that $f(x \vee y) = z_j$. Then

$$h(x \wedge y) + h(x \vee y) \geq 2^{j-1} = 2^{j-2} + 2j - 2 \geq h(x) + h(y).$$

□

Corollary 34. *Let $(X, \leq)$ be a finite lattice. If $f : X \to \mathbf{R}$ is strictly monotone increasing then there is $g : \mathbf{R} \to \mathbf{R}$ such that $g \circ f : X \to \mathbf{R}$ is strictly monotone increasing and supermodular.*

Proof. Note that if f is is strictly monotone increasing then it is quasi-supermodular. The result follows then from Theorem 33. □

Remark 35. A consequence of Corollary 34 is that supermodularity of utility is not testable using a finite collection of data from "downward-inclusive" budgets (this means that is x is a feasible choice in a budget, then any $y \leq x$ is also a feasible choice; budgets in neoclassical demand theory are an example). See Chambers and Echenique (2009).

8.1. **Infinite supermodularity.** A notion of infinite supermodularity has been used in the literature on cooperative games and decision theory.

Let $(X, \leq)$ be a lattice.

A function $f : X \to \mathbf{R}$ is ***infinitely supermodular*** if, for all $n \geq 2$, and all $x_1, \ldots, x_n \in X$,

$$\sum_{I \subseteq [n], I \neq \emptyset} (-1)^{|I|+1} f(\bigwedge_{i \in I} x_i) \leq f(\bigvee_{i=1}^{n} x_i)$$

Note that when $n = 2$ the equation above states that, for any $x_1, x_2 \in X$,

$$f(x_1) + f(x_2) - f(x_1 \wedge f(x_2) \leq f(x_1 \vee x_2).$$

Infinite supermodularity imposes this inequality, and many more.

Theorem 33 has been strengthened by Chateauneuf et al. (2017) as follows:

Theorem 36. *Let $(X, \leq)$ be a finite lattice. A function $f : X \to \mathbf{R}$ is monotone increasing and quasi-supermodular iff there is a strictly monotone increasing function $g : \mathbf{R} \to \mathbf{R}$ such that $g \circ f : X \to \mathbf{R}$ is monotone increasing and infinitely supermodular.*

I omit the proof of Theorem 36, but it is worth mentioning that their construction of an infinitely supermodular representation uses ideas from Kreps (1979) that I reproduce in Lemma 42.

Connection to belief functions - Dempster-Shafer. In particular see: Wong et al. (1991) discuss an axiomatization of infinitely supermodular capacities. These are used by Mukerji (1997) and Ghirardato (2001) to capture how unforseen contingencies give rise to ambiguity aversion.

8.2. **Preference for flexibility.** We can define quasi-submodularity, a "dual" to quasi-supermodularity, and an ordinal notion of submodularity, by

$$f(x') \leq f(x \wedge x') \Longrightarrow f(x \vee x') \leq f(x)$$
$$f(x') < f(x \wedge x') \Longrightarrow f(x \vee x') < f(x).$$

This notion was introduced by David Kreps in his paper on preferences for flexibility (Kreps (1979)). Kreps wanted f to represent preferences over subsets of some grand set of possible choices. So $(X, \leq) = (2^\Omega, \subseteq)$.[5] It is then natural to impose that f is monotone increasing: more choices must be at least as good as fewer choices. Since $x' \supseteq x \wedge x'$ and $x \vee x' \supseteq x$ then we have that quasi-submodularity is equivalent to

$$f(x') = f(x \wedge x') \Longrightarrow f(x \vee x') = f(x).$$

This is Kreps' axiom of preference for flexibility: If $\succeq$ is a preference relation (a weak order) on 2^Ω then

$$A \sim A \cup B \Longrightarrow A \cup C \sim A \cup C \cup B,$$

for all $A, B, C \subseteq \Omega$. The meaning of the axiom is that if adding the options in B to A is not valuable, then adding the options in B to the larger set $A \cup B$ is not valuable either.

Theorem 37. *(Kreps) Let Ω be finite and $f : 2^\Omega \to \mathbf{R}$. Then f is quasisubmodular and monotone increasing iff there is a finite set S and $u : \omega \times S \to \mathbf{R}$ such that f is a strictly monotone increasing transformation of*

$$v(A) = \sum_{s \in S} max\{u(\omega, s) : \omega \in A\}.$$

8.2.1. *Proof of Theorem 37.* I omit the proof of the less interesting "if" direction of the theorem. Let $A \succeq B$ iff $f(A) \geq f(B)$. When $A \succeq B$ and $B \succeq A$ we write $A \sim B$. When $A \succeq B$ and it is false that $B \succeq A$ we write $A \succ B$. Define a binary relation $\sqsupseteq$ on 2^Ω by $A \sqsupseteq B$ if $A \sim A \cup B$.

Lemma 38. $\sqsupseteq$ *is transitive.*

[5]Kreps restricts attention to the nonempty subset of Ω.

Proof. Let $A \sqsupseteq B$ and $B \sqsupseteq C$. Then $A \sim A \cup B$ and $B \sim B \cup C$. By quasisubmodularity, $B \sim B \cup C$ implies that $B \cup A \sim B \cup C \cup A$, while $A \sim A \cup B$ implies that $A \cup C \sim A \cup B \cup C$. So we have that

$$A \sim A \cup B \sim A \cup B \cup C \sim A \cup C.$$

Therefore transitivity of $\sim$ gives $A \sim A \cup C$, and thus $A \sqsupseteq C$. □

Define

$$g(A) = \bigcup_{\{B : A \sqsupseteq B\}} B.$$

Observe that if $A \supseteq B$ then $A \sqsupseteq B$. We shall see that $g(A)$ is the largest (in the sense of set inclusion) set B with the property that $A \sqsupseteq B$.

Lemma 39. *The function g is monotone increasing.*

Proof. Let $A \supseteq B$. For any C with $B \sqsupseteq C$, $A \sqsupseteq B \sqsupseteq C$, so transitivity of $\sqsupseteq$ implies that $C \subseteq g(A)$. Since C was arbitrary, $g(A) \supseteq g(B)$. □

Lemma 40. *The function g satisfies:*

(1) $A \sqsupseteq g(A)$ *and* $A \sim g(A)$.
(2) $g(A) = g(g(A))$.
(3) $A \sqsupseteq B$ *iff* $g(A) \supseteq g(B)$.

Proof. For (1) we show that if $A \sqsupseteq B$ and $A \sqsupseteq B'$ then $A \sqsupseteq B \cup B'$. Since B and B' are arbitrary, we conclude that $A \sqsupseteq g(A)$. In particular, $A \sqsupseteq B$ and $A \sqsupseteq B'$ mean that $A \sim A \cup B$ and $A \sim A \cup B'$. By quasisubmodularity, $A \sim A \cup B$ implies that $A \cup B' \sim A \cup B \cup B'$. Hence

$$A \sim A \cup B' \sim A \cup B \cup B'.$$

Thus $A \sqsupseteq B \cup B'$. Also, since $A \sqsupseteq A$, $g(A) \supseteq A$ so $g(A) \succeq A$ and $A \sim A \cup g(A)$ implies that $A \succeq g(A)$. Thus $A \sim g(A)$.

For property (2), note that if $g(A) \sqsupseteq B$ then $A \sqsupseteq g(A) \sqsupseteq B$. So transitivity of $\sqsupseteq$ implies that $A \sqsupseteq B$. Since $g(g(A))$ is the union of such B, we have that $A \sqsupseteq g(g(A))$ and thus $g(A) \supseteq g(g(A))$. For the converse set inclusion note that $A \subseteq g(A)$ and that g is monotone increasing (Lemma 39).

Finally, to prove (3) note that $A \sqsupseteq B$ iff $B \subseteq g(A)$ (since, as we noted above, $B \subseteq g(A)$ implies that $A \sqsupseteq g(A) \sqsupseteq B$). Now, $g(A) \supseteq g(B)$ implies that $g(A) \supseteq B$ (as $g(B) \supseteq B$), and hence that $A \sqsupseteq B$. Conversely, $B \subseteq g(A)$ implies that $g(B) \subseteq g(g(A)) = g(A)$ by monotonicity of g and (2). □

Using g we can define the set S of "states" on which the utility function will depend. These states give rise to a preference for flexibility. Let $S = \{g(A) : A \subseteq \Omega\}$.

First we prove an ordinal version of the theorem. This will involve two objects: u and w. Define

$$u(x, s) = \mathbf{1}_{x \notin s} \text{ and } w_s(A) = \max\{u(x, s) : x \in A\},$$

so that $w_s(A)$ is 1 if $A \setminus s \neq \emptyset$ and 0 if $A \subseteq s$.

Consider the vector $w(A) = (w_s(A))_{s \in S}$.

Lemma 41. *$w(A) \geq w(B)$ iff $A \sqsupseteq B$.*

Proof. If $A \sqsupseteq B$ and $w(A)_s = 0$ then $A \subseteq s = g(D)$. Then

$$g(A) \subseteq g(g(D)) = g(D) = s.$$

Since $B \subseteq g(A)$, we have that $B \subseteq s$, and therefore $w_s(B) = 0$. This means that $w(A) \geq w(B)$. Conversely, if it is not the case that $A \sqsupseteq B$ then $B \not\subseteq g(A)$ by Lemma 40, and hence $w_{g(A)}(B) = 0$ while $w_{g(A)}(A) = 1$ as $A \subseteq g(A)$. □

The proof requires the idea of an extension. If R is a binary relation and $\succeq$ a weak order (a preference relation) then we say that $\succeq$ ***extends*** R if $R \subseteq \succeq$ and $P_R \subseteq \succ$.[6]

First we argue that $\succeq$ extends the binary relation R defined as $A \; R \; B$ iff $w(A) \geq w(B)$. Note that if $w(A) = w(B)$ then $A \sqsupseteq B$ and $B \sqsupseteq A$, so $A \sim A \cup B \sim B$ and therefore $f(A) = f(B)$. On the other hand, if $w(A) > w(B)$ then $A \sqsupseteq B$, and it is false that $B \sqsupseteq A$. This means that

$$A \sim A \cup B \succ B,$$

by monotonicity of $\succeq$. Thus $A \succ B$.

Observe that at this point we have established an ordinal version of Theorem 37. Since $\succeq$ is an extension of R, it is possible to find a monotone increasing function $v : \mathbf{R}^S \to \mathbf{R}$ such that $f(A) = v(w(A))$.[7]

Finally, we turn to the additive representation in the statement of the theorem. The relevant "utility function" will be u^a below. As preparation we shall need the following result:

[6]Here $P_R \subseteq R$ is the strict part of R, defined as $x \; P_R \; y$ if $x \; R \; y$ and not $y \; R \; x$. Extensions are a fundamental idea in revealed preference theory: see Chambers and Echenique (2016).

[7]Incidentally, the extension property can be written as: $w(A) \geq w(B)$ implies that $f(A) \geq f(B)$ and $w(A) > w(B)$ implies that $f(A) > f(B)$.

Lemma 42. *Let $(X, \leq)$ be a PO set. If R is a weak order and an extension of the partial order $\geq$ on X. Then there exists a function $\alpha : X \to \mathbf{R}_-$ such that $x\ R\ y$ iff*

$$\sum_{x':x\leq x'} \alpha(x') \leq \sum_{x':y\leq x'} \alpha(x').$$

Proof. Denote by I_R and P_R the indifference and strict parts of R. The proof proceeds by induction. Let $X_1 = \{x \in X : (\forall x' \in X)(x\ R\ x')\}$ $X_2 = \{x \in X \setminus X_1 : (\forall x' \in X \setminus X_1)(x\ R\ x')\}$, and so on, until we have a partition $X_1, \ldots, X_K$ of X into equivalence classes of I_R. For each $x \in X_1$ let $\alpha(x) = -1$.

Next, suppose that $\alpha(x) < 0$ has been defined for all $x \in X_1 \ldots X_k$ with the property that if $x, y \in X_l$ then

$$\sum_{x':x\leq x'} \alpha(x') = \sum_{x':y\leq x'} \alpha(x'),$$

for $l = 1, \ldots, k$.

Let $x \in X_{k+1}$. If $x < x'$ then $x'\ P_R\ x$, as R extends $\leq$, and so $\alpha(x')$ has been defined. Let

$$v^{k+1} < \inf\{\sum_{x':x<x'} \alpha(x') : x \in X_{k+1}\},$$

which is well defined as X is finite. Now define $\alpha(x) = \sum_{x':x<x'} \alpha(x') - v_l$. Observe that for any $x \in X_{k+1}$, then,

$$\sum_{x':x\leq x'} \alpha(x') = \alpha(x) + \sum_{x':x<x'} \alpha(x') = v^{k+1},$$

as $\leq$ is reflexive.

By construction, $x\ R\ y$ iff $\sum_{x':x\leq x'} \alpha(x') \geq \sum_{x':y\leq x'} \alpha(x')$. □

Now to finish the proof of Theorem 37, apply Lemma 42 to the set S with the weak order $\succeq$ and the partial order $\subseteq$. Observe that, while we may in general have $A \subsetneq B$ and $A \sim B$, this cannot happen when $A, B \in S$. So let α be as defined in the lemma and consider

$$u^a(s, x) = \begin{cases} 0 & \text{if } x \notin S \\ \alpha(S) & \text{if } x \in S \end{cases}$$

so that $\max\{u^a(s, x) : x \in A\}$ is 0 if $A \setminus s \neq \emptyset$, and $\alpha(s) < 0$ if $A \subseteq s$. Then

$$\sum_{s\in S} \max\{u^a(s, x) : x \in A\} = \sum_{s\in S:s\supseteq A} \alpha(S) = \sum_{s\in S:s\supseteq g(A)} \alpha(S).$$

Finally, observe that $A \sim g(A)$ so that

$$\begin{aligned} A \succeq B \text{ iff } & g(A) \succeq g(B) \\ \text{iff } & \sum_{s \in S: s \supseteq g(A)} \alpha(S) \geq \sum_{s \in S: s \supseteq g(B)} \alpha(S) \\ \text{iff } & \sum_{s \in S} \max\{u^a(s,x) : x \in A\} \geq \sum_{s \in S} \max\{u^a(s,x) : x \in B\}. \end{aligned}$$

8.3. **Bibliographical notes.** Kreps' notion of quasi-submodularity was used by Epstein and Marinacci (2007) to study mutual absolute continuity in multiple priors model. The connections between these definitions and the ideas in monotone comparative statics are considered in Chambers and Echenique (2008).

9. Monotone comparative statics

Let $(X, \leq)$ be a lattice, $(T, \leq')$ a PO set, and $f : X \times T \to \mathbf{R}$ a function. Denote by

$$M(t,S) = \text{argmax}\{f(x,t) : x \in S\}$$

the set of maximizers of f over the set $S \subseteq X$ for fixed $t \in T$.

The next result is due to Milgrom and Shannon (1994)

Theorem 43. *(Milgrom and Shannon's Monotonicity Theorem) $M(t,S)$ is monotone increasing if and only if*

- $x \mapsto f(x,t)$ *is quasi-supermodular;*
- f *satisfies the single crossing property in* (x,t)

Proof. First prove the if direction. Let $t \leq t'$ and $S \sqsubseteq S'$. Let $x \in M(t,S)$ and $x' \in M(t',S')$. Since $S \sqsubseteq S'$, $x \wedge x' \in S$ and $x \vee x' \in S'$. Now, $x \wedge x' \in S$ and $x \in M(t,S)$ means that $f(x \wedge x', t) \leq f(x,t)$. By the single-crossing property, then, $f(x \wedge x', t') \leq f(x,t')$. Then quasi-supermodularity implies that $f(x',t') \leq f(x \vee x', t')$. Since $x' \in M(t',S')$ and $x \vee x' \in S'$ we obtain that $x \vee x' \in M(t',S')$. Now, $x', x \vee x' \in M(t',S')$ imply that $f(x',t') = f(x \vee x', t')$. So we must have, by single crossing and supermodularity, that $f(x \wedge x', t) = f(x,t)$. Thus $x \wedge x' \in M(t,S)$.

Conversely, suppose that $M(t,S)$ is monotone increasing. Let $x, x' \in X$ and $t, t' \in T$. First consider $S = \{x, x \wedge x'\}$ and $S' = \{x', x \vee x'\}$. Note that $S \sqsubseteq S'$. Suppose that $f(x \wedge x', t) \leq f(x,t)$; so $x \in M(t,S)$. Then $M(t,S) \sqsubseteq M(t,S')$ implies that we must have $f(x',t) \leq f(x \vee x', t)$ (as

$f(x',t) > f(x \vee x',t)$ would imply $x' \in M(t,S')$ and $x \vee x' \notin M(t,S')$)). Similarly, $f(x \wedge x',t) < f(x,t)$ implies $f(x',t) < f(x \vee x',t)$.

Let $x < x'$ and $t < t'$. Consider $S = \{x, x'\}$. If $f(x,t) \leq f(x',t)$, then $x' \in M(t,S) \sqsubseteq M(t',S)$ implies that $f(x,t') \leq f(x',t')$. Similarly, $f(x,t) \leq f(x',t)$ implies that $f(x,t') < f(x',t')$. ☐

Corollary 44. *Let S be a sublattice of X and $f : X \to \mathbf{R}$ be quasi-supermodular. Then argmax$\{f(x) : x \in S\}$ is either empty or a sublattice.*

The following result is essentially from Topkis (1978); it follows directly from the more general Milgrom and Shannon result.[8]

Corollary 45. *(Topkis's monotonicity theorem) Let $x \mapsto f(x,t)$ be supermodular and (x,t) satisfy increasing differences. Then $M(t,S)$ is monotone increasing.*

Theorem 46. *(Milgrom and Shannon's Monotone Selection Theorem) Let $t \mapsto S_t \in L(X)$ be monotone increasing, and $f : X \times T \to \mathbf{R}$ be quasi-supermodular in x and satisfy the strict single-crossing property in (x,t). Then every selection $x^*(t) \in M(t,S_t)$ is monotone increasing.*

Proof. Let $t < t'$, $x \in M(t,S_t)$ and $x' \in M(t',S_{t'})$. Since $x \wedge x' \in S_t$ we have that $f(x \wedge x',t) \leq f(x,t)$. Then quasi-supermodularity implies that $f(x',t) \leq f(x \vee x',t)$. But then $x' < x \vee x'$ and strict single crossing would imply that $f(x',t') < f(x \vee x',t')$, contradicting that $x' \in M(t',S_{t'})$ as $x \wedge x' \in S_{t'}$. So we must have $x' = x \vee x'$, and therefore that $x \leq x'$. ☐

Example 47. Labor demand slopes down. Lets consider again the example from the introduction.

Let $\pi(l,w) = pf(l) - wl$, where p,w,l are all real variables: p is price, w is wage and l is labor. $f : \mathbf{R} \to \mathbf{R}$ is a production function. Note that π satisfies the single-crossing property in $(-w,l)$. So labor demand is monotone decreasing. This result holds without any assumptions on f; it is purely a consequence of the interaction between wages and labor: at higher wages, any given increase in labor use gives a higher cost increase. Notice the contrast with the use of the implicit function theory in the first lecture.

Suppose you want to generalize the result to more than one factor. Let $\pi(z,w) = pf(z) - w \cdot z$, where $z = (z_1, z_2)$ is a vector of production factors and $w = (w_1, w_2)$ is a vector of factor prices. As before, p is price and f a production function.

[8] One may also want to obtain strict monotonicity of solutions in parameters, which turns out to be tricky. It is not true that strict versions of single-crossing and quasi-supermodularity are enough. See Edlin and Shannon (1998).

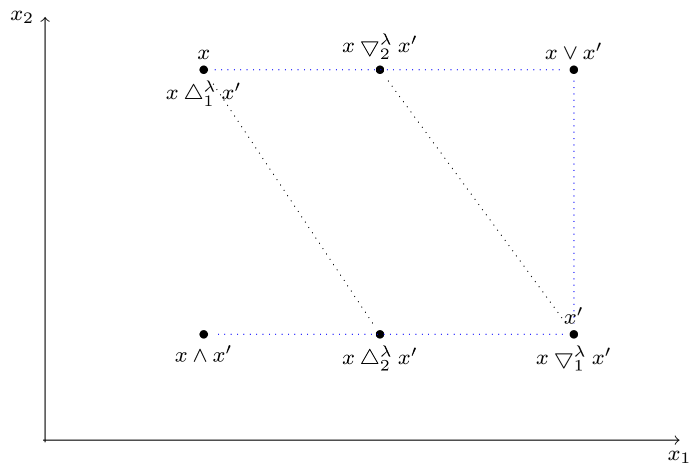

FIGURE 8. The elements $x \bigtriangleup_i^\lambda x'$ and $x \bigtriangledown_i^\lambda x'$

Now factor demand is decreasing when f is quasi-supermoular. In this case we do need some assumptions on production, but it is a natural assumption because we want the increase in one factor go "move" the demand for other factors in the same direction. The Milgrom-Shannon theorem formalizes this idea since it provides a necessary and sufficient condition for the monotonicity of factor demands.

Example 48. Let $\pi_i(p_i, p_{-i}, c_i) = (p_i - c_i)D_i(p_i, p_{-i})$ be the profit function of a firm i. D_i is the demand facing firm i when prices are $p = (p_1, \ldots, p_n) = (p_i, p_{-i})$ and c_i is the (constant) marginal cost. By a similar calculation to what we already did, we see that $\pi_i(p_i, p_{-i}, c_i)$ has increasing differences in (p_i, c_i) and in (p_i, p_{-i}). So the optimal price will be increasing in c_i and also in competitors' prices.

Shannon (1995) presents necessary and sufficient conditions for monotone comparative statics using both weaker and stronger set orders, which allow one to ensure the existence of a monotone increasing selection, or rule out the existence of a monotone decreasing selection, from the set of optimizers.

9.1. **Comparative statics of constrained optimization problems.** The theory is developed by Quah (2007). The punchline is that monotone comparative statics are possible when constraint sets are ordered using a weaker order than the strong set orden, in particular one that is compatible with constraint sets that are not sublattices. The cost will be a strengthening of the assumptions on objective functions. One sufficient condition will be supermodularity and concavity of the objective functions. (I follow Quah in outlining the theory with weaker ordinal property.)

If $x_i > x'_i$ then

$$x \bigtriangledown_i^\lambda x' = \lambda x + (1-\lambda)(x \vee x')$$
$$x \bigtriangleup_i^\lambda x' = \lambda x' + (1-\lambda)(x \wedge x'),$$

while $x \bigtriangledown_i^\lambda x' = x'$ and $x \bigtriangleup_i^\lambda x' = x$ when $x_i \leq x'_i$.

Observe in particular that when x and x' are unordered, $x \bigtriangledown_i^\lambda x' \neq x' \bigtriangledown_i^\lambda x$. The binary operations $\bigtriangledown_i^\lambda$ and $\bigtriangleup_i^\lambda$ are not symmetric.

For x and x' unordered, and with $x_i < x'_i$, the four points $\{x, x', x \bigtriangleup_i^\lambda x', x \bigtriangledown_i^\lambda x'\}$ define a parallelogram: see Figure 8. This addresses the problem we identified in Figure 4, whereby budget sets are not lattices. In fact, the main monotone comparative statics result here will deliver comparative statics for consumer choice problems: see Example 53.

The function $f : D \to \mathbf{R}$ is C_i***-supermodular*** if

$$f(x) + f(x') \leq f(x \bigtriangledown_i^\lambda x') + f(x \bigtriangleup_i^\lambda x')$$

for all $\lambda \in [0, 1]$. Say that it is C-supermodular if it is C_i-supermodular for all i. The notion is illustrated in Figure 9.

Remark 49. A C-supermodular function is supermodular. Indeed, $x \bigtriangledown_i^0 x'$ is either $x \vee x'$ or x', and $x \bigtriangleup_i^0$ is either $x \wedge x'$ or x. Then, for all $x, x' \in D$, the inequalities $f(x) + f(x') \leq f(x \bigtriangledown_i^0 x') + f(x \bigtriangleup_i^0 x')$ for all i imply $f(x) + f(x') \leq f(x \wedge x') + f(x \vee x')$.

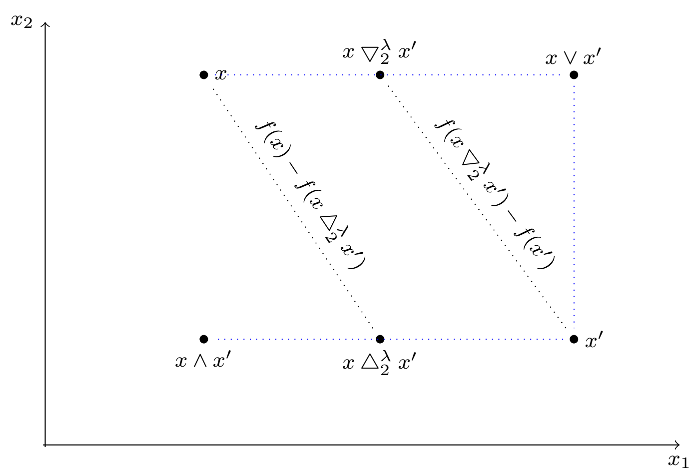

FIGURE 9. C_i-quasisupermodularity

The function $f : D \to \mathbf{R}$ is C_i***-quasisupermodular*** if

$$f(x) - f(x \triangle_i^\lambda x') \geq 0 \Longrightarrow f(x \triangledown_i^\lambda x') - f(x') \geq 0$$
$$f(x) - f(x \triangle_i^\lambda x') > 0 \Longrightarrow f(x \triangledown_i^\lambda x') - f(x') > 0$$

for all $\lambda \in [0,1]$. Say that it is C-quasisupermodular if it is C_i-quasisupermodular for all i.

Let $D \subseteq \mathbf{R}^n$ be convex. A function $f : D \to \mathbf{R}$ is ***concave in direction*** v, for $v \in \mathbf{R}^n$ if, for any $x \in D$, the mapping

$$\alpha \mapsto f(x + \alpha v)$$

is concave. The function is i-concave if it is concave in direction v, for all $v\mathbf{R}^n$ with $v_i = 0$. Finally, f is ***partially concave*** if it is i-concave for all i. Obviously a concave function is partially concave, but the latter is a strictly weaker property.

Observe that when $\lambda = 0$ the inequality in the definition of C_i-supermodularity requires f to be supermodular, so that any C_i-supermodular function is also supermodular. In fact, we have that

Proposition 50. *If $f : D \to \mathbf{R}$ is partially concave and supermodular, then it is C-supermodular.*

Proof. Let $x, x' \in D$ be unordered, and $x'_i < x_i$. Then we have:

$$\begin{aligned} f(x \triangledown_i^\lambda x') - f(x \vee x') &= f(x \vee x' - \lambda \underbrace{(x \vee x' - x)}_{=v}) - f(x \vee x') \\ &\geq f(x \vee x' - \lambda v - (1-\lambda)v) - f(x \vee x' - (1-\lambda)v) \\ &= f(x) - f(x + \lambda v), \end{aligned}$$

where the inequality follows from i-concavity, as $v > 0$ and $v_i = 0$, and the last equality because $x \vee x' = x + v$, so that $x \vee x' - (1-\lambda)v = x + \lambda v$.

Now $x + \lambda v = \lambda x \vee x' + (1-\lambda)x$ and $x \wedge x' + \lambda v = \lambda x' + (1-\lambda)x \wedge x'$. So that $(x + \lambda v) \vee x' = x \vee x'$ and $(x + \lambda v) \wedge x' = \lambda x \wedge x' + (1-\lambda)x' = x \wedge x' + \lambda v$. Hence supermodularity implies that

$$f(x \vee x') - f(x') \geq f(x + \lambda v) - f(x \wedge x' + \lambda v) = f(x + \lambda v) - f(x \triangle_i^\lambda x')$$

Thus,

$$\begin{aligned} f(x \triangledown_i^\lambda x') - f(x') &= f(x \triangledown_i^\lambda x') - f(x \vee x') + f(x \vee x') - f(x') \\ &\geq f(x) - f(x + \lambda v) + f(x + \lambda v) - f(x \triangle_i^\lambda x') \end{aligned}$$

□

Let D be a convex sublattice and suppose that $S, S' \subseteq D$. We say that S' ***dominates*** S ***in the*** C_i***-flexible set order*** if, for all $x \in S$ and $x' \in S'$ there exists $\lambda \in [0,1]$ such that $x \triangle_i^\lambda x' \in S$ and $x \bigtriangledown_i^\lambda x' \in S'$. When this happens we write $S' \geq_i S$.

We say that S' ***dominates*** S ***in the flexible set order*** if $S \geq_i S$ for all $i = 1, \ldots, n$. When this happens we write $S' \geq_F S$.

Example 51. Let $S = \{x \in \mathbf{R}^n_+ : p \cdot x \leq m\}$ and $S' = \{x \in \mathbf{R}^n_+ : p \cdot x \leq m'\}$, where $p \in \mathbf{R}^n_{++}$ is a ***price vector***, and $m, m' > 0$ are two different ***levels of income***. In this case, S and S' are two budget sets. Suppose that $m < m$", so that S' involves a larger income and therefore is strictly larger in the sense of set containment. The set S' is, however, not larger than S in the strong set order: if we take $x = (m/p_1, 0, \ldots, 0) \in S$ and $x' = (0, m'/p_2, 0, \ldots, 0) \in S'$, then $p \cdot x \vee x' > m'$ and so $x \vee x' \notin S'$. It is nevertheless true that S' dominates S in the flexible set order.

The budget set example illustrates the power of allowing λ to depend on x and x' in the definition of flexible set order.

When $f : D \to \mathbf{R}$ is given, we write Let

$$M(S) = \{x \in S : (\forall x' \in S) f(x) \geq f(x')\}$$

for the set of maximizers of f in the set $S \subseteq D$.

Theorem 52. *The function f satisfies the property that*

$$S' \geq_i S \Longrightarrow M(S') \geq_i M(S)$$

iff it is C_i-quasisupermodular, and the property that

$$S' \geq_F S \Longrightarrow M(S') \geq_F M(S)$$

iff it is C-quasisupermodular,

The proof of Theorem 52 is essentially the same as that of Theorem 43, accounting for the different objects involved here. I include a proof of the first statement of the theorem, not the second.

Proof. We prove the "if" direction first. Let $x' \in M(S')$ and $x \in M(S)$. If $x'_i \geq x_i$ then we are done and there is nothing to prove. So suppose that $x'_i < x_i$. Given that $S' \geq_i S$ there exists $\lambda \in [0,1]$ such that $x \triangle_i^\lambda x' \in S$ and $x \bigtriangledown_i^\lambda x' \in S'$.

Hence, $f(x) \geq f(x \triangle_i^\lambda x')$. The C_i-quasisupermodularity of f implies that $f(x \bigtriangledown_i^\lambda x') \geq f(x')$. Now, $x \in M(S')$ implies that $x \bigtriangledown_i^\lambda x' \in M(S')$. Finally, $x' \in M(S')$ in turn implies that $f(x) = f(x \triangle_i^\lambda x')$ and thus that

$x \triangle_i^\lambda x' \in M(S)$, as C_i-quasisupermodularity of would f would otherwise imply $f(x \triangledown_i^\lambda x') > f(x')$.

Now turn to the "only if" direction. Let $x, x' \in D$. If $x_i \leq x_i'$ then there is nothing to prove. So suppose that $x_i > x_i'$ and x and x' are unordered. Fix $\lambda \in [0, 1]$ Let $S = \{x, x \triangle_i^\lambda x'\}$ and $S' = \{x', x \triangledown_i^\lambda x'\}$ so that $S' \geq_i S$. If $f(x \triangle_i^\lambda x') \leq f(x)$ then $x \in M(S)$. The hypothesis then implies that $x \triangledown_i^{\lambda'} x' \in M(S')$ for some $\lambda' \in [0, 1]$. In fact, $x \triangledown_i^{\lambda'} x' \neq x'$, so we must have $\lambda = \lambda'$. Thus $f(x \triangledown_i^\lambda x') \geq f(x')$. On the other hand, $f(x \triangle_i^\lambda x') < f(x)$ rules out that $x' \in M(S')$ as this would imply that $x \triangledown_i^\lambda x' \in M(S)$. □

Example 53. When utility is partially concave and supermodular, demand is normal.

9.2. **Comparative statics "in t."** We now turn to comparative statics for a fixed constraint set S. The Milgrom-Shannon monotone comparative statics theorem provides necessary and sufficient conditions for monotone comparative statics in (S, t), but if we are only interested in comparative statics in t, then Milgrom and Shannon's conditions are too strong.

Figure 10 illustrates the problem. Consider an example with a totally ordered set X, and a fixed $S = X$. In the figure it is clear that $M(S, t) \leq_{SSO} M(S, t')$, but the single crossing property is violated because $x < x'$, $f_t(x) \leq f_t(x')$ while $f_{t'}(x) > f_{t'}(x')$.

Consider the POset $(X, \leq)$ with $X \subseteq \mathbf{R}$ and $\leq$ being the usual order on $\mathbf{R}$. The subset $I \subseteq X$ is an ***interval*** if, whenever $x, x' \in I$ and $x \in [x, x']$, then $x \in I$.

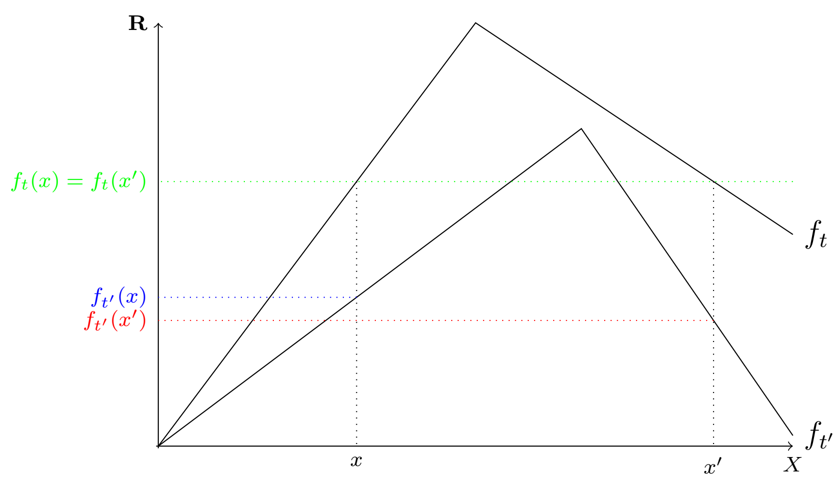

FIGURE 10. Failure of the single-crossing property.

A function $f : X \times T \to \mathbf{R}$ satisfies the ***interval single crossing property*** if, for all $x, x' \in X$, $t, t' \in T$ with the properties that

- $x < x'$ and $t < t'$,
- $(\forall \hat{x} \in [x, x'])f(\hat{x}, t) \leq f(x', t)$,

it holds that

$$f(x,t) \leq f(x',t) \Longrightarrow f(x,t') \leq f(x',t')$$
$$f(x,t) < f(x',t) \Longrightarrow f(x,t') < f(x',t')$$

The function $x \mapsto f(x,t)$ is ***regular over intervals*** if the problem $\max\{f(\hat{x},t) : \hat{x} \in [x,x']\}$ has a solution for any $x, x' \in X$ with $x < x'$. We say that f is regular over intervals in X if $x \mapsto f(x,t)$ is regular over intervals for all $t \in T$.

The following result is due to Quah and Strulovici (2009).

Theorem 54. *Let $X, T \subseteq \mathbf{R}$, and $f : X \times T \to \mathbf{R}$ be regular over intervals in X. Then f satisfies the interval single crossing property iff*

$$t < t' \Longrightarrow M(t, [x, x']) \leq_{SSO} M(t', [x, x'])$$

for all $x < x'$.

Proof. Let $t, t' \in T$, $x, x' \in X$ with $t < t'$ and $x < x'$. Suppose that $y \in M(t, [x, x'])$ and $y' \in M(t', [x, x'])$. If $y \leq y'$ then there is nothing to prove. Suppose then that $y' < y$. The interval crossing property and $f(y', t) \leq f(y, t)$ implies that $f(y', t') \leq f(y, t')$, as $f(\hat{x}, t) \leq f(y, t)$ for all $\hat{x} \in [y', y] \subseteq [x, x']$. Hence $y' \in M(t', [x, x'])$. On the other hand, $f(y', t) < f(y, t)$ would deliver $f(y', t') < f(y, t')$, by the strict inequality in the definition of interval crossing property, so we must have $f(y', t) = f(y, t)$ and thus that $y' \in M(t, [x, x'])$.

Conversely, let $t, t' \in T$, $x, x' \in X$ with $t < t'$, $x < x'$, and $(\forall \hat{x} \in [x, x'])(f(\hat{x}, t) \leq f(x', t))$. Then $x' \in M(t, [x, x'])$, so we know that $x' \in M(t', [x, x'])$. Therefore $f(x, t') \leq f(x', t)$. Now, if $f(x, t) < f(x', t)$ then we know that $x' \notin M(t, [x, x'])$. This implies that $f(x, t) < f(x', t)$, as otherwise we would obtain $x \in M(t', [x, x'])$ and thus $x \in M(t, [x, x'])$. $\square$

Proposition 55. *Let $X \subseteq \mathbf{R}$ be convex. Suppose that $x \mapsto f(x,t)$ is differentiable a.e. and that for all $t, t' \in T$ with $t < t'$ there exists a monotone increasing $\alpha : X \to \mathbf{R}_{++}$ such that*

$$(\forall x \in X) f'_{t'}(x) \geq \alpha(x) f'_t(x),$$

then f has the interval single crossing property.

Proof. The proof uses the following fact, which I will state without proof.

Lemma 56. *Suppose that* α *is monotone increasing and that* $\int_{\hat{x}}^{x'} h(t)dt \geq 0$ *for all* $\hat{x} \in [x, x']$ *then* $\int_x^{x'} \alpha(t)h(t)dt \geq \alpha(x)\int_x^{x'} h(t)dt$

Now let $x < x'$ and suppose that $(\forall \hat{x} \in [x, x'])(f_t(\hat{x}) \leq f_t(x'))$. Then, using the fundamental theorem of calculus, and Lemma 56, we have:

$$\begin{aligned} f_{t'}(x') - f_{t'}(x) &= \int_x^x f'_{t'}(\hat{x})d\hat{x} \\ &\geq \int_x^x \alpha(\hat{x})f'_t(\hat{x})d\hat{x} \\ &\geq \alpha(x)\int_x^x f'_t(\hat{x})d\hat{x} \\ &= \alpha(x)(f_t(x') - f_t(x)) \end{aligned}$$

□

9.3. **Supermodularity of value functions.** Let $(X, \leq)$ and $(T, \leq')$ be lattices. Let $S \subseteq X \times T$, and

$$S_t = \{x \in X : \exists t \in T \text{ s.t } (x, t) \in S\}$$

be the ***section*** of S at t. Suppose that $S_t \neq \emptyset$ for all t (just to save on notation).

Let $f : X \times T \to \mathbf{R}$ and $g : T \to \mathbf{R} \cup \{+\infty\}$ be

$$g(t) = \sup\{f(x, t) : x \in S_t\}.$$

Theorem 57. *If* f *is supermodular,* S *is a sublattice and* $g < +\infty$ *then* g *is supermodular.*

Proof. Let $t, t' \in T$ and let $x \in S_t$ and $x' \in S_{t'}$. Note that $x \wedge x' \in S_{t \wedge t'}$ and $x \vee x' \in S_{t \vee t'}$ as S is a sublattice. Then supermodularity of f and the definition of g imply that:

$$\begin{aligned} f(x, t) + f(x', t') &\leq f(x \vee x', t \vee t') + f(x \wedge x', t \wedge t') \\ &\leq g(t \vee t') + g(t \wedge t'). \end{aligned}$$

Since $x \in S_t$ and $x' \in S_{t'}$ were arbitary, we have that

$$g(t) + g(t') \leq g(t \vee t') + g(t \wedge t').$$

□

9.4. **Completeness and Existence of optima.**

Theorem 58. *Let* $(X, \leq)$ *be a lattice;* $X \subseteq \mathbf{R}^n$*. If* $f : X \to \mathbf{R}$ *is continuous and quasi-supermodular, and* S *is a subcomplete sublattice of* X*, then* $\text{argmax}_{x \in S} f(x)$ *is a (nonempty) subcomplete sublattice of* X*.*

10. Comparative statics under uncertainty

10.1. **Optimization under uncertainty.** We consider parameterized maximization problems under uncertainty. Let Θ be a set of ***states of the world***, assumed to be either an interval or a finite set in $\mathbf{R}$. Uncertainty is captured through a probability distribution on Θ, where we want to see how the solutions to the optimization problem change when we change the distribution. In other words, we want comparative statics with respect to the distribution over state. To this end, we parameterize the distributions over Θ. Each distribution is described through a ***density function***: a strictly positive function on S that is Lebesgue integrable and integrates to one. Specifically, let $T \subseteq \mathbf{R}$, and consider the family of density functions $\{h_t : t \in T\}$, with $h_t(s) > 0$ for all $\theta \in \Theta$ and $\int_\Theta h_t(\theta) ds = 1$.

An agent is choosing an element $x \in X \subseteq \mathbf{R}^n$ so as to maximize an objective $u(x, \theta)$, which depends on the (unknown) state of the world. The agent's objective is to optimize

$$f(x, t) = \int_\Theta u(x, \theta) h_t(\theta) ds.$$

We seek to understand when the optimal choice of x are monotone increasing in t. Here we continue to use the notation $M(S, t)$ for the set of maximizers $\text{argmax}\{f(x, t) : x \in S\}$, but note the underlying dependence on u and h_t.

Let $(X, \leq)$ be a lattice. A function $u : X \to \mathbf{R}_+$ is ***log-supermodular*** if, for all $x, x' \in X$

$$u(x)u(x') \leq u(x \wedge x')u(x \vee x').$$

A theory of log-supermodularity was first developed in statistics, see Karlin (1968).[9]

The family of densities h_t has ***monotone likelihood ratios*** if, for any $t < t'$,

$$\theta \mapsto \frac{h_{t'}(\theta)}{h_t(\theta)}$$

is (weakly) monotone increasing.

[9]For an early exposition of how these ideas relate to economics, see Jewitt (1991).

Observe that if $\Theta \subseteq \mathbf{R}$ then log-supermodularity of the function $(t, \theta) \mapsto h_t(\theta)$ is equivalent to h_t having monotone likelihood ratios. Indeed, for $t < t'$ and $\theta < \theta'$ we obtain:

$$\frac{h_{t'}(\theta)}{h_t(\theta)} \leq \frac{h_{t'}(\theta')}{h_t(\theta')} \text{ iff } h_t(\theta')h_{t'}(\theta) \leq h_t(\theta)h_{t'}(\theta')$$

The following result is due to Athey (2002):

Theorem 59. *Let $X \subseteq \mathbf{R}^n$ and $\Theta \subseteq \mathbf{R}^m$, with $n, m \geq 2$. Then $M(S,t)$ is monotone increasing for all a.e. log-supermodular u iff $(t, \theta) \mapsto h_t(\theta)$ is a.e. log-supermodular.*

Now let Θ be either finite or an interval in $\mathbf{R}$ The next result is due to Quah and Strulovici (2009).

Theorem 60. *Suppose that $\Theta \subseteq \mathbf{R}$, and that Θ is either finite or an interval. Let $u : X \times \Theta \to \mathbf{R}$ be regular over intervals in X, and satisfy the interval single crossing property. If the family of densities $\{h_t : t \in T\}$ has monotone likelihood ratios, then $f : X \times T \to \mathbf{R}$ defined by*

$$f(x,t) = \int_\Theta u(x,\theta) g_t(\theta) d\theta.$$

has the interval single crossing property.

A direct consequence of Theorem 60 is that if $t < t'$ then

$$\text{argmax}\{f(x,t) : x \in X\} \leq_{SSO} \text{argmax}\{f(x,t') : x \in X\}$$

If $(\forall \hat{x} \in [x, x'] f_{t'}(x) \geq f_{t'}(\hat{x})$ then

$$f_{t'}(x) \geq f_{t'}(x') implies f_{t'}(x') \geq f_{t'}(x) and f_{t'}(x) > f_{t'}(x') implies f_{t'}(x') > f_{t'}(x)$$

Proof. I give the proof for the case when $\Theta = [\underline{\theta}, \bar{\theta}]$. Suppose that $(\forall \hat{x} \in [x, x'] f(\hat{x}, t) \geq f(x', t)$.

$$f(x',t) - f(\hat{x},t) \geq 0 \Longrightarrow (\forall \hat{\theta} \in \Theta)(\int_{\hat{\theta}}^{\bar{\theta}} [u(x',\theta) - u(x,\theta)] h_t(\theta) d\theta \geq 0)$$

Suppose that this is not true. Let $\theta^* \in \Theta$ be such that $\int_{\theta^*}^{\bar{\theta}} [u(x', \theta) - u(x, \theta)] h_t(\theta) d\theta < 0$. Let x^* solve the problem of maximizing $u(\hat{x}, \theta^*)$ over $\hat{x} \in [x, x']$. Note that x^* is well defined due to the regularity of u over intervals. We shall prove that $f(x', t) - f(x^*, t) < 0$, a contradiction.

Consider first the case that $\theta \leq \theta^*$. Then $u(x^*, s^*) \geq u(\hat{x}, s^*)$ for all $\hat{x} \in [x, x']$. By interval single crossing and Lemma 2 $u(x^*, \theta) - u(x', \theta) \geq 0$.

Therefore

$$\int_{\underline{\theta}}^{\theta^*} [u(x^*, \theta) - u(x', \theta)] h_t(\theta) d\theta \geq 0$$

In second place, consider $\theta > \theta^*$. Then by definition of x^* and the interval single crossing we have that

$$\int_{\theta^*}^{\bar{\theta}} [u(x^*, \theta) - u(x', \theta)] h_t(\theta) d\theta \geq 0$$

$$\int_{\theta^*}^{\bar{\theta}} [u(x^*, \theta) - u(x', \theta)] h_t(\theta) d\theta = \int_{\theta^*}^{\bar{\theta}} [u(x^*, \theta) - u(x, \theta)] h_t(\theta) d\theta + \int_{\theta^*}^{\bar{\theta}} [u(x, \theta) - u(x', \theta)] h_t(\theta) d\theta > 0,$$

where we used that $\int_{\theta^*}^{\bar{\theta}} [u(x, \theta) - u(x', \theta)] h_t(\theta) d\theta > 0$ by assumption. This is a contradiction.

Let

$$V(\tilde{\theta}, h) = \int_{\underline{\theta}}^{\tilde{\theta}} [u(x', \theta) - u(x, \theta)] h_t(\theta) d\theta$$

$$V(\bar{\theta}, h_t) \geq V(\tilde{\theta}, h_t)$$

for all $\tilde{\theta} \in [\underline{\theta}, \bar{\theta}]$

$$V'(\tilde{\theta}, h_{t'}) = h_{t'}(\tilde{\theta}) / h_t(\tilde{\theta}) H'(\tilde{\theta}, h_t)$$

Since the family of densities has monotone increasing likelihood ratios, we can use Proposition 50, $(\tilde{\theta}, \tilde{t}) \mapsto V(\tilde{\theta}, h_{\tilde{t}}$ satisfies the interval single crossing on $\Theta \times \{t, t'\}$. In particular then $V(\bar{\theta}, h_t) \geq V(\underline{\theta}, h_t)$ implies that $V(\bar{\theta}, h_{t'}) \geq V(\underline{\theta}, h_{t'})$ and $V(\bar{\theta}, h_t) > V(\underline{\theta}, h_t)$ implies that $V(\bar{\theta}, h_{t'}) > V(\underline{\theta}, h_{t'})$.

$$V(\bar{\theta}, h) \geq V(\underline{\theta}, h) \Longrightarrow \int_{\underline{\theta}}^{\bar{\theta}} [u(x', \theta) - u(x, \theta)] h_t(\theta) d\theta \geq 0$$

□

10.2. **Single crossing and aggregation.** Case 1:

$$\max_{x\in X} V(x,t) = \int_\Theta u(x;\theta s,t)dF(\theta)$$

Suppose that v satisfies single-crossing differences so that argmax$\{u(x;(\theta,t) : x \in X\}$ is monotone increasing in (θ,t). This doesn't mean that $V(x,t)$ will satisfy single-crossing differences because single-crossing differences does not aggregate.

The solution would be increasing differences, but this notion is too strong.

For example let

$$u(x,\theta,t) = \phi(M(x,\theta,t)),$$

where $M(x,\theta,t)$ is a monetary payoff.

Typically, while $M(x,\theta,t)$ may obey increasing differences, u will obey single-crossing but not increasing differences. For example we may choose two increases $\delta > 0$ and $\delta' > 0$ in money, with $\delta < \delta'$ but we may have $\log(x+\delta) - \log(x) > \log(x'+\delta') - \log(x')$ for $x, x' \in \mathbf{R}$.[10]

The following is based on Quah and Strulovici (2012).

Let $(X,\leq)$ be a PO set.

A function $f : X \to \mathbf{R}$ has the ***single-crossing property*** if, for any $x, x' \in X$ with $x < x'$,

$$f(x) \geq 0 \Longrightarrow f(x') \geq 0$$
$$f(x) > 0 \Longrightarrow f(x') > 0$$

The function $u : X \times \Theta \to \mathbf{R}$ has the ***signed ratios property*** if, for any two $\theta, \theta' \in \Theta$, and any two $x, x' \in X$ with $x < x'$

$$u(x,\theta) < 0 \text{ and } u(x,\theta') > 0 \Longrightarrow \frac{-u(x,\theta)}{u(x,\theta')} \geq \frac{-u(x',\theta)}{u(x',\theta')}$$

and

$$u(x,\theta) > 0 \text{ and } u(x,\theta') < 0 \Longrightarrow \frac{-u(x,\theta')}{u(x,\theta)} \geq \frac{-u(x',\theta')}{u(x',\theta)}$$

Theorem 61. *Let $(X,\leq)$ be a PO set, and $(\Theta,\mathcal{T},\mu)$ be a probability space. Suppose that*

- *$u : X \times \Theta \to \mathbf{R}$ has the signed ratio property,*

[10] For example for any $x, \delta > 0$ there exists $\epsilon > 0$ so that if we let $x' = x + \delta$ and $\delta' = \delta + \epsilon$ the statement holds.

- *that $\theta \mapsto u(x,\theta)$ is bounded and integrable for all $x \in X$,*
- *and that $x \mapsto u(x,\theta)$ has the single-crossing property for all $\theta \in \Theta$.*

Then the function $f : X \to \mathbf{R}$ defined by

$$f(x) = \int_\Theta u(x,\theta) d\mu(\theta)$$

has the single-crossing property.

Proof. I provide a proof in the case that $\Theta = \{\theta, \theta'\}$. Suppose that $p = \mu(\{\theta\}) \in (0,1)$, otherwise there is nothing to prove.

Fix $x, x' \in X$ with $x < x'$, and suppose that $pu(x,\theta) + (1-p)u(x,\theta') \geq 0$. First, if $u(x,\theta) \geq 0$ and $u(x,\theta') \geq 0$ then $u(x',\theta) \geq 0$ and $u(x',\theta') \geq 0$ by the single-crossing property. So $pu(x',\theta) + (1-p)u(x',\theta') \geq 0$. Similarly, if $pu(x,\theta) + (1-p)u(x,\theta') > 0$, and $u(x,\theta) \geq 0$ and $u(x,\theta') \geq 0$, then at least one of the two functions must be > 0 and the single-crossing property again implies that $pu(x',\theta) + (1-p)u(x',\theta') > 0$.

Secondly, assume that $pu(x,\theta) + (1-p)u(x,\theta') \geq 0$. but that one of $u(x,\theta)$ or $u(x,\theta')$ is < 0. Since one has to be > 0 we may as well say that $u(x,\theta) < 0 < u(x,\theta')$. Then the signed ratio property implies that

$$\frac{1-p}{p} \geq \frac{-u(x,\theta)}{u(x,\theta')} \geq \frac{-u(x',\theta)}{u(x',\theta')},$$

where the first inequality follows from $pu(x,\theta) + (1-p)u(x,\theta') \geq 0$. Thus we conclude that $pu(x',\theta) + (1-p)u(x',\theta') \geq 0$. Similarly $pu(x,\theta) + (1-p)u(x,\theta') > 0$ implies that $(1-p)/0 > -u(x,\theta)/u(x,\theta')$ and therefore $pu(x,\theta) + (1-p)u(x,\theta') > 0$. □

Proof for finite Θ:

Partition Θ into three sets, $\Theta_1 = \{\theta : u(x,\theta) < 0\}$ $\Theta_2 = \{\theta : u(x,\theta) = 0\}$ $\Theta_3 = \{\theta : u(x,\theta) > 0\}$.

For all $\theta \in \Theta_3$, we know that $u(x',\theta) > 0$ by the single crossing property.

We may rewrite $\sum_{\theta \in \Theta_1 \cup \Theta_2} u(x,\theta)$ as $\sum v_j$ where each j is a positive linear combination of at most two states. Then apply the result for two states.

If $u(x,\theta) < 0$ for some θ we can partition Θ

A partial converse to Theorem 61 holds:

Proposition 62. *Let $(X, \leq)$ be a PO set, Θ a finite set, and $u : X \times \Theta \to \mathbf{R}$ be such that $x \mapsto u(x,\theta)$ has the single-crossing property for all $\theta \in \Theta$. If*

$f : X \to \mathbf{R}$ as defined in Theorem 61 has the single crossing property for all probabilities μ on $(\Theta, 2^{\Theta})$, then u has the signed ratio property.

Proof. Again, I only prove the result for the case when $\Theta = \{\theta, \theta'\}$. Let $x, x' \in X$ with $x < x'$ and suppose that $u(x, \theta) < 0$ while $u(x, \theta') > 0$. Set

$$\mu(\{\theta'\}) = \frac{-u(x,\theta)}{u(x,\theta') - u(x,\theta)},$$

so that $\mu(\{\theta'\})u(x,\theta') + (1 - \mu(\{\theta'\}))u(x,\theta) = 0$. Then single crossing implies that $\mu(\{\theta'\})u(x',\theta') + (1-\mu(\{\theta'\}))u(x',\theta) \geq 0$, which together with $u(x',\theta') > 0$ (by single crossing of u) gives us that

$$\frac{-u(x,\theta)}{u(x,\theta')} = \frac{\mu(\{\theta'\})}{1 - \mu(\{\theta'\})} \geq \frac{-u(x',\theta)}{u(x',\theta')}.$$

□

Theorem 63. *Let $(X, \leq)$ be a PO set, and $(\Theta_i, \mathcal{T}_i, \mu_i)$ be Borel probability spaces with $\Theta_i \subseteq \mathbf{R}$, for $i = 1, \ldots, n$. Let $\Theta = \prod_{i=1}^n \Theta_i$, and $(\Theta, \mathcal{T}, \mu)$ be the resulting product probability space. Suppose that $u : X \times \Theta \to \mathbf{R}$ satisfies the following assumptions.*

- *For any $\theta', \theta'' \in \Theta$ with $\theta' < \theta''$, $(x, \tilde{\theta}) \mapsto u(x, \tilde{\theta})$ has the signed ratio property for $\tilde{\theta} \in \{\theta', \theta''\}$;*
- *$(\theta_i, \theta_{-i}) \mapsto u(x, \theta_i, \theta_{-i})$ has the signed ratio property for each $x \in X$;*
- *and $\theta \mapsto u(x, \theta)$ is bounded and integrable for all $x \in X$.*

Then the function $f : X \to \mathbf{R}$ defined by

$$f(x) = \int_{\Theta} u(x,\theta) d\mu(\theta)$$

has the single-crossing property.

If $u : X \times \Theta \to \mathbf{R}$ satisfies (63) and (63) and $h : X \times \Theta \to \mathbf{R}$ is log-supermodular, then

$$\hat{u}(x,\theta) = u(x,\theta)h(x,\theta)$$

satisfies (63) and (63).

Corollary 64. *Let $X = \times_{i=1}^m X_i \subseteq \mathbf{R}^m$ and $Y = \times_{i=1}^n Y_i \subseteq \mathbf{R}^n$. Suppose that Y is a rectangle in $\mathbf{R}^n$. Let $u : X \times Y \to \mathbf{R}_{++}$ be a function and suppose that $y \mapsto u(x, y)$ is bounded and measurable for all $x \in X$. If u is log-supermodular, then the function $f : X \to \mathbf{R}$ defined by*

$$f(x) = \int_Y u(x,y) dy$$

is log-supermodular.

Proof. Let $x, x' \in X$. If x and x' are ordered there is nothing to prove. So suppose they are unordered, and without loss write $x = (x_1, x_2) \in \mathbf{R}^{m_1+m_2}$ and $x' = (x'_1, x'_2) \in \mathbf{R}^{m_1+m_2}$ with m_1 and m_2 such that $m = m_1 + m_2$, $x_1 > x'_1$ and $x'_2 > x_2$. Choose a constant Q such that $f(x_1, x_2) = Qf(x'_1, x_2)$. This means that

$$0 = f(x_1, x_2) - Qf(x'_1, x_2) = \int_Y u((x_1, x_2), y) - Qu((x'_1, x_2), y)dy.$$

Consider a function $G : \mathbf{R}^{m_1} \to \mathbf{R}$ defined as

$$G(\tilde{x}_2) = \int_Y u((x_1, \tilde{x}_2), y) - Qu((x'_1, \tilde{x}_2), y)dy = \int_Y [\frac{u((x_1, \tilde{x}_2), y)}{u((x'_1, \tilde{x}_2), y)} - Q]u((x'_1, \tilde{x}_2), y)dy.$$

Log-supermodularity implies that

$$(\tilde{x}_2, y) \mapsto \frac{u((x_1, \tilde{x}_2), y)}{u((x'_1, \tilde{x}_2), y)}$$

is monotone increasing. This implies that G has the single-crossing property by Theorem 63

$$o(\tilde{x}_2, y) = \frac{u((x_1, \tilde{x}_2), y)}{u((x'_1, \tilde{x}_2), y)}$$

Suppose that $o(\tilde{x}_2, y) - Q < 0$ and $o(\tilde{x}_2, y') - Q > 0$. Then

$$\frac{-(o(\tilde{x}_2, y) - Q)}{o(\tilde{x}_2, y') - Q} = \frac{Q - o(\tilde{x}_2, y)}{o(\tilde{x}_2, y') - Q} \geq \frac{Q - o(\tilde{x}'_2, y)}{o(\tilde{x}'_2, y') - Q},$$

as o is monotone increasing in $\tilde{x}_2$.

Now $G(x_2) = 0$, so $G(x'_2) \geq 0$ by the single-crossing property and $x_2 < x'_2$.

$$0 \leq G(x'_2) = f(x_1, x'_2) - Qf(x'_1, x'_2) = f(x_1, x'_2) - \frac{f(x_1, x_2)}{f(x'_1, x_2)} f(x'_1, x'_2)$$

□

10.3. **Affiliated random variables and the supermodular order.** Consider a collection $X_1, \ldots, X_n$ of random variables on a probability space $(\Theta, \mathcal{T}, \pi)$. Let G be the cumulative distribution function of the ***random vector*** $X = (X_1, \ldots, X_n)$ on $\mathbf{R}^n$, and suppose that G is absolutely continuous with density g.

We say that the random variables $X_1, \ldots, X_n$ are***affiliated*** if g is log-supermodular.

Consider now two random vectors $X = (X_1, \ldots, X_n)$ and $X' = (X'_1, \ldots, X'_n)$. Say that X' is larger than X is the ***supermodular order*** if $\mathbf{E}f(X) \leq \mathbf{E}f(X')$ for all supermodular and integrable functions $f : \mathbf{R}^n \to \mathbf{R}$.

Let μ, μ' be probability measures on $\mathbf{R}^n$, with cumulative distribution function F_μ and F'_μ. Suppose that μ and μ'

11. Information

Let Θ be a set of states of the world, and $(\Theta, \mathcal{T})$ a measure space. Denote by $\Delta(\Theta)$ the set of all probability measures on $(\Theta, \mathcal{T})$.

An ***information structure*** (or an ***experiment*** in the terminology of Blackwell) is a tuple $\mathcal{I} = ((Z, \mathbf{Z}), \{\mu_\theta : \theta \in \Theta\})$ where

- $(Z, \mathbf{Z})$ is a measure space of possible ***signal values***;
- $\mu_\theta \in \Delta(Z, \mathbf{Z})$ for all $\theta \in \Theta$.

The interpretation is that the experiment will produce a ***signal*** $z \in Z$. If the state of the world of $\theta \in \Theta$ then μ_θ is the probability law from which z is drawn. For example the state of the world could be a real number $\theta \in \mathbf{R}$, and the signal be determined by $z = \theta + u$, where u is normally distributed with zero mean and variance σ^2.

A Bayesian agent who cares about $\theta \in \Theta$ but not intrinsically about $z \in Z$ will only use the signal to update her beliefs about the state of the world. Specifically, given a prior $p \in \Delta(\Theta)$ and upon observing $z \in Z$, the agent will form posterior beliefs $\pi(.|z) \in \Delta(\Theta, \mathcal{T})$ for each possible $z \in Z$.

Finally, the prior π and the experiment $\mathcal{I}$ induce a marginal distribution $m \in \Delta(Z, \mathbf{Z})$ on signal values. Specifically, $m(A) = \int_\Theta \mu_s(A) d\pi(s)$.

So consider an agent that has to make a choice $x \in X$ without knowing the state of the world. The agent cares about the state of the world, as captured by a utility function $u : X \times \Theta \to \mathbf{R}$, and has access to an experiment, or information structure, $\mathcal{I}$. If the agent's prior is π then her expected utility is

$$
\begin{aligned}
V(u, \mathcal{I}, \pi) &= \int_Z \max \left\{ \int_\Theta u(x, \theta) d\pi(\theta|z) : x \in X \right\} dm(z) \\
&= \int_\Theta \int_Z u(x^*(z), \theta) d\mu(z)_\theta d\pi(\theta)
\end{aligned}
$$

The next results are due to Quah and Strulovici:

Suppose that $Z \subseteq \mathbf{R}$ is a compact interval, and that each μ_θ admits a density h_θ and a cumulative distribution function H_s. Let π be a prior of Θ.

Suppose that $\Theta \subseteq \mathbf{R}$ is either finite or a compact interval, and that S is the support of π. In the latter case we assume that π is absolutely continuous.

Consider another information structure $\mathcal{I}' = \{\mu'_\theta : \theta \in \Theta\}$ with associated cdfs H'_θ and densities h'_θ. Since the distributions admit a density, for each (z, θ) there is a unique $T(z, \theta) \in Z$ such that $H_s(T(z,\theta))_\theta = H'(z)_\theta$. Specifically, since H_s admits a strictly positive density, it has an inverse, and thus we can let

$$T(z, \theta) = H_\theta^{-1}(H'_\theta(z)).$$

Say that $\mathcal{I}$ is ***more accurate*** than $\mathcal{I}'$ if $\theta \mapsto T(z, \theta)$ is monotone increasing.

Theorem 65. *Let $u : X \times \Theta \to \mathbf{R}$ satisfy the interval single crossing property. Suppose that the family of densities h_θ have monotone likelihood ratios. Let π be a prior over Θ. If $\mathcal{I}$ is more accurate than $\mathcal{I}$ then $V(\mathcal{I}, \pi, u) \geq V(\mathcal{I}', \pi, u)$.*

Proof. The first observation is that posteriors have montone likelihood ratios. In consequence, we can choose an optimal x that is monotone in the signal z.

$$\frac{\pi(\theta|z')}{\pi(\theta|z)} = \frac{m(z)}{m(z')}\frac{h(z')_\theta}{h(z)_\theta}$$

When $\theta < \theta'$, we have that

$$\frac{h(z)_{\theta'}}{h(z)_\theta} \leq \frac{h(z')'_\theta}{h(z')_\theta} \implies \frac{h(z')_\theta}{h(z)_\theta} \leq \frac{h(z')'_\theta}{h(z)_{\theta'}}$$

Then by Theorem 60 there is a monotone selection $x^*(z) \in \operatorname{argmax} \int u(x, \theta) d(\theta)$.[fix this!] □

$$\int_\Theta \int_Z u(\phi(z), \theta) dH_\theta(z) d\pi(\theta) \int_\Theta \int_Z u(\psi(z), \theta) dH'_\theta(z) d\pi(\theta)$$

for any monotone increasing $\psi : Z \to X$.

Lemma 66. *For each monotone increasing $\psi : Z \to X$ there exists a monotone increasing $\phi : Z \to X$ such that for each $\theta \in \Theta$ the distribution of the random variable $z \mapsto u(\phi(z), \theta)$ with z drawn from H_θ first-order stochastically dominates the distribution of the random variable $z \mapsto u(\psi(z), \theta)$ with z drawn from H'_θ.*

Proof. Construct such that For all (z,θ)

$$u(\phi(T(z,\theta),\theta) \geq u(\psi(z),\theta)$$

Let $\zeta'_\theta \sim H'_\theta$ and $\zeta_\theta \sim H_\theta$.

$$P(u(\phi(\zeta),\theta) \leq \bar{u}|\theta) = P(u(\phi(T(\zeta',\theta)),\theta) \leq \bar{u}|\theta) \leq P(u(\psi(\zeta'),\theta) \leq \bar{u}|\theta),$$

where the inequality follows from the construction of ϕ. □

11.1. **Bayesian comparative statics.** This result is from Mekonnen and Leal-Vizcaíno (2018).

Let X and Θ be compact intervals in $\mathbf{R}$.

Let $Z \subseteq \mathbf{R}$ be a Borel set.

Suppose that if $z < z'$ then the posterior $\pi(.|z')$ first-order stochastically dominates $\pi(.|z)$. In this section we restrict attention to information structures that satisfy this monotonicity assumption.

Focus attention on information structures defined on the signal space Z and with the same marginal m. The assumption of a fixed marginal is wlog due to a trick provided by Lehmann.

Consider a class $\mathcal{U}^I$ of utility functions $u : X \times \Theta \to \mathbf{R}$ with the following properties

- u is C^2 and $x \mapsto u(x,\theta)$ strictly concave for all $\theta \in \Theta$.
- The maximization problem $\max\{u(x,\theta) : x \in X\}$ has an interior solution, for all $\theta \in \Theta$.
- $(x,\theta) \mapsto u(x,\theta)$ has increasing differences.
- $\tilde{x} \mapsto \partial_x u(\tilde{x},\theta)$ is convex, for all $\theta \in \Theta$ and $(\tilde{x},\theta) \mapsto \partial_x u(\tilde{x},\theta)$ has increasing differences.

Given a signal value z, let $x^*(z)$ be the (unique) optimal choice of $x \in X$ for the posterior $\pi(\cdot|z)$. That is,

$$\{x^*(z)\} = \operatorname{argmax}\{\int_S u(x,\theta)d\pi(\theta|z) : x \in X\}.$$

Uniqueness follows from assuming strict concavity.

An information structure $\mathcal{I}$ with marginal m on Z induces a probability distribution $G^X_{\mathcal{I}}$ on X. So that $G^X_{\mathcal{I}}(x) = \int_Z \mathbf{1}_{x^*(z)\leq x} dm(z)$. Say that ***the***

optimal action under $\mathcal{I}'$ ***is more responsive with higher mean than under*** $\mathcal{I}$ if $G^X_{\mathcal{I}'}$ dominates $G^X_{\mathcal{I}}$ in the increasing convex order.

Say that an information structure $\mathcal{I}'$ dominates $\mathcal{I}$ in the ***supermodular stochastic order*** if $G_{\mathcal{I}}(\theta, z) \leq G_{\mathcal{I}'}(\theta, z)$ for all $(\theta, z) \in \Theta \times Z$.

Theorem 67. *Consider two information structures* $\mathcal{I}$ *and* $\mathcal{I}'$. *For any* $u \in \mathcal{U}$ *the optimal action under* $\mathcal{I}'$ *is more responsive with higher mean than under* $\mathcal{I}$ *iff* $\mathcal{I}'$ *dominates* $\mathcal{I}$ *in the supermodular stochastic order.*

11.2. **Good news and bad.** Let:

- $\Theta \subseteq \mathbf{R}$ be a set of ***states of the world***.
- $Z \subseteq \mathbf{R}^n$ be a set of possible signal values.
- For each $\theta \in \Theta$, let $\mu_\theta \in \Delta(Z)$ be a probability distribution over signal values.

The tuple $(Z, \{\mu_\theta : \theta \in \Theta\})$ is an ***information structure***.

Suppose that each μ_θ is absolutely continuous with density h_θ.

Let $Z \subseteq \mathbf{R}$. The collection of densities $\{h_\theta : \theta \in \Theta\}$ has strictly monotone likelihood ratios if, for any $\theta < \theta'$ and $z < z'$

$$h_\theta(z')h_{\theta'}(z) < h_\theta(z)h_{\theta'}(z').$$

Equivalently, we may say that the function $(\theta, z) \mapsto h_\theta(z)$ is strictly log-supermodular.

Given a prior $\pi \in \Delta(\Theta)$ we may be Bayes' theorem compute a posterior distribution $\pi(\cdot|z) \in \Delta(\Theta)$ for each $z \in Z$. Then we say that the information structure has the ***good news property*** for prior π if

Theorem 68 is due to Milgrom (1981).

Theorem 68. *An information structure satisfies that*

$$(\forall z, z' \in Z)(z < z' \Longrightarrow \pi(.|z) \leq_{FOSD} \pi(\cdot|z)')$$

iff it has strictly monotone likelihood ratios.

Proof. Let $z < z'$. Fix $\bar{\theta} \in \Theta$ with $\pi(\{(-\infty, \bar{\theta}) \cap \Theta\}) \in (0, 1)$. For any $\theta < \bar{\theta}$ and $\theta' \geq \bar{\theta}$ we obtain that

$$h_\theta(z')h_{\theta'}(z) < h_\theta(z)h_{\theta'}(z') \Longrightarrow h_\theta(z') \int_{\bar{\theta}}^{\infty} h_{\theta'}(z) d\pi(\theta') < h_\theta(z) \int_{\bar{\theta}}^{\infty} h_{\theta'}(z') d\pi(\theta').$$ [11]

[11] For notational convenience we suppose that π is defined over all $\mathbf{R}$ in writing down these integrals.

Now integrating over $\theta < \bar{\theta}$ we have that

$$\int_{-\infty}^{\bar{\theta}} h_\theta(z')d\pi(\theta) \int_{\bar{\theta}}^{\infty} h_{\theta'}(z)d\pi(\theta') < \int_{-\infty}^{\bar{\theta}} h_\theta(z)d\pi(\theta) \int_{\bar{\theta}}^{\infty} h_{\theta'}(z')d\pi(\theta').$$

By Bayes' theorem:

$$\int_{-\infty}^{\bar{\theta}} d\pi(\theta|z') \int_{\bar{\theta}}^{\infty} d\pi(\theta|z) < \int_{-\infty}^{\bar{\theta}} d\pi(\theta|z) \int_{\bar{\theta}}^{\infty} d\pi(\theta|z'),$$

or

$$\int_{-\infty}^{\bar{\theta}} d\pi(\theta|z')(1 - \int_{-\infty}^{\bar{\theta}} d\pi(\theta|z')) < \int_{-\infty}^{\bar{\theta}} d\pi(\theta|z)(1 - \int_{-\infty}^{\bar{\theta}} d\pi(\theta|z)).$$

This means that $\int_{-\infty}^{\bar{\theta}} d\pi(\theta|z') < \int_{-\infty}^{\bar{\theta}} d\pi(\theta|z)$. Since $\bar{\theta}$ is arbitrary with $\pi(\{(-\infty, \bar{\theta}) \cap \Theta\}) \in (0,1)$ we are done. □

Chambers and Healy (2012) consider conditions under which Bayesian updating makes expected values move in the direction of the signal.

12. Tarski's fixed point theorem

Theorem 69 (Tarski's fixed point theorem)**.** *Let* $(X, \leq)$ *be a complete lattice and* $f : X \to X$ *be monotone increasing. Then*

$$\mathcal{E} = \{x \in X : x = f(x)\}$$

is nonempty and $(\mathcal{E}, \leq)$ *is a complete lattice.*

Remark 70. Note that the theorem does not say that $\mathcal{E}$ is a sublattice. Here is an example: Let $X = \{1,2,3\} \times \{1,2,3\}$ with the usual (componentwise) order. Let $f(x) = x$ for $x = (1,1), (1,2), (2,1), (3,3)$. Let $f(x) = (3,3)$ for $x = (1,3), (3,1), (2,2), (3,3)$.

Proof. We first establish the existence of a fixed point. Let

$$B = \{x \in X : x \leq f(x)\}$$

Note that $B \neq \emptyset$ because $\inf X \in B$. So there is $\bar{e} = \sup B \in X$, as X is a complete lattice. We shall prove that $\bar{e}$ is a fixed point. For any $x \in B$, $x \leq \bar{e}$ so the definition of B and the monotonicity of f implies

$$x \leq f(x) \leq f(\bar{e}).$$

Thus $f(\bar{e})$ is an upper bound on B, and therefore $\bar{e} \leq f(\bar{e})$. Then the monotonicity of f implies that $f(\bar{e}) \leq f(f(\bar{e}))$. So $f(\bar{e}) \in B$. But then $f(\bar{e}) \leq \bar{e}$ as $\bar{e}$ is an upper bound on B. This gives $\bar{e} = f(\bar{e})$. Note also that for any $e \in \mathcal{E}$, $e \in B$ and therefore $e \leq \bar{e}$. This implies that $\bar{e}$ is a fixed point and moreover that $\bar{e} = \sup \mathcal{E}$.

Similarly, we can show that $\underline{e} = \inf\{x \in X : f(x) \leq x\} \in \mathcal{E}$, and $\underline{e} = \inf \mathcal{E}$.

To show that $\mathcal{E}$ is a complete lattice, let $E \subseteq \mathcal{E}$ be nonempty. Let $\bar{x} = \sup E$ (in X). Consider $Y = \{x \in X : \bar{x} \leq x$. We show that f restricted to Y maps Y into Y. For any $e \in E$, $e \leq \bar{x}$, so $e = f(e) \leq f(\bar{x})$. Therefore $\bar{x} \leq f(\bar{x})$ as $\bar{x}$ is the least upper bound on E. So $f(\bar{x}) \in Y$. Moreover, for all $y \in Y$, $\bar{x} \leq y$, so $\bar{x} \leq f(\bar{x}) \leq f(y)$. Hence if g is the restriction of f to Y, we have $g : Y \to Y$. Y is a complete lattice (homework) and g is a monotone increasing funcion. So g has a smallest fixed point $\hat{e}$. Note that $\hat{e}$ must be a fixed point of f, so $\hat{e} \in \mathcal{E}$. Moreover, since $\bar{x} \leq \hat{e}$, $\hat{e}$ is a upper bound on E. And if $e' \in \mathcal{E}$ is an upper bound on E, we have $\bar{x} \leq e'$, so $e' \in Y$. Then e' is a fixed point of g and therefore $\hat{e} \leq e'$. Hence $\hat{e}$ is the least upper bound on E in $\mathcal{E}$.

The proof for $\inf E$ is analogous. □

We next turn to a generalization of Tarski's fixed point theorem to correspondences. The result is due to Zhou (1994) If X is a set and $\phi(x) \to 2^X \setminus \emptyset$ a correspondence on X, then the set of ***fixed points*** of ϕ is $\{x \in X : x \in \phi(x)\}$.

Theorem 71. *Let $(X, \leq)$ be a complete lattice, and $\phi : X \to 2^X \setminus \emptyset$ be a monotone increasing correspondence (when the range of ϕ is ordered by the strong set order). If $\phi(x)$ is a subcomplete sublattice for all x, then the set of fixed points of ϕ is a nonempty complete lattice.*

Theorem 72. *Let $(X, \leq)$ be a complete lattice and $(T, \leq')$ a PO set. Let $\phi : X \times T \to 2^X \setminus \emptyset$ be a monotone increasing correspondence (when the range of ϕ is ordered by the strong set order). If $\phi(x, t)$ is a subcomplete sublattice for all x, then the set*

$$\mathcal{E}(t) = \{x \in X : x \in \phi(x, t)\}$$

is a nonempty complete lattice for each t, and

$$\inf \mathcal{E}(t) \leq \mathcal{E}(t')$$
$$\sup \mathcal{E}(t) \leq \sup \mathcal{E}(t'),$$

whenever $t < t'$.

Proof. We'll do the proof for the case of a single-valued correspondence, a function $f : X \times T \to X$. Let $t < t'$. Note that if $x \leq f(x, t)$ then $x \leq f(x, t')$, as $f(x, t) \leq f(x, t')$. So,

$$\{x \in X : x \leq f(x, t)\} \subseteq \{x \in X : x \leq f(x, t')\}$$

Thus

$$\sup \mathcal{E}(t) = \sup\{x \in X : x \leq f(x, t)\} \leq \sup\{x \in X : x \leq f(x, t')\} = \sup \mathcal{E}(t').$$

The proof for $\inf \mathcal{E}(t) \leq \inf \mathcal{E}(t')$ is analogous. □

12.1. **Application: Cournot oligopoly.** The results here are from Amir and Lambson (2000).[12]

Consider n firms competing by setting quantities. Firm i chooses a quantity $q_i \in \mathbf{R}_+$. Firm i's profits are

$$\pi_i(q_1, \ldots, q_n) = q_i P(\sum_{j=1}^{n} q_j) - c(q_j),$$

where P is an (inverse) demand and c is a cost function. Note that all firms are identical.

A quantity vector $q = (q_1, \ldots, q_n)$ is a Cournot-Nash equilibrium if $q_i \in \text{argmax}\{\pi_i(\tilde{q}_i, q_{-1}) : \tilde{q}_i \in \mathbf{R}_+\}$ for all i.

Let

$$b(x) = \text{argmax}\{(y - x)P(y) - c(y - x) : y \in \mathbf{R}_+, x \leq y\}.$$

Note that if

$$x = \frac{n-1}{n} b(x)$$

then $q_i = x/(n-1)$ is a Cournot-Nash equilibrium. So we're interested in studying the fixed points of

$$x \mapsto \frac{n-1}{n} b(x).$$

We want to say that b is monotone increasing, so that we can use Tarski's fixed point theorem. The function b gives solutions to parameterized maximization problems, and we want to use Topkis or Milgrom and Shannon's theorems to obtain monotonicity. Suppose (to simplify the exposition) that b is singleton valued and that P and c are smooth. The The objective function in the definition of b has cross partial derivatives (with respect to x and y) given by:

$$\frac{\partial}{\partial y}[-P(y) + c'(y - c)] = -P'(y) + c''(y - c).$$

And $-P'(y) + c''(y - c) > 0$ if demand slopes down and c is convex. So in that case, we have increasing differences in (x, y), and b is monotone increasing by Topkis's, or Milgrom and Shannon's, theorem. Note that, even if c is not convex, increasing differences can be satisfied.

[12] See also Amir (1996).

Suppose also that we constrain choices to lie in some compact interval $[0, M] \subseteq \mathbf{R}_+$. Then by Tarski's fixed point theorem there is a Cournot-Nash equilibrium.

Theorem 73. *The set of Cournot-Nash equilibria is nonempty and has a smallest and largest element. The smallest and largest equilibria are monotone increasing in the number of firms.*

Proof. Let $\phi(x, n) = \frac{n-1}{n} b(x)$. Under the assumptions of the model ϕ is monotone increasing. The interval $[0, M]$ is a complete lattice. So the set of fixed points is a nonempty complete lattice, and the extremal (smallest and largest) equilibria are monotone increasing in n.

Let $\bar{e}(n)$ be the largest fixed point as a function of n. Then the equilibrium total market quantity is $b(\bar{e}(n)$. We have seen that b is monotone increasing. So the equilibrium quantity is monotone increasing. $\square$

Remark 74. One implication of this result is that the equilibrium price corresponding to the smallest and largest equilibria are monotone decreasing in the number of firms. This result makes a lot of economic sense, but there are instances of the Cournot model that have price increases after an increase in the number of firms. Obviously such instances violate that $-P' + c'' > 0$. See Amir and Lambson (2000).

12.2. **Application: stable matching.** The following approach to stable matching started with Adachi (2000).[13] For more on stable matchings, see Roth and Sotomayor (1990).

A ***marriage market*** is a tuple $(M, W, (\succeq_m)_{m \in M}, (\succeq_w)_{w \in W})$, where

- M and W are disjoint finite sets. The elements of M are called ***men*** and the elements of W are ***women***.
- For each $m \in M$, $\succeq_m$ is a linear order (a ***strict preference***) over $W \cup \{\emptyset\}$.
- For each $w \in W$, $\succeq_w$ is a linear order (a ***strict preference***) over $M \cup \{\emptyset\}$.

We use the symbol $\emptyset$ to represent that an agent is single, so $m \succeq_w \emptyset$ means that women w ranks a match with $m \in M$ over remaining single. In this case we say that m is ***acceptable*** to w.

[13]See Fleiner (2003), Hatfield and Milgrom (2005), and Echenique and Oviedo (2004, 2006) for variations on this approach. The result on the lattice structure stable matchings is usually attributed to John Conway; see Knuth (1997).

A ***fantasy*** is a function $\nu : M \cup W \to M \cup W$ for which

$$\forall m \in M, \nu(m) \in W \cup \{\emptyset\}; \text{ and } \forall w \in W, \nu(w) \in M \cup \{\emptyset\}.$$

Let V be the set of all fantasies. Note that V is a product set. For any two fantasies ν and ν', say that ν is ***less than*** ν' ($\nu \leq \nu'$) if

$$(\forall m \in M) \ \ \nu'(m) \succeq_m \nu(m)$$
$$(\forall w \in W) \ \ \nu(w) \succeq_w \nu'(w).$$

Observe that $\leq$ is a product order, and that $(V, \leq)$ is a lattice by virtue of being the product of linearly ordered sets.

A ***matching*** is a fantasy μ with the property that $w = \mu(m)$ iff $m = \mu(w)$. A matching μ is ***individually rational*** if $\mu(a) \succeq_a \emptyset$ for all $a \in M \cup W$. A pair $(m, w) \in M \times W$ is a ***blocking pair*** for μ if $m \succ_w \mu(w)$ and $w \succ_m \mu(m)$. A matching is ***stable*** if it is individually rational and admits no blocking pair.

For any fantasy ν, and any m and w, define the following sets:

$$A(m, \nu) = \{w \in W : m \succeq_w \nu(w)\}$$
$$A(w, \nu) = \{m \in M : w \succeq_m \nu(m)\}$$

So that $A(m, \nu)$ is the set of women who are willing to match with m, given who their match is in the fantasy ν.

Define a function $T : V \to V$, mapping fantasies to fantasies, by $(T\nu)(m)$ being the optimal choice for $\succeq_m$ in $A(m, \nu) \cup \{\emptyset\}$, for any $m \in M$; and similarly by $(T\nu)(w)$ being the optimal choice for $\succeq_w$ in $A(w, \nu) \cup \{\emptyset\}$, for any $w \in W$.

Lemma 75. *T is monotone increasing.*

Proof. Let $\nu \leq \nu'$. Then $A(m, \nu) \subseteq A(m, \nu')$ and $A(w, \nu) \supseteq A(w, \nu')$, for all m and w: to see this, note that if $w \in A(m, \nu)$ then

$$m \succeq_w \nu(w) \succeq_w \nu'(w),$$

so $w \in A(m, \nu')$. Similarly for $A(w, \nu) \supseteq A(w, \nu')$.

Since the best element from a larger set cannot be worse than from a smaller, $(T\nu')(m) \succeq_m (T\nu)(m)$ and $(T\nu)(w) \succeq_w (T\nu')(w)$. Thus $T\nu \leq T\nu'$. □

Lemma 76. *Any fixed point of T is a stable matching, and if μ is a stable matching then it is a fixed point of T.*

Proof. Suppose that $\nu \in V$ is a fixed point of T. So $\nu = T\nu$. Suppose, by way of contradiction, that there is (m, w) with $w = \nu(m)$ and $m \neq \nu(w)$.

Now $w = \nu(m)$ means that $m \in A(w, \nu)$, by definition of A. So $m \neq \nu(w) = (T\nu)(w)$ means that $\nu(w) \succ_w m$. But then $m \notin A(w, \nu)$, which contradicts that $w = \nu(m) = (T\nu)(m)$.

Now we show that ν is not only a matching but it is also stable. By construction, ν is individually rational, as $\nu(a) = (T\nu)(a) \succeq_a \emptyset$. Let $(m, w) \in M \times W$ be arbitrary. If $w \succ_m \nu(m)$ then $w \in A(m, \nu)$. Then $(T\nu)(m) = \nu(m) \succeq_m w$. So (m, w) cannot be a blocking pair. Hence ν is a stable matching.

Conversely, let μ be a stable matching. We must show that $\mu = T\mu$. Suppose, by way of contradiction, that there is an element of $A(m, \mu) \cup \{\emptyset\}$ that is strictly better than $\mu(m)$ for $\succeq_w$. By individual rationality, $\mu(a) \succeq_a \emptyset$ for all $a \in M \cup W$. So this element cannot be $\emptyset$. Suppose then that $w \in A(m, \mu)$ is such that $w \succ_m \mu(m)$. We show that this implies that (m, w) is a blocking pair, which is a contradiction. Now, $w \neq \mu(m)$ implies that $m \neq \mu(w)$ as μ is a matching. Then $w \in A(m, \mu)$ implies that $m \succ_w \mu(w)$ because $m \succeq_w \mu(w)$ and $\succeq$ is a strict preference. Then (m, w) would form a blocking pair, a contradiction. □

We obtain he following as an application of Tarski's fixed point theorem:

Theorem 77. *The set $S \subseteq V$ of stable matching is nonempty and $(S, \leq)$ is a lattice.*

Corollary 78. *There are two stable matchings, μ_W and μ_M with the property that for any stable matching μ,*

$$\mu_M(m) \succeq_m \mu(m) \succeq_m \mu_W(m)$$
$$\mu_W(w) \succeq_w \mu(w) \succeq_w \mu_M(w)$$

Theorem 79. *The set of stable matchings is a sublattice of $(V, \leq)$, and for any two stable matchings μ and μ', $(\mu \vee \mu')(m)$ is the best for $\succeq_m$ of $\mu(m)$ and $\mu'(m)$, while $(\mu \wedge \mu')(m)$ is the worst for $\succeq_m$ of $\mu(m)$ and $\mu'(m)$.*

Proof. Let ν be the fantasy defined by giving each men the best partner out of μ and μ', and each woman the worst. Then ν is actually a matching: $w = \nu(m)$ and $\nu(w) \neq m$ would imply that m and w would agree as to which is the better matching, μ or μ'. Then the other matching could not be stable; (m, w) would be a blocking pair.

Just to be explicit: if $w = \nu(m) = \mu(m)$, say, and $\nu(w) \neq m$, then $w \succ_m \mu'(m)$ (as $\mu(m) \neq \nu'(m)$ because otherwise we couldn't have $\nu(w) \neq \mu(w)$). Also $\nu(w) \neq \mu(w)$ implies that $m \succ_w \mu'(w)$. Then (m, w) is a blocking pair for μ'. □

13. Games of strategic complements

Games of strategic complements were introduced by Topkis (1979) and Vives (1990). I'll present some results from Milgrom and Roberts (1990) and Milgrom and Shannon (1994).

A n-player ***normal-form game*** is a tuple $(S_i, u_i)_{i=1}^n$, where S_i is a set of ***strategies*** for player i, and

$$u_i : \times_{i=1}^n S_i \to \mathbf{R}$$

is the ***payoff function*** for player i.[14]

The set $S = \times_{i=1}^n S_i$ is the set of ***strategy profiles*** $s = (s_1, \ldots, s_n)$ of the game. We often write $s = (s_i, s_{-i})$, for $s_i \in S_i$ and $s_{-i} \in S_{-i} = \times_{j \neq i} S_j$.

The ***best response correspondence of player*** i is the correspondence $\beta_i : \times_{j\neq i} S_j \to 2^{S_i}$ defined by

$$\beta_i(s_{-i}) = \text{argmax}\{u_i(\tilde{s}_i, s_{-i}) : \tilde{s}_i \in S_i\}.$$

The ***best response correspondence*** (of the game) is the correspondence $\beta : S \to 2^S$ defined by

$$\beta(s) = \times_{i=1}^n \beta_i(s_{-i}).$$

A ***Nash equilibrium*** is a profile $s \in S$ with $s \in \beta(s)$.

An n-player game $(S_i, u_i)_{i=1}^n$ is a ***supermodular game*** if, for all i,

(1) $(S_i, \leq_i)$ is a (nonempty) complete lattice;
(2) $s_i \mapsto u_i(s_i, s_{-i})$ is supermodular, for all $s_{-i} \in S_{-i}$;
(3) $s_i \mapsto u_i(s_i, s_{-i})$ is upper-semicontinuous, for all $s_{-i} \in S_{-i}$;
(4) $(s_i, s_{-i}) \mapsto u_i(s_i, s_{-i})$ has increasing differences.

An n-player game $(S_i, u_i)_{i=1}^n$ is a ***game of (ordinal) strategic complementarities*** if for all i,

(1) $(S_i, \leq_i)$ is a (nonempty) complete lattice;
(2) $s_i \mapsto u_i(s_i, s_{-i})$ is quasi-supermodular, for all $s_{-i} \in S_{-i}$;
(3) $s_i \mapsto u_i(s_i, s_{-i})$ is upper-semicontinuous, for all $s_{-i} \in S_{-i}$;
(4) $(s_i, s_{-i}) \mapsto u_i(s_i, s_{-i})$ has the single-crossing property.

Games of strategic complements have a number of appealing properties: Nash equilibria in pure strategies always exist (and under a slight strengthening of the complementarity assumption, properly mixed equilibria can be

[14]For convenience I assume a finite number of player, but it is easy to generalize many of these results to games with an infinite number of players.

dismissed as ustable: see Echenique and Edlin (2004)). The serially undominated strategies are bounded by the smallest and largest Nash equilibria, and one can easily draw comparative statics conclusions about the set of Nash equilibria of the game.

Example 80. Bertrand competition with (imperfect) substitutes. Suppose that there are n firms, each firm i setting a price $s_i \in [0, M]$ for its product. Payoffs are

$$u_i(s) = (s_i - c_i)D_i(s),$$

where D is the demand function facing firm i, s_i is the price set by firm i, and c_i is its (constant) marginal cost.

If demand is log supermodular, then this is a supermodular game.

Example 81. An arms race game (see Milgrom and Roberts (1990)). Suppose two countries: $n = 2$. Each chooses an amount of military spending $s_i \in [0, M]$, an interval in $\mathbf{R}$; payoffs are given by

$$u_i(s) = h_i(s_i - s_{-i}) - c_i(s_i),$$

where h_i is a monotone increasing and concave function.

Lemma 82. *Let G be a game of strategic complementarities. Then β is monotone increasing (when its range is ordered by the strong set order), and takes subcomplete, sublattice values.*

Proof. Immediate from previous results. □

Theorem 83. *The set of Nash equilibria of a game of strategic complementarities is a nonempty complete lattice.*

Proof. Follows from Lemma 82 and Theorem 71. □

Note that Theorem 83 does not say that the set of Nash equilibria is a sublattice of the product space of agents' strategies. It is easy to construct counterexamples. It is, however, a sublattice for certain two-player games:

Proposition 84. *Let $G = ((S_1, u_1), (S_2, u_2))$ be a two player-game of strategic complements in which each strategy space $(S_i, \leq_i)$ is totally ordered. Then the set of Nash equilibria of G is a nonempty complete sublattice of $S_1 \times S_2$.*

Proof. Let $s = (s_1, s_2), s' = (s'_1, s'_2) \in S_1 \times S_2$ be two Nash equilibria of G, and suppose that they are not ordered in the product order on $S_1 \times S_2$. Without loss of generality, suppose that $s_1 <_1 s'_1$ while $s'_2 \leq_2 s_2$. Then s_2 being a best response to s_1, and s'_2 being a best response to s'_1, by strategic complements, implies that $s_2 = s_2 \vee s'_2$ is a best response to s'_1.

Similarly, $s_1' = s_1 \vee s_1'$ is a best response to $s_2 \geq s_2'$. Thus, (s_1', s_2) is a Nash equilibrium. By analogous reasoning, (s_1, s_2') is a Nash equilibrium. But $(s_1', s_2) = (s_1, s_2) \vee (s_1', s_2')$ and $(s_1, s_2') = (s_1, s_2) \wedge (s_1', s_2')$, using the $\wedge$ and $\vee$ operations in the product lattice $S_1 \times S_2$. So the set of Nash equilibria of G is a sublattice. □

Proposition 84 is due to Echenique (2003).

13.1. **Cournot dynamics.** Let G be a game of strategic complements, and suppose in addition that $s_{-i} \mapsto u_i(s_i, s_{-i})$ is continuous.

Lemma 85. *β has a closed graph.*

Let $\{\underline{s}^k\}_{k=0}^{\infty}$ and $\{\bar{s}^k\}_{k=0}^{\infty}$ be two sequences in S defined by $\underline{s}^0 = \inf \beta(\inf S)$ and $\bar{s}^0 = \sup \beta(\sup S)$, and $\underline{s}^k = \inf \beta(\underline{s}^k)$ and $\bar{s}^k = \sup \beta(\bar{s}^k)$ for $k \geq 1$.

Theorem 86.

$$\inf \mathcal{E} = \lim_{k\to\infty} \underline{s}^k \text{ and } \sup \mathcal{E} = \lim_{k\to\infty} \bar{s}^k.$$

Proof. Note that $\bar{s}^1 \leq \bar{s}^0$, by definition of $\bar{s}^0$. Then

$$\sup \beta(\bar{s}^1) = \bar{s}^2 \leq \bar{s}^1 = \sup \beta(\bar{s}^0),$$

as $\sup \beta$ is monotone increasing. By induction, then $\bar{s}^k$ is a monotone decreasing sequence.

Similarly, $\underline{s}^k$ is a monotone increasing sequence.

Since S is a complete lattice, $\{\underline{s}^k\}_{k=0}^{\infty}$ and $\{\bar{s}^k\}_{k=0}^{\infty}$ are convergent sequences.

For each k,

$$(\bar{s}^k, \bar{s}^{k+1}) \in \{(s, s') : s' \in \beta(s)\},$$

the graph of β. Say that $\bar{s}^* = \lim_{k\to\infty} \bar{s}^k$ Then

$$(\bar{s}^k, \bar{s}^{k+1}) \to (\bar{s}^*, \bar{s}^*),$$

so $s^* \in \beta(s^*)$ as the graph of β is closed. Then $s^* \in NE$.

Moreover, for any $e \in \mathcal{E}$, $e \leq \bar{s}^0$. And therefore $e \leq \sup \beta(e) \leq \sup \beta(s^0) = \bar{s}^1$. By induction then $e \leq \bar{s}^k$. So $e \leq \bar{s}^*$. Therefore $\bar{s}^* = \sup \mathcal{E}$.

The proof that $\inf \mathcal{E} = \lim \underline{s}^*$ is analogous. □

Theorem 86 is due to Milgrom and Roberts (1990). It is easy to interpret this result as an algorithm for finding the smallest and largest Nash equilibria in a game of strategic complements.

The ideas in these results can be used (without assuming continuity) to give a constructive proof of Tarski's theorem (Echenique, 2005), and (in finite games) an algorithm for finding all Nash equilibria in games of strategic complements (Echenique, 2007). The latter application relies on "pruning" the game so as to force Cournot dynamics to move inside the interval between the largest and smallest Nash equilibrium. The iteration of pruning and Cournot dynamics leads one to find all Nash equilibria.

13.2. **Serially undominated strategies.** Let $A \subseteq S$, and

$$U_i(A) = \{s_i \in S_i : \exists \tilde{s} \in A \text{ s.t. } u_i(s_i, \tilde{s}_{-i}) \geq u_i(s_i', \tilde{s}_{-i}) \text{ for all } s_i' \in S_i\}$$

denote the set of best responses by player i to some belief in A. Let $U(A) = \times_{i=1}^n U_i(A)$ and

$$U^I(A) = [\inf U(A), \sup U(A)].$$

Let $\mathcal{U}$ be the set of serially undominated strategies, defined as

$$\mathcal{U} = \cap_{k=0}^{\infty} U^k(S),$$

where $U^k(S) = U(U^{k-1}(S))$, and $U(S) = U(S)$.

Let G be a game of strategic complements, and suppose in addition that $s_{-i} \mapsto u_i(s_i, s_{-i})$ is continuous. The main result we obtain is then the following. The next result is from Milgrom and Roberts (1990) and Milgrom and Shannon (1994).

Theorem 87.

$$\mathcal{U} \subseteq [\inf \mathcal{E}, \sup \mathcal{E}]$$

Proof. We show that

$$U^k(S) \subseteq [\underline{s}^k, \bar{s}^k].$$

This is obviously true for $k = 0$. So suppose that the statement is true for $k - 1$. Then

$$U^k(S) = U(U^{k-1}(S)) \subseteq U([\underline{s}^{k-1}, \bar{s}^{k-1}]) \subseteq [\underline{s}^k, \bar{s}^k],$$

where the first set containment uses the inductive hypothesis and the fact that U is monotone increasing, while the second set containment follows from Lemma 88. Thus

$$\mathcal{U} \subseteq \cap_{k=0}^{\infty} [\underline{s}^k, \bar{s}^k] = [\inf \mathcal{E}, \sup \mathcal{E}].$$

□

Lemma 88. *If $\underline{s}, \bar{s} \in S$ and $\underline{s} < \bar{s}$ then*

$$U^I([\underline{s}, \bar{s}]) = [\inf \beta(\underline{s}), \sup \beta(\bar{s})]$$

Proof. Clearly, $\inf \beta(\underline{s}) \in \beta(\underline{s}) \subseteq U([\underline{s}, \bar{s}])$ and $\sup \beta(\bar{s}) \in \beta(\bar{s}) \subseteq U([\underline{s}, \bar{s}])$. So $\inf U([\underline{s}, \bar{s}]) \leq \inf \beta(\underline{s})$ and $\sup \beta(\bar{s}) \leq \sup U([\underline{s}, \bar{s}])$. Hence,

$$U^I([\underline{s}, \bar{s}]) \supseteq [\inf \beta(\underline{s}), \sup \beta(\bar{s})]$$

We shall prove that $\sup \beta(\bar{s})$ is an upper bound on $U([\underline{s}, \bar{s}])$, thereby showing that $\sup \beta(\bar{s}) = \sup U([\underline{s}, \bar{s}])$.

Let $\bar{\beta}(s) = \sup \beta(s)$.

So let $s \not\leq \bar{\beta}(\bar{s})$ we shall prove that $s \notin U([\underline{s}, \bar{s}])$. There is i with $s_i \not\leq \bar{\beta}_i(\bar{s}_{-i})$. Suppose, towards a contradiction, that $u_i(s'_i, \tilde{s}_{-i}) \leq u_i(s_i, \tilde{s}_{-i})$ for all $s' \in S_i$, for some $\tilde{s} \in [\underline{s}, \bar{s}]$. Then

$$u_i(s_i \wedge \bar{\beta}_i(\bar{s}), \tilde{s}_{-i}) \leq u_i(s_i, \tilde{s}_{-i}).$$

The single-crossing property then implies that

$$u_i(s_i \wedge \bar{\beta}_i(\bar{s}), \bar{s}_{-i}) \leq u_i(s_i, \bar{s}_{-i}).$$

Quasi-supermodularity implies that

$$u_i(\bar{\beta}_i(\bar{s}), \bar{s}_{-i}) \leq u_i(s_i \vee \bar{\beta}_i(\bar{s}), \bar{s}_{-i}).$$

Thus

$$s_i \vee \bar{\beta}_i(\bar{s}) \in \beta_i(\bar{s}_{-i});$$

a contradiction, as $\bar{\beta}_i(\bar{s}) < s_i \vee \bar{\beta}_i(\bar{s})$ when $s_i \not\leq \bar{\beta}_i(\bar{s}_{-i})$.

So we have shown that $s \leq \bar{\beta}(\bar{s})$ for all $s \in U([\underline{s}, \bar{s}])$, and therefore that $\bar{\beta}(\bar{s}) = \sup U([\underline{s}, \bar{s}])$. The proof that $\inf \beta(\underline{s}) = \inf U([\underline{s}, \bar{s}])$ is analogous. □

Remark 89. Many results on games of strategic complements rely only on the monotonicity of best response correspondences.[15] Theorem 87 is an exception.

13.3. **Comparative statics of equilibria.** The results on comparative statics of extremal equilibria are from Sobel (1988) and Milgrom and Roberts (1990). The correspondence principle (an idea originally due to Paul Samuelson) is from Echenique (2002).

Let $(T, \leq')$ be a PO set. For each t, let $G_t = (S_{i,t}, u_{i,t})_{i=1}^n$ be a ***parameterized family of games***. For each $t \in T$, let β_t be the best-response correspondence of game G_t, and

$$\mathcal{E}(t) = \{s \in S_t = \times_{i=1}^n S_{i,t} : s \in \beta_t(s)\}$$

be the set of Nash equilibria of G_t.

[15]This was Vives' original notion of strategic complements in the WP version of his paper; see Echenique (2004) for a discussion and characterization of all games that have this property.

For each i, let $(S_i, \leq_i)$ be a nonempty lattice. Say that G_t is an ***increasing family of games*** if

- $S_{i,t}$ is a sublattice of S_i;
- $((S_{i,t}, \leq_i), u_{i,t})_{i=1}^n$ is a game of strategic complements;
- $t \mapsto S_{i,t}$ is monotone increasing;
- $(s_i, t) \mapsto u_{i,t}(s_i, s_{-i})$ satisfies the single-crossing property.

Theorem 90. *Let $(G_t)_{t \in T}$ be an increasing family of games. Then*

$$t \mapsto \inf \mathcal{E}(t) \text{ and } t \mapsto \sup \mathcal{E}(t)$$

are monotone increasing.

Proof. This is an application of Theorem 72. □

A sequence $\{s_k\}$ is ***Cournot dynamics*** starting at s for game G_t if $s^0 = s$ and $s^k = \beta_t(s^{k-1})$, for $k \geq 1$.

Lemma 91. *Let $\{s_k\}$ be Cournot dynamics starting at s for game G_t. If $s \leq \inf \beta_t(s)$, then $s \leq s^k$ for all $k \geq 0$.*

Proof. The result follows by induction on k. Suppose that $s \leq s^{k-1}$, then

$$s \leq \inf \beta_t(s) \leq \inf \beta_t(s^{k-1}) \leq s^k,$$

as $s^k \in \beta_t(s^{k-1})$ (and $\inf \beta_t$ is monotone increasing). □

A Nash equilibrium e of G_t is ***unstable*** if for every neighborhood V of e there is $s \in V$ and a nighborhood W of e such that for any Cournot dynamics $\{s^k\}$ starting at s we have $s^k \notin W$ for all k.

An increasing family of games $(G_t)_{t \in T}$ is a ***strictly increasing family of games*** if $(s_i, t) \mapsto u_t(s_i, s_{-i})$ satisfies the strict single crossing property.

Let $T \subseteq \mathbf{R}^n$, endowed with the usual order. Suppose that T is an open and convex set. A ***continuous equilibrium selector*** is a function $e : T \to S$ such that $e(t) \in \mathcal{E}(t)$ for all t.

Say that a continuous selectr is ***nowhere monotone*** of $t < t'$ implies that $e(t) \not\leq e(t')$.

Theorem 92 (Correspondence Principle)**.** *If e is a continuous selector, and nowhere monotone. Then $e(t)$ is unstable for every $t \in T$.*

Proof. Fix $t \in T$ and V a neighborhood of $e(t)$. Since e is continuous, there is $t' < t$ such that $e(t') \in V$. Since e is nowhere monotone, $e(t') \not\leq e(t)$.

The family (G_t) is strictly increasing, and

$$e(t') \in \beta_{t'}(e(t'));\ \text{so}\ e(t') \leq \inf \beta_t(e(t'))$$

by the Milgrom-Shannon monotone selection theorem. By Lemma 91 we have that $e(t') \leq s^k$ for any Cournot dynamics starting at $e(t')$ for game G_t. Let $W = \{z \in S : e(t') \leq z\}^c$; note that W is open and $e(t) \in W$. Then $e(t)$ is unstable. □

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